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IMPROVED LOWER BOUNDS TO THE TRAVELLING SALESMAN PROBLEM (*)

by Gianfranco D'ATRI [1]

Abstract. — *In this paper we study the dual of the Travelling Salesman Problem as a source of lower bounds for the primal problem. We consider an extended set of constraints and design a general iterative procedure in the space of the multipliers, i.e. dual variables, which also provides new simple bounds.*

0. INTRODUCTION

The Travelling Salesman Problem (TSP) has been widely studied in order to find optimal as well as approximate solutions, but no satisfactory algorithm yet exists despite its quite elegant integer programming formulation.

In recent years emphasis has been put on its dual program which provides good bounds to sub-problems generated during a Branch and Bound procedure; moreover techniques have been introduced for optimizing, or sub-optimizing, the dual by means other than the Simplex Method, unpractical for the size of (TSP) and of its dual [2]. The first use of such lower bounds has to be ascribed to Little *et al.* [18] indeed they used an approximate solution to the related Assignment Problem Dual.

A more sophisticated and computationally simple bound was introduced by Held and Karp [12, 13] and, in a slightly different way, by Christofides [3]: this method solves related Spanning Tree Problem during an iterative procedure and has been shown to approach the optimal value of the dual, i. e. the best possible bound in this context, in a short amount of computer time.

In [3] another technique was also presented solving a sequence of Assignment Problems: for it good computational results are reported.

In this paper we study the dual of (TSP) in the symmetric case and give a general "bound generator" framework in which preceding procedures are imbedded or extended to the symmetric (TSP).

(*) Received June 1977.
[1] Institut de Programmation, Équipe "Graphes et Optimisation Combinatoire".
[2] We must cite Miliotis [19] who resumed the Symplex for (TSP).

We use an extended set of constraints and a very large dual with multipliers associated with each constraint; but they are handled implicitly, exploiting results due to Edmonds and Johnson [6, 7].

This work is not a computational report or the description of a particular algorithm, but an effort to understand the structure of a special combinatorial problem, providing, as by-product, tools to be used in the design of algorithms.

For a general presentation of (TSP) see [2]; while for concepts and results about Lagrangean relaxation [9] can be consulted.

The paper is so organized: in section 1 the structure of (TSP) and related problems is studied; in section 2 a first lower bound procedure is presented; in section 3 the general one is discussed.

SECTION 1

1.1. Let $G=(N, E)$ be a graph with vertex set $N=\{1, 2, \ldots, n\}$ and let the non-negative (3) vector $c=(c_e)_{e \in E}$ specify the costs assigned to the edges. Each partial graph of G is assigned a cost equal to the sum of the costs of its edges. A *tour* is a cycle passing through each vertex exactly once.

The *Symmetric Travelling Salesman Problem* is that of finding a tour in G of minimum cost.

The following notation is used: for any set of vertices S in a graph, $\omega(S)$ is the set of edges linking S to $N-S$; for a problem $(.)$, $v(.)$ is its optimal value; for a matrix A and an index i, A^i is the column i of A and A_i is the row i of A; $|X|$ denotes the cardinality of set X or the absolute value of number X; $\mathbf{R}_+$ is the set of all non-negative real numbers.

If we associate to each edge **e** the binary variable x_e, the vector $x=(x_e)_{e \in E}$ represents the partial graph G_x with edge set $\{e \in E / x_e = 1\}$; then the problem can be formulated as the integer linear program

$$
\text{(TSP)} \quad \begin{cases} \text{MIN} \quad cx, & \\ \sum_{e \in \omega(S)} x_e \geqq 2 \quad \text{for} \quad S \subset N, & (1) \\ \sum_{e \in E} x_e = n, & (2) \\ x_e = 0 \quad \text{or} \quad 1 \quad \text{for} \quad e \in E, & \end{cases}
$$

indeed any tour has n edges and enters and leaves each subset of vertices at least once.

(3) But this is not a restriction.

In this paper we are concerned with relaxations of (TSP), that is, problems with optimal value less than that of (TSP) and, possibly, solvable by efficient algorithms: two families of them can be constructed by relaxing some constraints.

For a given partition $\mathbf{P}=\{S_1, \ldots, S_r\}$ of N, let us define the $\mathbf{P}$-*contraction* of G to be the graph $G'=(V, E)$ with the same edge set as G and vertices $v_1, v_2, \ldots, v_r$ such that the edge e links v_i to $v_j \neq v_i$ if $e \in \omega(S_i) \cap \omega(S_j)$, or it is a loop of v_i if it links two vertices of S_i.

An example is given in figure 1 where $S_1=\{1, 2, 3\}$, $S_2=\{4, 5\}$, $S_3=\{3\}$.

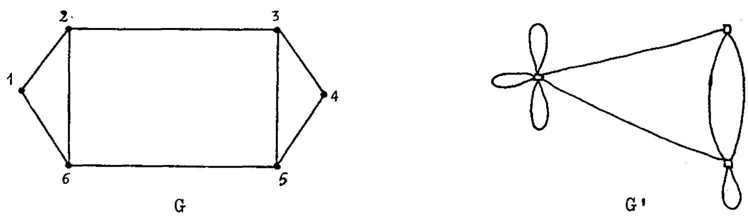

Figure 1.

For a given partition $\mathbf{P}$ and a non-negative cost vector f, we call:

$\mathbf{P}$-*Bicovering* problem, the problem of finding a $(2, 2, \ldots, 2)$-*covering* in G' of minimum cost, that is

$$(\mathrm{B_P}) \qquad \begin{cases} \text{MIN} \quad fx, \\ \sum_{e \in \omega(v)} x_e \geqq 2 \quad \text{for} \quad v \in V, \\ x_e = 0 \quad \text{or} \quad 1 \quad \text{for} \quad e \in E, \end{cases} \tag{3}$$

$\mathbf{P}$-*Spanning* problem, the problem of finding a 1-*tree* in G' of minimum cost, that is

$$(\mathrm{T_P}) \qquad \begin{cases} \text{MIN} \quad fx, \\ \sum_{e \in \omega'(S)} x_e \geqq 1 \quad \text{for} \quad S \subset V - \{v_1\}, & (4) \\ \sum_{e \in \omega(v_1)} x_e = 2, & (5) \\ x_e = 0 \quad \text{or} \quad 1 \quad \text{for} \quad e \in E, \end{cases}$$

where $\omega'(S)=\omega(S)-\omega(v_1)$.

Some comment is in order:

(*a*) in (TSP) there are obvious redundant constraints, e. g. $\omega(S)=\omega(N-S)$; but their reduction is not in the scope of the paper.

(*b*) in (B_P) and (T_P), the variables associated to loops in G' appear only in the objective function: in an optimal solution they take value 0 whenever the cost is non-zero, so they doesn't affect the optimal value;

(*c*) for the special partition $\mathbf{N} = \{\{i\}/i \in N\}$, ($T_N$) is a 1-tree problem in G and (B_N) plus the constraint (2), denoted (B'_N), is a 2-Matching problem in G.

1.2. We have the obvious:

PROPOSITION 1: *Let P-Bicovering, P-Spanning and 2-Matching problems have cost vector $f = c$, then*

$$v(\mathrm{TSP}) \geqq \max(\max_{\mathbf{P}} v(\mathrm{B_P}); \max_{\mathbf{P}} v(\mathrm{T_P}); v(\mathrm{B'_N})),$$

It says that any of the cited problems are relaxations of (TSP) and so an improved lower bound can be obtained by solving as many as possible of them. Among them there is no best one, as showed in the following examples.

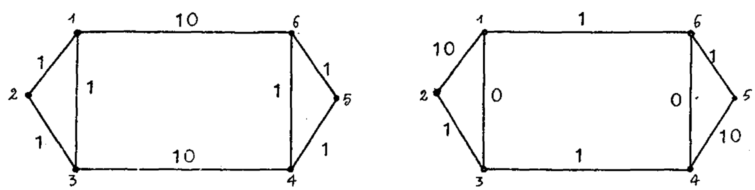

Figure 2.

In all the figures the numbers written near the edges are the costs.

Let us consider the graphs of figure 2, with

$$\mathbf{P} = \{S_1, S_2\} \quad \text{and} \quad S_1 = \{1, 2, 3\}, \quad S_2 = \{4, 5, 6\}.$$

For graph A:

$$v(\mathrm{TSP}) = 24; \quad v(\mathrm{T_N}) = 15; \quad v(\mathrm{B'_N}) = 6; \quad v(\mathrm{B_P}) = v(\mathrm{T_P}) = 20.$$

For graph B:

$$v(\mathrm{TSP}) = 24; \quad v(\mathrm{T_N}) = 4; \quad v(\mathrm{B'_N}) = 22; \quad v(\mathrm{B_P}) = v(\mathrm{T_P}) = 2.$$

There are better ways of using **P**-contractions than that suggested by proposition 1, but this is the scope of section 2.

The introduction of these relaxations is justified by the fact that we know efficient algorithms solving them and giving, as by-product, a dual associated solution. Indeed, if we denote by $B^P x \geqq b^P$ and $T^P x \geqq t^P$ the linear systems defining the convex hulls of the integer solutions to (B_P) and (T_P), respectively, we have:

THEOREM 1: *For any problem* (Q) = (B_P) – resp. (T_P) – *there exists a multiplier vector* $\lambda = \eta^P$ – resp. μ^P – *such that an optimal solution to the Lagrangean relaxation of* (Q), MIN $fx + \lambda (q - Qx)$, $x_e = 0$ *or* 1, *is optimal to* (Q); *where* $Qx \geqq q$ *is the convex hull of integer points in* (Q).

Furthermore, the multipliers are computable in polynomial time.

The first assertion derives from the duality theory for linear programming, while the multipliers can be obtained by solving (B_P) with Blossom algorithm [6, 7] and (T_P) with some modified version of Kruskal's one [15, 20].

The full characterization of the convex hulls of integer solutions for (B_P) and (T_P) has been given by Edmonds [7] and Held-Karp [12], respectively.

Three remarks:

(*d*) theorem 1 is valid also for (Q) = (B'_N);

(*e*) the reduced costs $f' = f - \lambda Q$, which appear in the Lagrangean relaxation, are non-negative and zero for variables with value 1;

(*f*) despite the large number of constraints in (Q) of theorem 1, the multipliers with non-zero value are less than the cardinality of the edge set, $|E|$.

The next theorem gives the tools for exploiting the results of theorem 1, as will be shoxn in the next section. Let be

$$(\mathrm{P}) \quad \begin{cases} \mathrm{MIN} \quad dy, \\ Hy \geqq h, \\ y_j = 0 \quad \text{or} \quad 1, \qquad j \in J, \end{cases}$$

a linear integer problem and, for a conformable non-negative multipliers vector λ,

$$(\mathrm{RP}_\lambda) \quad \begin{cases} \mathrm{MIN} \quad dy + \lambda (h - Hy), \\ y_j = 0 \quad \text{or} \quad 1, \qquad j \in J, \end{cases}$$

its Lagrangean relaxation then:

THEOREM 2, [4]: *If* $\hat{y}$ *is a solution to* (P) *and* y' *to* (RP_λ), *set* $I = \{ j \in J / \hat{y}_j \neq y'_j \}$ *and* $l_j = |d_j - \lambda H^j|$, *then:*

$$v(\mathrm{P}) \geqq v(\mathrm{RP}_\lambda) + \sum_{j \in I} l_j. \tag{6}$$

SECTION 2

2.1. Let us rewrite problem (TSP) by introducing all the constraints needed for the continuous characterization of **P**-contraction problems, but maintaining the integrality condition on the variables, that is

$$
\text{(TSP)} \quad \begin{cases} \text{MIN} \quad cx, & \\ \mathrm{B}^{\mathbf{P}} x \geqq b^{\mathbf{P}} \quad \text{for any partition } \mathbf{P} \text{ of } N & (7\,\mathrm{i}) \\ \mathrm{T}^{\mathbf{P}} x \geqq t^{\mathbf{P}} \quad \text{for any partition } \mathbf{P} \text{ of } N & (7\,\mathrm{ii}) \\ \sum_{e \in E} x_e = n, & (7\,\mathrm{iii}) \\ x_e = 0 \quad \text{or} \quad 1, \qquad \text{for} \quad e \in E, & \end{cases}
$$

then a Lagrangean relaxation of (TSP) is

$$
(\mathrm{TSP}_\lambda) \quad \begin{cases} \text{MIN} \quad cx + \lambda (a - A x), \\ x_e = 0 \quad \text{or} \quad 1, \qquad \text{for} \quad e \in E, \end{cases}
$$

where $A x \geqq a$ is a synthetic representation of constraints (7) and λ a non-negative vector of multipliers associated to all the constraints ([4]).

The optimal value of a particular **P**-contraction problem is the optimal value of a relaxation with all zero multipliers except those associated to the constraints of this problem which are determined as specified in theorem 1.

Now, let us suppose to have solved $(\mathrm{B}'_{\mathrm{N}})$, i.e. selected a multipliers vector λ according to theorem 1; the primal solution $\hat{x}$ solves (TSP_λ) and $(\mathrm{B}'_{\mathrm{N}})$ but not necessarily all the other **P**-contraction problems. If this is the case an optimal, and then feasible, solution $\bar{x}$ to (TSP) must have some components different from $\hat{x}$. We can try to evaluate the summation of (6), or a lower bound to it, for improving the simple bound $v\,(\mathrm{TSP}_\lambda)$.

If the partial graph $G_{\hat{x}}$ is a not tour, then it is constituted by $k > 1$ connected components whose vertex sets are $S_1, \ldots, S_k$; in the latter case, the optimal tour $G_{\bar{x}}$ must enter and leave any of them at least once, so the edges of $F' \subset F$, with

$$
\mathrm{F}' = \{ e \in E / \hat{x}_e = 0 \text{ and } \bar{x}_e = 1 \} \qquad \text{and} \qquad F = \{ e \in E / \hat{x}_e \neq \bar{x}_e \},
$$

are *a* (2, 2, . . . , 2)-covering of the contracted graph with $\mathbf{P} = \{ S_1, \ldots, S_k \}$.

If v_1 is the optimal value of $(\mathrm{B}_{\mathbf{P}})$ with costs $l_e = c_e - \lambda A^e$ ($\geqq 0$, *see* remark *e*), using theorem 2, we have

$$
v(\mathrm{TSP}) \geqq v(\mathrm{TSP}_\lambda) + \sum_{e \in F} l_e \geqq v(\mathrm{TSP}_\lambda) + v_1. \tag{8}
$$

Let us remark, now, that the solution x' of $(\mathrm{B}_{\mathbf{P}})$ produces a partial graph $G_{\hat{x}} \cup G_{x'}$ which is connected or constituted by new connected components

([4]) For equality constraints in (TSP), written as two inequalities, we could take multipliers unrestricted in sign.

whose vertex sets are $S'_1, \ldots, S'_{k'}$, so the same considerations induce us to improve the bound of (8) via the solution of problem ($B_{P'}$) with objective function coefficients $l'_e = l_e - \lambda' A^e$, where $\mathbf{P}' = \{S'_1, \ldots, S'_{k'}\}$ and λ' is the multipliers vector associated to the primal solution x' of (B_P).

More generally, the following is a bounding procedure:

BOUND 1:

Step 0: let $\mathbf{P}$ be a starting partition; $\mathbf{bound} = 0$; $\mathbf{l} = c$; $\mathbf{G} = (N, \emptyset)$;

Step 1: solve (B_P), using cost vector $\mathbf{l}$: let be $v = v(B_P)$, x the primal solution and λ the associated multipliers vector;

Step 2: $\mathbf{bound} = \mathbf{bound} + v$; $\mathbf{l} = \mathbf{l} - \lambda A$; $\mathbf{G} = \mathbf{G} \cup G_x$;

Step 3: If $\mathbf{G}$ is connected STOP, otherwise $\mathbf{P} = \{S_1, \ldots, S_k\}$, where $S_1, \ldots, S_k$ are the vertex-sets of the connected components in $\mathbf{G}$, and GO TO step 1.

PROPOSITION 2: *At any iteration* **bound** *is a valid lower bound to* (TSP).

Moreover, the procedure can be stopped before a connected partial graph is obtained, and then the current bound improved as follows:

after step 2: Solve (T_P) with cost vector $\mathbf{l}$; $\mathbf{bound} = \mathbf{bound} + v(T_P)$, where P is the partition of connected components in $\mathbf{G}$; STOP.

Indeed, a tour contains a 1-tree for the contracted graph, so a lower bound to the reduced cost of edges in $F'' = \{e \in E / x_e \neq \bar{x}_e\}$ is the optimal value of (T_P), (here x is such that $G_x = \mathbf{G}$). Four comments:

(*a*) BOUND 1 is the generalization and extension to the symmetric case of the procedure due to Christofides [3];

(*b*) it is easily realized that the enormous number of constraints (7) is only implicitly considered, for the reduced costs are computed by the mentioned algorithms;

(*c*) the number of variables decreases at each iteration of Bound 1 if multiple edges in contracted graphs are reduced to two for each pair of vertices;

(*d*) in the first iteration, if the initial partition is $\mathbf{N}$, (B'_N) can be used as well as (B_N); but, trivially, $v(B'_N) \geqq v(B_N)$.

An example: Let us consider graph A of figure 2, the solution of (B'_N) gives $w = v(B'_N) = 6$ and the two connected components $S_1 = \{1, 2, 3\}$ and $S_2 = \{4, 5, 6\}$ linked by two edges (1, 6) and (3, 4), of reduced cost $l = 9$ ([5]). Now, for this example, problems (B_P) and (T_P) are equivalent and

([5]) The convex hull of integer points of (B'_N) satisfies the following constraints $x_{12} + x_{23} + x_{31} + x_{16} \geqq 3$ and $x_{45} + x_{56} + x_{64} + x_{34} \geqq 3$, and the optimal solution is obtained assigning them multipliers $\lambda_1 = 1$ and $\lambda_2 = 1$, so $l_{16} = l_{34} = 10 - 1 = 9$.

their solution gives $v_1=18$ and the improved bound for (TSP) becomes $w+v_2=6+18=24$, indeed the optimal value.

2.2. A procedure, analogous to Bound 1, can be conceived starting with the solution of a 1-tree problem: here it is sketched.

Let us suppose that x' solves (T_N) and then a certain Lagrangean relaxation (TSP_λ), but not necessary the other **P**-contraction problems. The graph $G_{x'}$ or is a tour or it contains $k \geqq 1$ vertices, say $i_1, \ldots, i_k$, of degree 1; in the latter case, the optimal tour $G_{\bar{x}}$ must meet each of them with at least one edge other than that of the tree, so the edges of $F \cup \bar{F}$, with

$$F=\{ e\in E/x'_e \neq x_e \} \qquad \text{and} \qquad \bar{F}=\{ e\in E/x'_e=1 \}$$

are a $(2, 2, \ldots, 2)$-covering of G; if v_2 is the optimal value of (B_N) with reduced costs $l_e=c_e-\lambda A^e$,

then
$$v_2 \leqq \sum_{e\in F\cup\bar{F}} l_e.$$

Furthermore, $l_e=0$, for any $e\in\bar{F}$, and so

$$\sum_{e\in F\cup\bar{F}} l_e=\sum_{e\in F} l_e,$$

then

$$v(TSP) \geqq v(TSP_\lambda)+v_2. \tag{10}$$

Now or the solution x'' of (B_N) produces a partial graph $G_{x''}\cup G_{x'}$ without cut-edges, or there are $k' \geqq 1$ sub-sets of vertices $S_1, \ldots, S_{k'}$ linked to $I=N-S_1-\ldots-S_{k'}$ by only one edge and not linked among them; the optimal tour meets each of them at least with another edge, so these edges and those of $G_{x''}\cup G_{x'}$ are a $(2, 2, \ldots, 2)$-covering in the **P**-contracted graph with $\mathbf{P}=\{ I, S_1, \ldots, S_{k'} \}$.

As before, an improved bound can be found by solving (B_P) with reduced costs $l'_e=l_e-\lambda' A^e$, where λ' are the multipliers associated to the optimal solution of (B_N).

Obviously, the procedure can be iterated until a partial graph is found with no cut-edge.

An example: Let us consider figure 3, a minimum cost 1-tree in graph C is the partial graph C_a, with associated multipliers $\lambda_e=0$ for all $e\in E$, and giving the bound $v(TSP_\lambda)=0$; then a minimum reduced cost $(2, \ldots, 2)$-covering (or **N**-Bicovering) is C_b, giving the bound $v(TSP_\lambda)+v_2=0$.

In C_b there are two cut-edges, (2, 7) and (7, 5), so we take $S_1=\{ 1, 2, 3 \}$, $S_2=\{ 4, 5, 6 \}$ and $I=\{ 7 \}$: C_c is the contracted graph in which the minimum

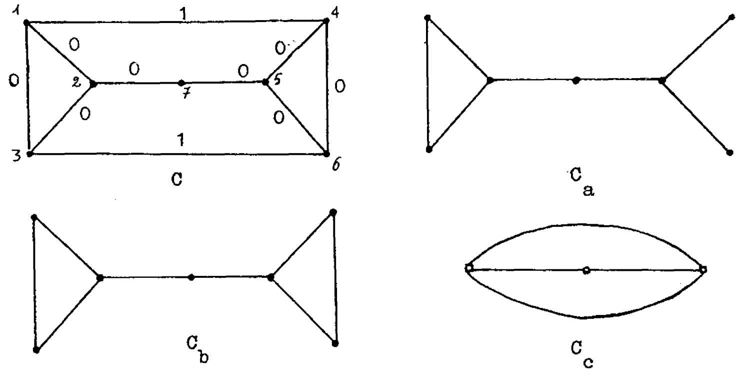

Figure 3.

reduced cost (2, . . ., 2)-covering has value $v'_2=1$, and so the final lower bound is $v(\mathrm{TSP}_\lambda)+v_2+v'_2=1$.

Two remarks:

(*e*) it is easily realized that the solution of a $(\mathrm{T_P})$-problem can be inserted as an intermediate step (only once) in Bound 1, as well as first or terminating one;

(*f*) there are problems for which whatever version of Bound 1 doesn't provide a good bound: for graph in figure 4 the output of Bound 1 is 0, while $v(\mathrm{TSP})=1$ and $v(\mathrm{B'_N})=1$.

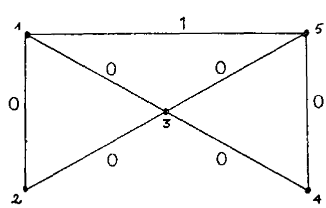

Figure 4.

SECTION 3

3.1. In this section we describe a method for obtaining lower bounds without graph structure considerations, and we show how it is related to those of section 2.

Let be $v(\lambda)=v(\mathrm{TSP}_\lambda)$, $p=$ the number of constraints (7) in (TSP), SUPP(·) = the index-set of non-zero components of vector (·); we call *dual* problem of (TSP):

$$(\mathrm{D}) \qquad \begin{cases} \mathrm{MAX}\ v(\lambda), \\ \lambda\in\mathbf{R}^p_+. \end{cases}$$

The main properties of function $v(\lambda)$ are summarized in the following:

THEOREM 3 [9]: *$v(\lambda)$ is continuous, concave and piece-wise linear;* $v(\mathrm{D}) \leqq v(\mathrm{TSP})$.

A method, called sub-gradient relaxation, has been introduced by Held-Karp [13] and Held *et al.* [10] for problems similar to (D): the idea behind is to generate a sequence $\{\lambda^r\}$ of non-negative multipliers vectors by the rule $\lambda_i^{r+1} = \max(0, \lambda_i^r + \theta^r g_i^r)$, $i=1, \ldots, p$; with g^r a sub-gradient of $v(\cdot)$ at λ^r and θ^r a step-length, chosen according to a rule guaranteeing convergence. In pratice the method is used only for finding approximate solution to (D) which are, obviously, lower bounds to (TSP).

However it is difficult to construct such a sequence for the Travelling Salesman Dual considered by us, due to the enormous number of dual variables, i.e. multipliers, which we hope to treat only implicitly.

The technique described below constructs a sequence of sub-spaces of $\mathbf{R}_+^p$, say $\{Z_r\}$, such that $Z_r \cap Z_{r+1} \neq \emptyset$ and $v(\cdot)$ is optimized over each Z_r, obtaining a sequence λ^r for which $v(\lambda^{r+1}) \geqq v(\lambda^r)$.

Let be G a matrix and ρ a vector of conformable dimensions, for a fixed $\hat{\lambda} \geqq 0$ the problem:

$$(\mathrm{D}_{\hat{\lambda}, G}) \qquad \begin{cases} \mathrm{MAX}\ v(\lambda), \\ \lambda = \hat{\lambda} + \rho\, G, \\ \rho \geqq 0, \end{cases} \tag{11}$$

can be considered the dual of

$$(\mathrm{L}_{\hat{\lambda}, G}) \qquad \begin{cases} \mathrm{MIN}\,(c - \hat{\lambda}\, A)\, x + \hat{\lambda}\, a, & (12) \\ (\mathrm{GA})\, x \geqq G\, a, & (13) \\ 0 \leqq x_e \leqq 1, \end{cases}$$

indeed the Lagrangean relaxation of $(\mathrm{L}_{\hat{\lambda}, G})$ with multipliers ρ is the same as that of (TSP) with multipliers $\lambda = \hat{\lambda} + \rho\, G$.

If we get a multipliers vector ρ' optimal for the dual of $(\mathrm{L}_{\hat{\lambda}, G})$ and satisfying $\lambda' = \hat{\lambda} + \rho'\, G \geqq 0$, then λ' optimizes $v(\cdot)$ over the sub-space

$$Z' = \{\lambda \in \mathbf{R}_+^p / \lambda = \hat{\lambda} + \rho G,\ \rho \geqq 0\}. \tag{14}$$

Otherwise, set

$$\theta'' = \max\{\theta / 0 \leqq \theta \leqq 1,\ \hat{\lambda} + \theta(\lambda' - \hat{\lambda}) \geqq 0\} \tag{15}$$

then $\lambda''=\lambda+\theta''(\lambda'-\hat{\lambda})$ maximizes $v(\cdot)$ over the segment

$$Z''=\{\lambda\in\mathbf{R}^p_+/\lambda=\hat{\lambda}+\theta(\lambda'-\hat{\lambda}),\ 0\leqq\theta\leqq 1\}. \tag{16}$$

For a given sequence $\{G^r\}$ of matrices, whose selection will be discussed later, the procedure can be described as follows:

BOUND 2:

Step 0: Let $\lambda^0\geqq 0$ be a starting multipliers vector; **bound** $=v(\lambda^0)$; $r=0$;

Step 1: Solve (L_{λ^r,G^r}) and set $\lambda^{r+1}=\lambda'$ or λ'', as discussed above;

Step 2: **bound** $=v(\lambda^{r+1})$; $r=r+1$ and GO TO step 1.

PROPOSITION 3: *At any iteration* **bound** *is a valid lower bound to* (TSP) *and* $v(\lambda^r)$ *is a non decreasing sequence of bounds.*

The computational complexity of Bound 2 depends on the dimensions of λ^0 and G^r as well as the complexity of $(L_{\lambda,G})$ problems, but special classes of matrices G^r exist providing an overall polynomial behaviour for Bound 2.

As first, let us point out that multipliers are only implicity handled at the same time; indeed at any iteration the coefficients of (12) – apparently requiring the tremendous product λA – are simply updated by the following recursion

$$(c-\lambda^{r+1}A)=(c-\lambda^r A)-\rho^r G^r A, \tag{17 i}$$

$$\lambda^{r+1}a=\lambda^r a+\rho^r G^r a, \tag{17 ii}$$

where ρ^r is the multipliers vector associated to the optimal solution of (L_{λ^r,G^r}).

Moreover, the determination of θ'' in (15) – apparently requiring the solution of a system of p linear inequalities – reduces to the search for the minimum among $|\,\mathrm{SUPP}(\lambda^r)\cap\mathrm{SUPP}(\lambda^{r+1})\,|$ numbers.

3.2. Let us now study special matrices producing easily solvable sub-problems of the $(L_{\lambda,G})$-type.

Case 1: If G is an $m\times p$ matrix – remember that A is $p\times|E|$ – the resulting problem is an $m\times|E|$ linear program; the computation of its constraints, defined in (13), requires $m\times|E|$ scalar products, each of them obtained with $\max_i|\,\mathrm{SUPP}(G_i)\,|$ multiplications at most.

So, for moderate values of m and $|\,\mathrm{SUPP}(G_i)\,|$, $(L_{\lambda,G})$ is easily solvable.

A remarkable case is $m=1$, then the linear program reduces to a continuous knapsack problem, which is solved by very simple algorithms.

An example: Let be G an $1 \times p$ matrix such that SUPP(G) is the index-set of constraints (3) in (B_P), with $\mathbf{P} = \{S_1, \ldots, S_k\}$, then $(L_{\lambda,G})$ is

$$\begin{cases} \text{MIN}\, c' x + a' \\ \sum_{e \in E} x_e \geqq k, \\ 0 \leqq x_e \leqq 1, \end{cases}$$

and it is optimal value, a lower bound to (TSP), is the summation of the k smallest reduced costs $c'_e = c_e - \lambda A^e$ plus $a' = \lambda a$.

For constructing a sequence of $1 \times p$ matrices to be used in Bound 2, the following general strategy is useful.

At iteration r, let be $\bar{x}$ the solution so defined

$$\bar{x}_e = 1 \quad \text{if } c'_e \leqq 0, \qquad \bar{x}_e = 0 \quad \text{if } c'_e > 0,$$

where $c'_e = c_e - \lambda^r A_e$; select a constraint, say the i-th one, which is not satisfied by $\bar{x}$ and set $G^r_i = 1$ and $G^r_j = 0$ for $j \neq i$.

If the selected constraint is one of (1), say $\sum_{e \in \omega(S)} x_e \geqq 2$, let be c^1 and c^2 the two smallest reduced costs in $\{c'_e / e \in \omega(S)\}$, then

$$v(\lambda^{r+1}) = v(\lambda^r) + c^2 + \max(c^1; 0) > v(\lambda^r),$$

$$\lambda^{r+1}_j = \lambda^r_j \quad \text{if } j \neq i \qquad \text{and} \qquad \lambda^{r+1}_i = \lambda^r_i + c^2.$$

Remark:

(*a*) The previous strategy, in its version for the directed (TSP), was implicitly used in [18], exploiting the constraints (3) of (B_N).

Case 2: If G is an all 0-1 matrix such that:

(i) $|\text{SUPP}(G_j)| = 1$;

(ii) $\text{SUPP}(G_j) \neq \text{SUPP}(G_i)$ for all $i \neq j$;

(iii) $\bigcup \text{SUPP}(G_j)$ = the index set of all the constraints defining the convex hull of integer points of a **P**-contraction problem;

then $GAx \geqq Ga$ defines this convex hull.

So, the linear program $(L_{\lambda,G})$ can be considered implicitly and solved by the algorithms cited in section 1.

Now it is evident the following:

THEOREM 4: *Bound* 1 *is a special case of Bound* 2.

Indeed, at each iteration of Bound 1 the objective function coefficients are the reduced costs of the preceding iteration and the constraints can be obtained from A by an appropriate choice of matrix G^r.

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