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AN $O(n^3)$ WORST CASE BOUNDED SPECIAL *LP* KNAPSACK $(0-1)$ WITH TWO CONSTRAINTS (*)

by Ruy E. CAMPELLO [1] and Nelson MACULAN [2]

Abstract. — *In this paper it is shown that a linear knapsack* $(0-1)$ *problem amended with a nontrivial multiple-choice constraint can be solved by an algorithm requiring running time* $O(n^3)$. *Though not being a new result the approach of the proof is worth reporting for it is based on the nice ideas of geometric complexity and efficient median-finding.*

Keywords : Linear programming; knapsack $(0-1)$.

Résumé. — *Dans cet article on montre que le problème de programmation linéaire du sac-à-dos* $(0-1)$ *dans lequel le nombre de variables valant* 1 *est fixé peut être résolu par un algorithme de complexité temporelle en* $O(n^3)$. *Son intérêt réside dans sa preuve géométrique et l'utilisation de l'algorithme de la médiane.*

Mots clés : Programmation linéaire; sac-à-dos $(0-1)$.

1. THE LINEAR $(0-1)$ KNAPSACK PROBLEM

The problem (*LKP-k*) addressed here is a special linear knapsack with an additional nontrivial constraint. This is an important mathematical programming problem for it arises as a subproblem in various integer programming problem.

Let problem (*KLP-k*) be defined as follows:

$$\text{maximize } x_0 = \sum_{j=1}^{n} q_j x_j \tag{1.1}$$

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subject to:

$$\sum_{j=1}^{n} a_j x_j \leqq T \tag{1.2}$$

$$\sum_{j=1}^{n} x_j = k \tag{1.3}$$

$$0 \leqq x_j \leqq 1, \qquad j=1, 2, \ldots, n \tag{1.4}$$

where q_j, a_j $(j=1, 2, \ldots, n)$ and T are real constants and $0<k<n$ is an integer. We shall assume that the feasible set to (*LKP-k*) is non-empty and thus $v(LKP\text{-}k)<\infty$, where $v(.)$ stands for the value of (1.1) at the optimal solution.

This problem has received much attention in recent literature. Linear $(0-1)$ knapsack problems with generalized upper bound constraints (GUB) and special ordered sets are treated by Johnson and Padberg (1981) where several properties of the convex hull of the associated $0-1$ polytope are derived. Campello and Maculan (1983) report on a subgradient technique for generating a lower bound to (*LKP-k*) when (1.4) is replaced by $x_j \in \{0, 1\}$, $j=1, 2, \ldots, n$. This problem is known to be *NP*-hard although it can generally be solved in pseudo-polynomial time by dynamic programming.

The knapsack *LP* has been shown to have time complexity $O(n)$ using linear median-finding techniques while the multiple-choice knapsack *LP* is $O(n\log n)$ using efficient planar convex hull algorithms. It is also worth mentioning that the multiple-choice knapsack with two linear programming constraints is $O(n \log n + m^2 \log^3 (n/m))$, where m is the number of multiple-choice constraints. The details of these results are described in Dyer (1983 and 1984). For a survey of the literature see Zemel (1980) and Dudzinski and Walukiewicz (1987).

In this paper we report on an algorithm requiring worst case time bound $O(n^3)$ for solving (*LKP-k*). Though not being a new result the approach is worth reporting for it is based on the nice ideas of geometric complexity and efficient median-finding. A complete numerical example with four variables is included.

2. AN $O(n^3)$ ALGORITHM FOR SOLVING (*LKP-k*)

It is known that any linear programming problem with n variables and a fixed number of constraints, say m, is solved in time $O(n^m)$. Problem (*LKP-k*) has $(2+n)$ constraints and therefore, according to this result, could be solved in running time $O(n^{2+n})$. In this case, however, one can do much better than that for the upper bound constraints $0 \leqq x_j \leqq 1$, $j \in N$ on each variable can be replaced by generalized upper-bounded constraints (GUB). The other two constraints are left as before. As such, the complexity reduces to $O(n^2)$.

Yet, in this particular case an interesting approach due to Muller and Preparata (1978) and Dyer (1983) based on the ideas of geometric complexity can be developed so as to prove that there exists an $O(n^3)$ time algorithm.

THEOREM: *There exists an $O(n^3)$ time algorithm for solving problem (LKP-k).*

Proof: Let $(LKD\text{-}k)$ be the dual of $(LKP\text{-}k)$:

$$(LKD\text{-}k)\ \text{minimize}\ w_0 = Tu + ky + \sum_{j=1}^{n} v_j$$

subject to:

$$a_j u + y + v_j \geqq q_j, \quad \text{for all } j \in N$$
$$u \geqq 0 \text{ and } v_j \geqq 0, \text{ for all } j \in N$$

Define $f_j(u, y) = \text{maximum}\ \{0, q_j - a_j u - y\}$, $j \in N$ and $f_0(u, y) = Tu + ky$. Hence, problem $(LKD\text{-}k)$ can be written as follows:

$$\underset{u \geqq 0}{\text{minimize}}\ w_0 = \sum_{j=0}^{n} f_j(u, y)$$

Once $(LKD\text{-}k)$ has been solved the correspondent solution to $(LKP\text{-}k)$ can be retrieved, applying standard linear programming duality techniques in $O(n)$ time, therefore leaving the complexity of any algorithm for $(LKP\text{-}k)$ unaffected. Thus, the solution of the dual rather than the primal will be examined.

On solving $(LKD\text{-}k)$, it suffices to notice that one has to minimize a sum of convex functions because f_j for all $j \in N$ are convex polyhedral functions of (u, y) $(u \geqq 0)$, defining two regions of linearity, see Rochafellar (1970).

The tree-dimensional graph of f_j $(j \in N)$ is a polyhedral surface having two plane faces, one edge and one vertex $(u \geqq 0)$ (see *Fig.* 1).

As such, $E_j = \{(u, y, 0) \mid u \geqq 0 \text{ and } q_j - a_j u - y = 0\}$ is the edge of f_j and $V = (0, q_j, 0)$ its vertex. Notice that both are unbounded though this difficulty can be overcome.

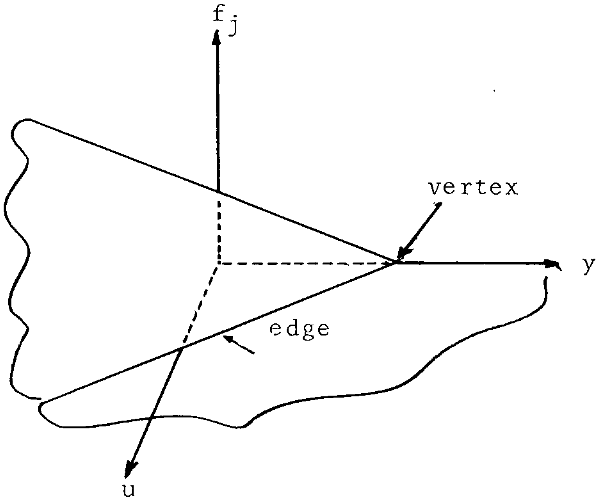

Figure 1

Muller and Preparata (1978) have shown that an optimal solution to $\max_{u \geqq 0} w_0 = \sum_{j=0}^{n} f_j(u, y)$ is either a vertex to one $f_j(j \neq 0)$ or a "pseudovertex" whose cordinates u and y are defined by the intersection of at least two edges.

We do have one vertex for each $f_j(j \neq 0)$ and there are n of such vertices. Since the maximum number of intersections of the n edges is $\binom{n}{2}$, it follows that $(LKD\text{-}k)$ can be solved by evaluating $\binom{n}{2} + n$ times its objective function. Each evaluation requires $O(n)$ time. Therefore, finding the solution to $(LKD\text{-}k)$ can be accomplished by an $O(n^3)$ time algorithm.

Giving an optimal solution to $(LKD\text{-}k)$, the optimal solution to $(LKP\text{-}k)$ can be easily retrieved in $O(n)$ time by standard linear programming duality without increasing the overall complexity.

3. NUMERICAL EXAMPLE

Let $(LKP\text{-}k)$ and its dual $(LKD\text{-}k)$ be as follows:

$$(LKP\text{-}k): \text{maximize } x_0 = 4x_1 + x_2 + 7x_3 + 8x_4$$

subject to: $4x_1+2x_2+3x_3+x_4 \leqq 7$

$$x_1+x_2+x_3+x_4=3$$

$$0 \leqq x_j \leqq 1, \qquad j=1, 2, 3, 4$$

$$(LKD\text{-}k): \text{minimize } w_0=\sum_{j=0}^{4} f_j(u, y)$$

subject to: $u \geqq 0$

where: $f_0(u, y)=7u+3y$

$$f_1(u, y)=\text{maximum}\{0, 4-4u-y\}$$

$$f_2(u, y)=\text{maximum}\{0, 1-2u-y\}$$

$$f_3(u, y)=\text{maximum}\{0, 7-3u-y\}$$

$$f_4(u, y)=\text{maximum}\{0, 8-u-y\}$$

The vertices and edges of f_j are depicted in figure 2.

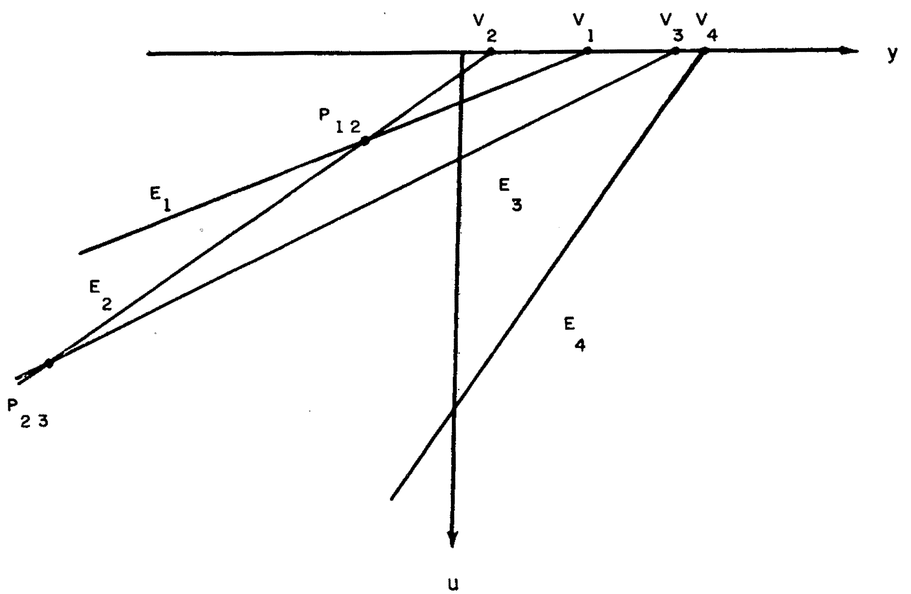

Figure 2

One evaluates w_0 in V_1, V_2, V_3, V_4, P_{12} and P_{23}:

V_j	f_0	f_1	f_2	f_3	f_4	$w_0(V_j)$
(0,4)	12	0	0	3	4	19
(0,1)	3	3	0	6	7	19
(0,7)	21	0	0	0	1	22
(0,8)	24	0	0	0	0	24

	f_0	f_1	f_2	f_3	f_4	$w_0(.)$
$P_{12} \equiv (3/2), -2)$	4,5	0	0	4,5	8,5	17,5
$P_{23} \equiv (6, -11)$	9	0	0	0	13	22

An optimal solution to (*LKD-k*) is then: $u=3/2$, $y=-2$, $v_1=v_2=0$, $v_3=4{,}5$, $v_4=8{,}5$ and $v(LKD\text{-}k)=17{,}5$. Therefore, the primal solution will be: $v(LKP\text{-}k)=17{,}5$, $x_1=x_2=1/2$ and $x_3=x_4=1$.

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