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MULTIPERIOD MEDIANS ON NETWORKS (*)

by Juan A. MESA (1)

Abstract. — *In this paper we consider multiperiod median problems on networks, in which multiple facilities are to be located. In order to reduce the choices available for the location of the facilities a number of important properties are studied. For the absolute multiperiod median the vertex optimality property is established, and for continuous multiperiod problems, the set of vertices and middle points contains at least one median.*

Keywords : Location on networks; multiperiod location; median.

Résumé. — *Dans cet article nous considérons des problèmes des médianes multipériodes sur réseaux, où il faut localiser des facilités multiples. Afin de réduire les élections possibles dans la localisation des facilités, quelques propriétés importantes ont été étudiées. Nous établissons la propriété d'optimalité de sommets pour des médianes multipériode absolues, tandis que pour des médianes multipériode continues nous généralisons la propriété d'optimalité de l'ensemble de sommets et points moyens d'arcs.*

Mots clés : localisation sur réseaux; localisation multipériode; médianes.

1. INTRODUCTION

The absolute p-median problem is to determine the optimum location for p facilities on a network which minimizes the sum of the distances from the vertices to their closest facility.

Since the papers of L. Hakimi [4, 5]; the so called vertex optimality, (*i.e.* the existence of a p-median consisting entirely of vertices) has been studied and extended in different ways. The value of this result is to assist in reducing to a finite set the number of choices available for the facilities.

Some of the generalizations of this property are in the papers of J. Levy [8], A. Goldman [3], L. Hakimi and S. Maheshwari [6], and R. Wendell and A. Hurter [14]; and a concise abstract of these extensions can be consulted

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in the paper of B. Tansel, R. Francis and T. Lowe [12]. Furthermore, that property has been extended to stochastic networks by P. Mirchandani and A. Odoni [11] and to absolute conditional problems by E. Minieka [10]. On the other hand, P. Hansen and M. Labbé [7] consider continuous p-medians which were introduced as absolute general medians by E. Minieka [9], and were defined so that they minimize the sum of the distances from each link to the closest facility. The point-link distances are the distances between the facility and the most distant point on each link allocated to this facility. In that paper they prove a *vertex and middle point of the links* optimality property.

Two kinds of multiperiod or dynamic location problems have been frequently treated: those for which the set of possible location is a finite one [2, 13]; and those which are defined on networks with probabilistic link demands [1].

In this paper we consider multiperiod median problems with at first, non finite location space and without a defined demand structure. The only fact initially assumed is the metric structure of the networks.

While a myopic strategy is implicitly applied when a conditional approach is used to solve the multiperiod problem in this case we consider a global strategy for the duration of a known number of periods of time. Applications to the problem of determining the optimal location of public facilities when a schedule for the opening times is known, can be studied.

Let $N=(V, E)$ be a connected, simple and undirected network, in which $V=\{v_i, \ldots, v_n\}$ is the vertex set and $E=\{(v_i, v_j), \ldots, (v_k, \mathrm{v}_l)\}$ is the edge set.

In order that it makes sense to speak about points on the graph different than nodes, we suppose that there exists an embedding of N in some space S, such that the nodes correspond to distinct points $z_1, \ldots, \mathrm{z}_n$ of S and each edge (v_i, v_j) corresponds to a subset S_{ij} of S. For each subset S_{ij} there exists and injective mapping T_{ij} from the interval [0,1] into S with $T_{ij}(0)=z_i$ and $T_{ij}(1)=z_j$. Denote by Θ_{ij} the inverse function of T_{ij}, such that each point in a set S_{ij} corresponds to a unique point $\Theta_{ij}(x)$ in [0,1]. Then, for each $z \in S_{ij}$ and by using the mapping T_{ij} it is possible to define the embedding sublink $[z, v_i]$ and the shortest path between any two points z and w, with the length of this shortest path $d(z, w)$ called the distance between z and w.

If B is a subset (proper or improper) of the set $\mathcal{N}$ of points of N, α is a nonnegative integer and $|.|$ denote the cardinality, let

$$\mathscr{P}_\alpha(B) := \{ C \subset B;\ |C| = \alpha \}$$

the set of sets of B of cardinality α.

Let now $X \in \mathscr{P}_{\alpha_1}(B) \times \ldots \times \mathscr{P}_{\alpha_T}(B)$, $X = (Z_1, \ldots, Z_T)$ and let $X^{(t)} = (Z_1, \ldots, Z_t)$ be the section of X up to the period t, $t = 1, \ldots, T$. Then, let us define

$$\Delta(v_i, X^{(t)}) := \min_{z \in Z_1 \cup \ldots \cup Z_t} d(v_i, z)$$

DEFINITION 1: The vector $X \in \mathscr{P}_{\alpha_1}(V) \times \ldots \times \mathscr{P}_{\alpha_T}(V)$; $\alpha_1 + \ldots + \alpha_T = p$, $\alpha_1, \ldots, \alpha_T \in \{0, 1, \ldots, p\}$ is a vertex T-period p-median if

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \Delta(v_i, X^{(t)}) \leqq \sum_{t=1}^{T} \sum_{i=1} \Delta(v_i, Y^{(t)})$$

for all $Y \in \mathscr{P}_{\alpha'_1}(V) \times \ldots \times \mathscr{P}_{\alpha'_T}(V)$; such that $\alpha'_1 + \ldots + \alpha'_T = p$, and $\alpha'_1, \ldots, \alpha'_T \in \{0, 1, \ldots, p\}$.

The vertex T-period, p-median problem can be reduced to the determination of the 1-period p-median, because the most efficient way to optimize is to locate all the facilities in the first period. However, this is often not the case because of budgetary restrictions or other causes. This suggests the following definition:

DEFINITION 2: If $\alpha_1, \ldots, \alpha_T$ are nonnegative integers, $X \in \mathscr{P}_{\alpha_1}(V) \times \ldots \times \mathscr{P}_{\alpha_T}(V)$ is called vertex multiperiod $(\alpha_1, \ldots, \alpha_T)$-median if

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \Delta(v_i, X^{(t)}) \leqq \sum_{t=1}^{T} \sum_{i=1} \Delta(v_i, Y^{(t)})$$

for all $Y \in \mathscr{P}_{\alpha_1}(V) \times \ldots \times \mathscr{P}_{\alpha_T}(V)$.

In particular, if $\alpha_1 = \ldots = \alpha_T = 1$, we identify the corresponding X, as vertex multiperiod uniform p-median.

In order to find out this type of p-median, it is necessary to compute the value of the objective function F, $n(n-1)\ldots(n-p+1)$ times in the worst case when $n \geqq p$. However, for the vertex multiperiod $(\alpha_1, \ldots, \alpha_T)$-median problem, we can apply the procedures used in dynamic location as well as those considered in the paper of D. Erlenkotter [2], and the algorithm afterthat established by T. Van Roy and D. Erlenkotter [13].

2. ABSOLUTE MULTIPERIOD MEDIAN

In this section, after defining the absolute multiperiod median, we establish the vertex optimality property for this median, so that the problem is transformed into a vertex one.

DEFINITION 3: *The vector* $X \in \mathscr{P}_{\alpha_1}(\mathscr{N}) \times \ldots \times \mathscr{P}_{\alpha_T}(\mathscr{N})$ *is an absolute multiperiod* $(\alpha_1, \ldots, \alpha_T)$-median if

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \Delta(v_i, X^{(t)}) \leqq \sum_{t=1}^{T} \sum_{i=1} \Delta(v_i, Y^{(t)})$$

for all $Y \in \mathscr{P}_{\alpha_1}(\mathscr{N}) \times \ldots \times \mathscr{P}_{\alpha_T}(\mathscr{N})$.

In particular, If this condition is true when $\alpha_1 = \ldots = \alpha_T = 1$, then Y is called the absolute multiperiod uniform p-median.

THEOREM 1: *For each distribution* $(\alpha_1, \ldots, \alpha_T)$ *of the number of facilities to be located in all the periods, there exists an absolute multiperiod* $(\alpha_1, \ldots, \alpha_T)$-*median consisting entirely of vertices.*

Proof. – Let $X \in \mathscr{P}_{\alpha_1}(\mathscr{N}) \times \ldots \times \mathscr{P}_{\alpha_T}(\mathscr{N})$, $X = (Z_1, \ldots, Z_T)$ be an absolute multiperiod $(\alpha_1, \ldots, \alpha_T)$-median. In order to decompose the objective function, we now define the following sets:

If $z_{\beta\tau} \in Z_\tau$, $1 \leqq \beta \leqq \alpha_\tau$, $1 \leqq \tau \leqq T$, then for $t \geqq \tau$

$V_t(z_{\beta\tau}) := \{ v_i \in V : \Delta(v_i, \mathrm{X}^{(t)}) = d(v_i, z_{\beta\tau})$ and if for some $y \in Z_r$, $1 \leqq r \leqq t$ $d(v_i, y) = \Delta(v_i, X^{(t)})$ then $v_i \notin V_t(y) \}$

Therefore $V_t(z_{\beta\tau})$ is the set of vertices which are assigned to the facility located at $z_{\beta\tau}$, which is at least as close from each vertex in $V_t(z)$ as any other facility already located.

Then, by separating

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \Delta(v_i, X^{(t)}) = \sum_{\beta=1}^{\alpha_1} \sum_{t=1}^{T} \sum_{v_i \in V_t(z_{\beta 1})} d(v_i, z_{\beta 1})$$
$$+ \sum_{\beta=1}^{\alpha_2} \sum_{t=2}^{T} \sum_{v_i \in V_t(z_{\beta 2})} d(v_i, z_{\beta 2})$$
$$+ \ldots + \sum_{\beta=1}^{\alpha_T} \sum_{v_i \in V_i(z_{\beta T})} d(v_i, z_{\beta T})$$

and defining the following weights,

$$h_i(z_{\beta t})=\begin{cases} T-\tau+1, & \text{if} \quad v_i\in V_T(z_{\beta\tau}) \\ T-\tau, & \text{if} \quad v_i\in V_{T-1}(z_{\beta\tau})\setminus V_T(z_{\beta\tau}) \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots \\ 1, & \text{if} \quad v_i\in V_\tau(z_{\beta\tau})\setminus V_{\tau+1}(z_{\beta\tau}) \\ 0, & \text{if} \quad v_i\notin V_\tau(z_{\beta\tau}). \end{cases}$$

we can consider for each $z_{\beta\tau}$; $1\leqq\beta\leqq\alpha_\tau$; $\tau=1, \ldots, T$, the problem, minimize:

$$\sum_{t=\tau}^{T}\sum_{v_i\in V_t(z_{\beta\tau})} d(v_i, x)=\sum_{v_i\in V} \mathrm{h}_i(z_{\beta\tau})\, d(v_i, x), \qquad x\in\mathcal{N}$$

which is a standard absolute (static) 1-median problem.

For this problem, there exists (Ref. [4]) a vertex $v_{\beta\tau}$ which is an optimal solution. Hence,

$$\sum_{v_i\in V} \mathrm{h}_i(z_{\beta\tau})\, d(v_i, z_{\beta\tau})\geqq\sum_{v_i\in V} \mathrm{h}_i(z_{\beta\tau})\, d(v_i, v_{\beta\tau})$$

Therefore, for each $z_{\beta\tau}$ there exists a vertex $v_{\beta\tau}$, such that if

$$V_*=(V_1, \ldots, V_T), \qquad V_\tau=\{v_{1\tau}, \ldots, v_{\alpha_\tau\tau}\}, \qquad \tau=1, \ldots, T$$

then

$$\begin{aligned}\sum_{t=1}^{T}\sum_{i=1}^{n}\Delta(v_i, X^{(t)}) &=\sum_{\beta=1}^{\alpha_1}\sum_{v_i\in V} h_i(z_{\beta 1})\, d(v_i, z_{\beta 1}) \\ &\quad+\sum_{\beta=1}^{\alpha=2}\sum_{v_i\in V} h_i(z_{\beta 2})\, d(v_i, z_{\beta 2}) \\ &\quad+\ldots+\sum_{\beta=1}^{\alpha_T}\sum_{v_i\in V} \mathrm{h}_i(z_{\beta T})\, d(v_i, z_{\beta T}) \\ &\geqq\sum_{\beta=1}^{\alpha_1}\sum_{v_i\in V} \mathrm{h}_i(z_{\beta 1})\, d(v_i, v_{\beta 1}) \\ &\quad+\sum_{\beta=2}^{\alpha_2}\sum_{v_i\in V} h_i(z_{\beta 2})\, d(v_i, v_{\beta 2})\end{aligned}$$

$$+\ldots+\sum_{\beta=1}^{\alpha_T}\sum_{v_i\in V} h_i(z_{\beta T})\,d(v_i, z_{\beta T})$$

$$=\sum_{\beta=1}^{\alpha_1}\sum_{t=1}^{T}\sum_{v_i\in V_t(z_{\beta 1})} d(v_i, v_{\beta 1})$$

$$+\sum_{\beta=1}^{\alpha_2}\sum_{t=2}^{T}\sum_{v_i\in V_t(z_{\beta 2})} d(v_i, v_{\beta 2})$$

$$+\ldots+\sum_{\beta=1}^{\alpha_T}\sum_{v_i\in V_i(z_{\beta T})} d(v_i, v_{\beta T})$$

$$\geqq\sum_{t=1}^{T}\sum_{i=1}^{n}\Delta(v_i, V_*^{(t)}).$$

The last inequality follows by regrouping the vertices allocated to each facility. ■

3. CONTINUOUS MULTIPERIOD MEDIAN

The distance between the point $x\in\mathcal{N}$ and the link (v_k, v_l) is defined as,

$$\delta(x, (v_k, v_l)) := \max_{y\in(v_k, v_l)} d(x, y)$$

and it is called point-edge distance.

If $X\in\mathcal{P}_{\alpha_1}(\mathcal{N})\times\ldots\times\mathcal{P}_{\alpha_T}(\mathcal{N})$, $X=(Z_1, \ldots, Z_T)$, then define

$$D(X^{(t)}, (v_k, v_l)) := \min_{z\in Z_1\cup\ldots\cup Z_t} \delta(z, (v_k, v_l))$$

Definition 4: *The vector* $X\in\mathcal{P}_{\alpha_1}(\mathcal{N})\times\ldots\times\mathcal{P}_{\alpha_T}(\mathcal{N})$ is a continuous multiperiod $(\alpha_1, \ldots, \alpha_T)$-median if for all $Y\in\mathcal{P}_{\alpha_1}(\mathcal{N})\times\ldots\times\mathcal{P}_{\alpha_T}(\mathcal{N})$,

$$F(X) := \sum_{t=1}^{T}\sum_{(v_k, v_l)\in E} D(X^{(t)}, (v_k, v_l)) \leqq \sum_{t=1}^{T}\sum_{(v_k, v_l)\in E} D(Y^{(t)}, (v_k, v_l))$$

and, in particular X is a continuous multiperiod uniform p-median if the last inequality is true when $\alpha_1=\ldots=\alpha_T=1$.

THEOREM 2: *For each* $(\alpha_1, \ldots, \alpha_T)$, *in which* $\alpha_1, \ldots, \alpha_T$ *are nonnegative integers, there exists a continuous multiperiod* $(\alpha_1, \ldots, \alpha_T)$-*median consisting entirely of vertices and/or middle points of the links.*

Proof. – We extend the proof given in (7) for static medians to multiperiod median.

Let $X=(Z_1, \ldots, Z_T)$ be a continuous multiperiod $(\alpha_1, \ldots, \alpha_T)$-median and $z_{\beta\tau} \in Z_\tau$ $1 \leqq \beta \leqq \alpha_\tau$, $1 \leqq \tau \leqq T$, $z_{\beta\tau} \in (v_k, v_l)$.

Let us denote

$$E^t(z_{\beta\tau}) := \{ (v_i, v_j) \in E : D(X^{(t)}, (v_i, v_j)) = \delta(z_{\beta\tau}, (v_i, v_j))$$
$$\text{and if for some } y \in Z_r,\ 1 \leqq r \leqq t$$
$$D(X^{(t)}, (v_i, v_j)) = \delta(y, (v_i, v_j)) \text{ then } (v_i, v_j) \notin E^t(y) \}$$

and let $E^t_k(z_{\beta\tau})$, $E^t_l(z_{\beta\tau})$, and $E^t_{kl}(z_{\beta\tau})$ be the subsets of $E^t(z_{\beta\tau})$ of links accessed through v_k only, v_l only, and both v_k, v_l respectively.

When the facility located in $z_{\beta\tau}$ is moved to v_j, the $|E^t_k(z_{\beta\tau})|$ point-link distances for $(v_i, v_j) \in E^t_k(z_{\beta\tau})$ are reduced by the distance d between $z_{\beta\tau}$ and v_k, the $|E^t_l(z_{\beta\tau})|$ point-link distances for $(v_i, v_j) \in E^t_l(z_{\beta\tau})$ augment by d at most, the $|E^t_{kl}(z_{\beta\tau})|$ point-link distances for $(v_i, v_j) \in E^t_{kl}(z_{\beta\tau})$ do not augment; while the distance $\delta(z_{\beta\tau}, (v_i, v_j))$ does not augment by more than d.

Replacing $z_{\beta\tau}$ by v_k if

$$\sum_{t=i}^{T} |E^t_k(z_{\beta\tau})| > \sum_{t=i}^{T} |E^t_l(z_{\beta\tau})|$$

by v_l if

$$\sum_{t=i}^{T} |E^t_k(z_{\beta\tau})| < \sum_{t=i}^{T} |E^t_l(z_{\beta\tau})|$$

and by m_{kl} the middle-point of the link (v_k, v_l) if

$$\sum_{t=i}^{T} |E^t_k(z_{\beta\tau})| = \sum_{t=i}^{T} |E^t_l(z_{\beta\tau})|,$$

applying this argument to each other facility, and due to the fact that the optimal allocation of the links to the closest facility in each step, does not cause any increasing of the objective function (in the same way as in the theorem 1), the result follows. ■

However, we find that the above theorem is not true if we consider multiperiod multimedian in continuous *location-allocation* problems, in which each point of the graph, is allocated to the closest median, so that different points in the same link can be allocated to different facilities.

For instance, let $G=(V, A)$ be a network, in which

$$V=\{v_1, v_2, v_3, v_4\}, A=\{(v_2, v_1), (v_3, v_1), (v_4, v_1)\}$$

and

$$d(v_2, v_1)=d(v_3, v_1)=d(v_4, v_1)=1$$

and M the set of the middle-points of the links. Let m_{kl}^{13} be the 1/3-point of the link (v_k, v_l); then the pair (v_1, m_{kl}^{13}), $k \neq l$ is better than any pair (x, y) with $x, y \in V \cup M$ for the continuous 2-period uniform location-allocation problem.

In order to eliminate possible candidates to be components of the medians, the following theorem gives a necessary condition for a middle point to be in a multiperiod multifacility.

If $z=z_{\beta\tau} \in (v_k, v_l)$, $z_{\beta\tau} \in Z_\tau$ and $X=(Z_1, \ldots, Z_\tau, \ldots, Z_T)$, let denote by $\underline{X}_{\beta\tau}$ and $\bar{X}_{\beta\tau}$ the vectors obtained from X by replacing $z_{\beta\tau}$ by v_k and v_l respectively.

THEOREM 3: *If the vector $X=(Z_1, \ldots, Z_\tau, \ldots, Z_T)$ in which $z_{\beta\tau}=m_{kl}$ the middle-point of the link (v_k, v_l); is a continuous multiperiod $(\alpha_1, \ldots, \alpha_T)$-median, then*

$$|F(\underline{X}_{\beta\tau})-F(\bar{X}_{\beta\tau})| \leqq (T-\tau+1)\, d_{kl}$$

in which $d_{kl}:=d(v_k, v_l)$

Proof:

$$F(\underline{X}_{\beta\tau}) \leqq F(X)+\left(-\sum_{t=\tau}^{T}|E_k^t(m_{kl})|+\sum_{t=\tau}^{T}|E_l^t(m_{kl})|+(T-\tau+1)\right)\frac{d_{kl}}{2}$$

$$F(\bar{X}_{\beta\tau}) \leqq F(X)+\left(-\sum_{t=\tau}^{T}|E_l^t(m_{kl})|+\sum_{t=\tau}^{T}|E_k^t(m_{kl})|+(T-\tau+1)\right)\frac{d_{kl}}{2}.$$

Since X is a continuous multiperiod median and therefore the second terms of the right hand sides are nonnegative, the condition follows. ■

As a particular case, a similar condition for a p-median in a static context, can be obtained.

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