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EXACT TRANSIENT EXPECTATIONS FOR QUEUEING NETWORKS (*)

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Abstract. – *It is stated, via generating function, that expectations of some generalized birth-and-death processes are solutions of a differential linear system. That holds for BCMP's setting when the number of servers in each station and for each class is infinite. All the rates may depend on a markovian modulator.*

Keywords: Transient, Jackson's theorem, queueing network, birth-and-death process, fictious state.

Résumé. – *On montre, en utilisant la fonction génératrice, que les espérances, pour certains processus de naissances et de morts généralisés, sont solutions d'un système différentiel linéaire. Les hypothèses considérées recouvrent celles du théorème BCMP sous réserve de supposer qu'il y a une infinité de serveurs dans chaque station et pour chaque classe. Tous les taux peuvent dépendre de l'état d'un modulateur (extérieur, état fictif, etc.).*

Mots clés : Régime transitoire, théorème de Jackson, réseau de files d'attente, processus de naissances et de morts, états fictifs.

1. INTRODUCTION

In queueing networks theory there are only a few exact results on the transient or equilibrium distributions when the number of stations is not very small. The most spectacular results are associated with product form solutions for steady-state probabilities, namely Jackson's and BCMP's theorems and their generalizations (*see* [2], [8], [11], [15], etc.). When there is one server in each station, an exciting generalization of Jackson's theorem may be found in [7].

The hypotheses considered here are more restrictive inasmuch as the number of servers in each station and for each class of customers is assumed to be infinite. Moreover, the results stated allow us to compute the exact

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expectations and other moments but they do not allow us to compute all the probabilities associated with the considered network.

On the other hand, our hypotheses are more general than BCMP's ones in the following sense (*see* section 4.2 below for more details):

• all the rates may depend on a modulator (outside, fictious state, etc.);

• there are some birth rates;

• the exact results stated concern not only the steady-state setting but also the transient moments.

Of course, with such hypotheses local balance and product form solution do not hold.

This paper is organized as follows: the notations and the associated hypotheses are given in section 2. Let us recall that our objective is to study a generalized birth-and-death process X: the hypotheses concerning the evolution of this homogeneous Markov process X are specified in section 3.

Some examples are given in section 4. Theorem 1 (section 5) states that the generating function fulfills a partial differential equation. In general, it is not possible to solve this equation; however, theorem 2 (section 7) states that this equation implies that the expectations fulfill an elementary differential linear system, the solution of which being easily computable. The behaviour of X when t goes to the infinity is studied in section 8.

2. NOTATIONS AND HYPOTHESES

For every function λ, μ, etc., we will assume that

$$\lambda_{j,i,k} = \lambda(j, i, k), \qquad \mu_{j,k} = \mu(j, k), \qquad \text{etc.}$$

The choice between these two notations will be only for typing convenience.

Let J and K be two finite sets (actually J and K could be countable with some additional hypotheses). Let $T = \mathbb{R}^+$ be the set of times.

Let (K', K'') be a partition of K. For every element k of K'' let $d(k)$ be a positive integer. Let F be the part of $\mathbb{N}^K$ defined by $n := (n_k)_{k \in K}$ belongs to F is and only if:

– for k element of K', $n(k)$ is nonnegative;

– for k element of K'', $0 \leqq n(k) \leqq d(k)$.

For the use of the elements of K'' (*see* [5]).

Let φ, λ, ν', ν'', ν, μ, τ and ρ be eight real mappings which fulfil the following assumptions:

- φ, λ, ν'', μ, τ and ρ are nonnegative;
- φ is defined on $(J \times J)$ and for every element j of J, $\varphi(j, j) = 0$;
- λ is defined on $(J \times J \times K)$; for every element (i, j, k) of $(J \times J \times K'')$ with $i \neq j$, $\lambda(i, j, k)=0$;
- ν' is defined on $(J \times K)$; if (j, k) belongs to $(J \times K')$, $\nu'(j, k)$ is nonnegative; if (j, k) belongs to $(J \times K'')$, $\nu'(j, k)=-\lambda(j, j, k)/d(k)$;
- ν'' is defined on $(J \times K \times K)$; if (j, k, h) belongs to $(J \times K'' \times K)$, $\nu''(j, k, h)=0$; if (j, k) belongs to $(J \times K)$, $\nu''(j, k, k)=0$;
- ν is defined on $(J \times K \times K)$; if (j, k, h) belongs to $(J \times K \times K)$ with $h \neq k$, $\nu(j, k, h)=\nu''(j, k, h)$ and $\nu(j, k, k)=\nu'(j, k)$;
- μ is defined on $(J \times K)$;
- τ is defined on $(J \times K)$;
- ρ is defined on $(J \times K \times K)$; for every element (j, k, h) of $(J \times K \times K'')$, $\rho(j, k, h)=0$; moreover for every element (j, k) of $(J \times K)$, one has:

$$\sum_{h \in K'} \rho_{j,k,h} = 1 \qquad \text{and} \qquad \rho_{j,k,k} = 0$$

Let us recall that F is defined above; let us put $F' := \mathbb{Z}^K$ where $\mathbb{Z}$ is the set of "all the integers" (negative or nonnegative) and let $F'' := F' \backslash F$.

As in [6], for every element (n, k) of $(F' \times K)$ we will denote by $n+e_k$ the element n' of F' which is defined as follows:

$$n'_k = 1 + n_k \qquad \text{and} \quad \text{for } j \neq k, \ n'_j = n_j$$

The elements $n-e_k$ and $n+e_j-e_k$ of F' are defined in the same way.

3. EVOLUTION OF X

Let us recall that our goal is to compute the expectations associated with a given homogeneous Markov process X. This process X is a $(J \times F)$-valued process. Let a be the nonnegative function defined on $(J \times F, J \times F)$ and associated with the evolution of X (for example *see* [15]), that is: for every pair (e, e') of elements of $(J \times F)$ and for every time t, one has:

$$a(e, e') := \lim_{dt \downarrow 0} \text{Proba}\,[X_{t+dt} = e' | X_t = e]$$

This mapping a is defined as follows; for every element n of F and for every element (i, j, k, h) of $(J \times J \times K \times K)$, one has:

- if $i \neq j$, $a((i, n), (j, n)) = \varphi_{i,j}$;
- if $i \neq j$, $a((i, n), (j, n+e_k)) = \lambda_{i,j,k}$;
- $a((j, n), (j, n+e_k)) = \lambda_{j,j,k} + \nu'_{j,k} n_k + \sum_{h \in K} \nu''_{j,k,h} n_h$
  $= \lambda_{j,j,k} + \sum_{h \in K} \nu_{j,k,h} n_h$;
- $a((j, n), (j, n-e_k)) = \mu_{j,k} n_k$;
- if $h \neq k$, $a((j, n), (j, n-e_k+e_h)) = \tau_{j,k} \rho_{j,k,h} n_k$;

We note that previous hypotheses imply that $a(e, e') = 0$ for every element (e, e') of $((J \times F) \times (J \times F''))$. Actually, for every e element of $(J \times F'')$, the probability for the process X of being in state e is always zero: it is only to facilitate the writing of the equations below that set F' is introduced.

Let p be the nonnegative function defined on $(T \times J \times F')$ by

$$p_t(j, n) = \text{Proba}[X_t = (j, n)]$$

In particular, according to the previous paragraph, if n belongs to $F' \backslash F$, one has $p_t(j, n) = 0$.

Before studying the process X, let us give some examples for which previous assumptions on X are satisfied.

4. EXAMPLES

4.1. Biological examples (*see* [3])

In the setting considered here, the most restrictive hypothesis is that the evolution rates are of the form $\alpha n(k)$ where α is a constant and $n(k)$ is an integer associated with the state n. This hypothesis is very often fulfilled in biology.

In such a context, K'' is usually empty (but this is not necessary). Set J is associated with the outside: its evolution is assumed to be markovian and independent from the inside. A state of the inside is assumed to be characterized by the number $n(k)$ of people in each class k, k belonging to K.

Of course there are a lot of possible classes: children and adults, ill and cured people, male and female, susceptible and infected individuals, etc. The considered populations may be animal, vegetable, bacterian or human populations.

All the rates may depend on the fictious state j and J; in biological examples, j is associated with the outside. The rates may be interpreted as follows: φ is associated with the markovian evolution of the outside; $\lambda(i, j, k)$ is the immigration rate in class k: such an immigration may change the outside state from i to j.

Rates ν' and ν'' are birth rates; the growing speed of a class k may depend on the size of the class k (rate ν') or/and on the size of other classes: for example, birth rate does not depend on the number of children but on the number of adults. μ is a death rate.

Finally, $(\tau\rho)(i, j, k)$ is the rate associated with the evolution of the population from the class k to the class h when the outside is in state j: the decomposition of the function $\tau\rho$ as a product is not a restriction but only to facilitate the writing of the equations below.

4.2. Jackson's or BCMP's network

Consider a Jackson's or BCMP's queueing network (*see* [2], [8], [11], [15], etc.) the family of stations (or classes) of which is the set K. Let us assume that, in every station, the number of servers is infinite : thus the hypotheses given in section 3 are satisfied.

On the other hand, these hypotheses are more general than Jackson'ones (or BCMP'ones) inasmuch as:

- the rates, specially the incoming rate λ, may depend on a "fictious state j" (as in the PH/M/r queue);
- there is a birth rate ν which is not available for BCMP networks;
- this birth rate ν in class k may be proportional to the size of this class k (rate ν') or to the size of another class h (rate ν''); for example, classes may be associated with jobs and tasks; thus, the "birth rate" of new tasks to do is proportional to the number of jobs; of course, in general, there are several kinds of jobs and tasks;
- the study is not only a study of the steady-state law but it gives the exact expectations of the transient law;

At last, let us remark that, when k belongs to K'', the incoming rate in station k is of the form (*see* [5]).

$$\lambda_{j,j,k} \frac{d(k) - n(k)}{d(k)}$$

which is proportional to the "available room" in station k.

4.3. A special case

Let us consider the following setting for which there are four classes: 1. healthy children, 2. ill children, 3. healthy adults, 4. ill adults. Even for such an elementary setting theorem 2 (*see* section 7) seems to be new; this theorem claims that the expectations m fulfill the differential linear system $m' = \lambda + mv$ where λ is defined as above and v is defined as follows:

$$v_{h,h} = \nu'_h - \mu_h - \tau_h \qquad \text{and} \quad \text{for} \quad h \neq j, \quad v_{j,h} = \nu''_{h,j} + \mu_j \, \rho_{j,h}$$

Let us assume that there is one healthy adult at time $t=0$ and let us define the values of the parameters; all parameters are equal to zero except:

$$\lambda_1 = \lambda_3 = 1, \qquad \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1,$$
$$\nu''_{1,3} = 0.50, \qquad \nu''_{2,4} = 0.10,$$
$$\rho_{2,1} = \rho_{2,4} = 0.50, \qquad \rho_{3,4} = \rho_{4,3} = 1,$$
$$\rho_{1,2} = 0.10, \qquad \rho_{1,3} = 0.90,$$
$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1,$$

Then we get

t	m_1	m_2	m_3	m_4
0.50	0.453	0.015	0.829	0.273
1.00	0.623	0.034	0.888	0.376
1.50	0.699	0.047	0.965	0.440
2.00	0.740	0.055	1.025	0.487
		...		
5.00	0.802	0.069	1.149	0.587
7.00	0.807	0.070	1.160	0.597
9.00	0.808	0.070	1.163	0.599

5.

THEOREM 1: *Let us introduce the generating function g (the definition of g is recalled below) and let us put:*

$$s_{j,k}(t, z) = \sum_{i \in J} \lambda_{i,j,k} \, z_k \, g(i) - \sum_{i \in J} \lambda_{j,i,k} \, g(j)$$
$$+ \sum_{h \in K} \nu_{j,k,h} \, z_h \, (z_k - 1) \frac{\partial g(j)}{\partial z(h)}$$

$$+ \mu_{j,k} (1 - z_k) \frac{\partial g(j)}{\partial z(k)}$$

$$+ \sum_{h \in K} \tau_{j,k} \rho_{j,k,h} (z_h - z_k) \frac{\partial g(j)}{\partial z(k)}$$

One has:

$$\frac{\partial}{\partial t} g(j) = \sum_{h \in K} s_{j,k}(t, z) - \sum_{i \in J} \varphi_{j,i}\, g(j) + \sum_{i \in J} \varphi_{i,j}\, g(i)$$

Proof: For every element (j, n) of $(J \times F)$, the forward Kolmogorov (or Chapman-Kolmogorov) equations may be written as follows:

$$\frac{d}{dt} p_t(j, n) = \sum_{k \in K} b_{j,k}(t, n) - \sum_{i \in J} \varphi_{j,i}\, p_t(j, n) + \sum_{i \in J} \varphi_{i,j}\, p_t(i, n)$$

where

$$\begin{aligned} b_{j,k}(t, n) := & -\Big\{\sum_{i \in J} \lambda_{j,i,k} + \sum_{h \in K} \nu''_{j,k,h}\, n_h \\ & + (\nu'_{j,k} + \mu_{j,k} + \tau_{j,k})\, n_k\Big\}\, p_t(j, n) \\ & + \sum_{i \in J} \lambda_{i,j,k}\, p_t(i, n - e_k) \\ & + \Big\{\sum_{h \in K} \nu''_{j,k,h}\, n_h + \nu'_{j,k}(n_k - 1)\Big\}\, p_t(j, n - e_k) \\ & + \mu_{j,k}\, p_t(j, n + e_k)(1 + n_k) \\ & + \sum_{h \in K} \tau_{j,k} \rho_{j,k,h} (1 + n_k)\, p_t(j, n + e_k - e_h) \end{aligned}$$

For every (j, n, z) with j element of J, n element of F and z element of $\mathbb{C}^K$, where $\mathbb{C}$ is the set of the complex numbers, let:

$$c(n, z) := \prod_{k \in K} z_k^{n(k)} \qquad \text{and} \qquad f(t, j, n, z) := p_t(j, n)\, c(n, z)$$

In the sequel we will write $f(j, n)$ instead of $f(t, j, n, z)$.

Multiplying relation above by $c(n, z)$ we get:

$$\frac{\partial}{\partial t} f(j, n) = \sum_{k \in K} r_{j,k}(t, n, z) - \sum_{i \in J} \varphi_{j,i}\, f(j, n) + \sum_{i \in J} \varphi_{i,j}\, f(i, n)$$

where

$$\begin{aligned} r_{j,k}(t,\,n,\,z) \;:=& -\sum_{i\in J}\lambda_{j,i,k}\,f(j,\,n)\\ & -\sum_{h\in K}\nu_{j,k,h}\,z_h\frac{\partial}{\partial z(h)}\,f(j,\,n)\\ & -(\mu_{j,k}+\tau_{j,k})\,z_k\frac{\partial}{\partial z(k)}\,f(j,\,n)\\ & +\sum_{i\in J}\lambda_{i,j,k}\,z_k\,f(i,\,n-e_k)\\ & +\sum_{h\in K}\nu''_{j,k,h}\,z_h\,z_k\,\frac{\partial}{\partial z(h)}f(j,\,n-e_k)\\ & +\nu'_{j,k}\,z_k\,z_k\,\frac{\partial}{\partial z(k)}f(j,\,n-e_k)\\ & +\mu_{j,k}\frac{\partial}{\partial z(k)}f(j,\,n+e_k)\\ & +\sum_{h\in K}\tau_{j,k}\,\rho_{j,k,h}\,z_h\frac{\partial}{\partial z(k)}f(j,\,n+e_k-e_h)\end{aligned}$$

For every element (t, j) of $(T\times J)$ let us introduce the generating function defined by: $g(t,\,j,\,z) \;:=\sum_{n\in F} f(t,\,j,\,n,\,z)$. In the sequel we will write $g(j)$ instead of $g(t, j, z)$.

For every element (t, j) of $(T\times J)$ by summing relation above over all the elements n of F it is easily seen [in particular, let us recall that $\rho(j, k, h)=0$ when $h\in K''$] that we get exactly:

$$\frac{\partial}{\partial t}\,g(j)=\sum_{k\in K} w_{j,k}(t,\,z)-\sum_{i\in J}\varphi_{j,i}\,g(j)+\sum_{i\in J}\varphi_{i,j}\,g(i)$$

where

$$\begin{aligned} w_{j,k}(t,\,z) \;:=& -\sum_{i\in J}\lambda_{j,i,k}\,g(j)-\sum_{h\in K}\nu_{j,k,h}\,z_h\frac{\partial g(j)}{\partial z(h)}\\ & -(\mu_{j,k}+\tau_{j,k})\,z_k\frac{\partial g(j)}{\partial z(k)}\end{aligned}$$

$$+\sum_{i\in J}\lambda_{i,j,k}\,z_k\,g(i)+\sum_{h\in K}\nu_{j,k,h}\,z_h z_k\frac{\partial g(j)}{\partial z(h)}$$
$$+\mu_{j,k}\frac{\partial g(j)}{\partial z(k)}+\sum_{h\in K}\tau_{j,k}\,\rho_{j,k,h}\,z_h\frac{\partial g(j)}{\partial z(k)}$$

Let us recall that we have, for every element (j, k) of $(J \times K)$:

$$\sum_{h\in K'}\rho_{j,k,h}=1$$

This property implies $w=s$ and that ends the proof.

6.

Corollary (Transient behaviour of the fictious state): *Let θ be the element of $\mathbb{C}^K$ such that $\theta(k)=1$ for every element k of K. For every element (t, j) of $(T \times J)$ let us define*

$$\psi(t, j) := g(t, j, z=\theta)=\sum_{n\in F}p(t, j, n)$$

One has:

$$\frac{d}{dt}\psi(t, j)=-\sum_{i\in J}\varphi'_{j,i}\,\psi(t, j)+\sum_{i\in J}\varphi'_{i,j}\,\psi(t, i)$$

where φ' is the nonnegative function defined on $(J \times J)$ by

$$\varphi'_{ij} := \varphi_{ij}+\sum_{k\in K}\lambda_{i,j,k}$$

Proof and remarks: Let us remark that $\psi(t, j)$ is the probability at time t associated with the state j in J.

For $z=\theta$, a lot of terms cancels in relations above and below: actually this fact is crucial in this study. Namely, the relation given above is an immediate corollary of theorem 1.

Thus the family of functions $\psi(t, j)$, where t is the variable and j is an indice, fulfils a classical differential linear system with constant coefficients: the general solution of such a system is well known, perfectly defined and easily computable (if J is not too large).

Actually this result is not surprising: the evolution of the state j in J (outside or fictious state) does not depend on the state n in F.

7.

THEOREM 2 (expectations): *For every element (t, j, m) of $(T \times J \times K)$ let $\eta(t, j, m)$ be the value for $z=\theta$ of the derivative with respect to $z(m)$ of $g(t, j)$:*

$$\eta(t, j, m) := \frac{\partial}{\partial z(m)} g(t, j, z=\theta)$$

This function η fulfills the following differential linear system:

$$\begin{aligned}
\frac{d}{dt}\eta(t, j, m) = &-\sum_{i\in J}\varphi_{j,i}\,\eta(t, j, m) + \sum_{i\in J}\varphi_{i,j}\,\eta(t, i, m)\\
&+\sum_{i\in J}\lambda_{i,j,m}\,\psi(i) + \sum_{i\in J}\sum_{k\in K}\lambda_{i,j,k}\,\eta(t, i, m)\\
&-\sum_{i\in J}\sum_{k\in K}\lambda_{j,i,k}\,\eta(t, j, m)\\
&+\sum_{h\in K}\nu_{j,m,h}\,\eta(t, j, h) - \mu_{j,m}\,\eta(t, j, m)\\
&+\sum_{k\in K}\tau_{j,k}\,\rho_{j,k,m}\,\eta(t, j, k)
\end{aligned}$$

Proof: Let us remark that $\eta(t, j, m)$ is the expectation, at time t, for state j in J, of the size of population m (with m element of K).

By derivating, theorem 1 gives:

$$\begin{aligned}
\frac{\partial}{\partial z(m)}\left\{\frac{\partial}{\partial t} g(t, j, z)\right\} = &\sum_{k\in K}\frac{\partial}{\partial z(m)} s_{j,k}(t, z)\\
&-\sum_{i\in J}\varphi_{j,i}\frac{\partial g(t, j, z)}{\partial z(m)} + \sum_{i\in J}\varphi_{i,j}\frac{\partial g(t, i, z)}{\partial z(m)}
\end{aligned}$$

For $z=\theta$, this relation implies theorem 2 (there are several cancellations):

Let us recall that ψ may be computed as explained in section 6: then, the above relations are a classical differential linear system with constant coefficients and "with a second member where ψ appears". Thus the conclusion is the same one as in the previous section. Moreover the other moments may be studied exactly in the same way.

8. STEADY-STATE RELATIONS

What happens in previous relations when t goes to the infinity? The hypotheses given in section 2 being very general it is not possible to list all the families of hypotheses for which X is ergodic and its steady-state expectations are well defined.

But if one assumes that X is ergodic, all relations in section 5 are available in the following way: $p(t, j, n)$ has to be replaced by the steady-state probability $q(j, n)$ and $dp(t, j, n)/dt$ has to be replaced by zero. For example, in particular, let us put:

$$\xi(j) := \lim_{t \to \infty} \psi(t, j) = \sum_{n \in F} q(j, n)$$

Then the corollary in section 6 imply that, for every element j of J, one has:

$$\xi(j) \sum_{i \in J} \varphi'_{j,i} = \sum_{i \in J} \xi(i) \varphi'_{i,j}$$

Moreover, of course, $\sum_{i \in J} \xi(i) = 1$.

If the matrix φ' is irreducible, this linear system has one and only one solution (for example, *see* [15]).

In the same way, if X is ergodic and if the expectations are well defined when t goes to the infinity (that is the associated families are summable), a classical study on analytical functions tells us that theorem 2 is still available in the following way: $d\eta(t, j, m)/dt$ has to be replaced by zero and $\eta(t, j, m)$ has to be replaced by the steady-state expectation of class m when the outside (or fictious state) is in state j.

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