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# APPLICATION OF MARKOV CHAINS TO THE ANALYSIS OF INTERZONAL FLOWS IN A NETWORK (*)

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Abstract. – *In this work we consider the motion of items between the nodes of a network as a Markov Chain. Consequently, we show that the distribution of the trips (matrix O-D) can be obtained from estimated transition flows rather than, as usually, from imprecise census data or by maximizing global utility functions, whose meaning may be not very clear.*

*The proposed model, by means of an appropriate choice of the states, fits the case in which the transition probability to any node depends on the nodes reached at the two previous times.*

*We also show how to obtain several interesting quantities, as the stationary distribution of the items among the nodes, the flows and the speed of convergence to the stationarity.* © Elsevier, Paris

Keywords: Markov chains, flows.

Résumé. – *Dans cet article le mouvement de plusieurs objets, qui se déplacent parmi les nœuds d'un réseau, est traité comme une chaîne de Markov. De cette façon la distribution des voyages (matrice « Origin-Destination ») peut être dérivée à partir des flots de déplacement estimés. Une telle méthode est préférable aux méthodes habituelles, qui consistent à utiliser des données de recensement (nécessairement imprécises) ou bien à maximiser des fonctions globales d'utilité, dont la signification n'est pas toujours très claire.*

*Le modèle proposé, avec un choix approprié des états, permet de représenter le cas où la probabilité de transition d'un nœud à l'autre dépend des nœuds atteints aux deux instants précédents.*

*On obtient notamment plusieurs quantités intéressantes, comme les flots, la distribution stationnaire des objets parmi les nœuds et la vitesse de convergence vers cette distribution.* © Elsevier, Paris

Mots clés : Chaînes de Markov, flots.

## 1. INTRODUCTION

This paper presents a model for the traffic flow of $N$ items among the nodes of a network; the system is observed at equidistant times $t$, $t+1$, $t+2, \ldots$ We suppose that the transitions between nodes are instantaneous.

The aim of this study is the vehicular traffic. We will refer to some division into zones of the area to be analized; consequently the items stand for the

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vehicles and the nodes for the zones. Nevertheless the results obtained can be applied to whatever flow.

In the past years a lot of Authors treated of the problem of the traffic flow [2, 3, 4]. Generally they solved the problem by calculating the trip distributions, which is represented by the so-called origin-destination (in short, 0-D) matrix. Such a matrix is obtained by maximizing a global trip function or by making census. Nevertheless, in the former case the results may be not very significant, because usually the maximized function is defined in a little arbitrary way; in the latter one, since the census does not consider the occasional (namely, not foreseeable) trips, the resulting matrix is not very close to the real traffic.

These drawbacks can be avoided by representing the traffic flow by means of a stochastic process which, rather than to resort to the trip distribution, refer to the probabilities of the shifts between contiguous nodes. This has been done in a recent work [6], where the traffic flow is represented by a Markov Chain model, so that the construction of the classic 0-D matrix is avoided. This model nevertheless is subject to a heavy limitation, because the node reached at the time $t$ is supposed to depend only on that reached at the time $t-1$, by neglecting the dependence from those reached at the times $t-2$, $t-3$, etc.

Here we show that, by means of a simple transformation, the case of dependence on the nodes at times $t-1$ and $t-2$ can be treated by means of an usual Markov Chain. In this way we reach the aim of both avoiding the O-D matrices and obtaining more precise results. Furthermore, by this model we can also draw the O-D matrices perhaps by a more satisfactoring way, because they are derived from transition probabilities, estimated from observed flows.

This procedure can be extended to the dependence of the node from the ones reached not only at the times $t-1$ and $t-2$, but also at the times $t-3$, $t-4$, etc.; nevertheless this has little interest, because the formulas and the parameters estimation would be very heavy.

## 2. THE MODEL

Let us suppose that the $N$ circulating items are moving between $n$ nodes $Z_1, Z_2, \ldots, Z_n$. We will say that an item is in the state $s_{ij}$, iff at present is in the node $Z_j$ and at the preceding time was in the node $Z_i$. Therefore an item is in the state $s_{ii}$ iff it is staying in the node $Z_i$ (namely it is in $Z_i$ at the present and at the preceding time).

We observe that $s_{hk}$ is not a state for our process, iff the nodes $Z_h$ and $Z_k$ are not communicating (*i.e.*, iff it is not possible to shift from $Z_h$ to $Z_k$ in a single step); therefore the different states can be less than $n^2$.

Let $S(t)$ denote the state of the item at the time $t$; the sequence of the states $\{S(t)\}_{t=-\infty}^{+\infty}$ is a stochastic process, which we denote by $\Pi$. We suppose this process is stationary; therefore the transition probabilities do not depend on $t$.

Let $\pi_{ihj}$ denote the transition probability from $s_{ih}$ to $s_{hj}$ in a single step (namely, in an unitary step of time). If $Z(t)$ denotes the node reached by the item at the time $t$, we have:

$$\begin{aligned}\pi_{ihj} &= P[S(t) = s_{hj} | S(t-1) = s_{ih}]\\ &= P[Z(t) = Z_j,\ Z(t-1) = Z_h | Z(t-1) = Z_h,\ Z(t-2) = Z_i]\\ &= P[Z(t) = Z_j,\ Z(t-1) = Z_h,\ Z(t-2) = Z_i]/\\ &\quad P[Z(t-1) = Z_h,\ Z(t-2) = Z_i]\\ &= P[Z(t) = Z_j | Z(t-1) = Z_h,\ Z(t-2) = Z_i].\end{aligned}$$

Similarly we can show that, whatever the indices $j$, $h$, $i$, $k, \ldots$ may be, we have:

$$\begin{aligned}&P[S(t) = s_{hj} | S(t-1) = s_{ih},\ S(t-2) = s_{ki}, \ldots]\\ &\quad = P[Z(t) = Z_j | Z(t-1) = Z_h,\ Z(t-2) = Z_i,\ Z(t-3) = Z_k, \ldots].\end{aligned}$$

If $\Pi$ is a Markovian process, namely if the conditional transition probability from $s_{ih}$ to $s_{hj}$ $(\forall\, i,\ h,\ j)$ does not depend on the previous states, we find:

$$\begin{aligned}&P[S(t) = s_{hj} | S(t-1) = s_{ih},\ S(t-2) = s_{ki}, \ldots]\\ &\quad = P[S(t) = s_{hj} | S(t-1) = s_{ih}].\end{aligned}$$

Because of the preceding formulas we have:

$$\begin{aligned}&P[Z(t) = Z_j | Z(t-1) = Z_h,\ Z(t-2) = Z_i,\ Z(t-3) = Z_k, \ldots]\\ &\quad = P[Z(t) = Z_j | Z(t-1) = Z_h,\ Z(t-2) = Z_i].\end{aligned}$$

Let us now denote by $\Pi^*$ the process in which as the state at the time $t$ we take the node $Z(t)$; this process is defined by the sequence $\{Z(t)\}_{t=-\infty}^{+\infty}$.

The result just obtained shows that, if $\Pi$ is a Markov Chain, then in $\Pi^*$ the state at the time $t$ depends on the two previous states, and vice-versa.

Therefore, if the node reached at the time $t$ depends on those reached at the times $t-1$ and $t-2$, $\Pi$ is a Markov Chain, differently from $\Pi^*$.

In the next Section we will prove that, by referring us to the properties of the process $\Pi$, we can recognize several properties of the process $\Pi^*$.

Let us now suppose that, $\forall\, i,\, j,\, h,\, k$, the state $s_{hk}$ of $\Pi$ can be reached from the state $s_{ij}$, in a finite number of steps; this implies that all the states are communicating; therefore the process $\Pi$ admits a stationary density of state, which we denote by $\underline{\pi}$ (column vector).

The components of $\underline{\pi}$ are the stationary probabilities $\pi_{ij}$ of the states $s_{ij}$ $(i,\, j = 1, 2, \ldots, n)$. The order of the components of $\underline{\pi}$ is arbitrary, but that of the transition matrix $\mathbf{P}$ depends on it: if $\pi_{jh}$ is the $r$-th component of $\underline{\pi}$ and $\pi_{ij}$ the $c$-th one, then $\pi_{ijh}$ must be the component of $\mathbf{P}$ in the $r$-th row and $c$-th column.

With these positions, the equation for the stationary density is:

$$\underline{\pi} = \mathbf{P}\,\underline{\pi}. \tag{1}$$

Let $\omega_{ij}(t)$ denote the probability that an item, chosen at random, reaches at the time $t$ the state $s_{ij}$ in the process $\Pi$; let also $\omega_i(t)$ denote the probability that it reaches $Z_i$ in $\Pi^*$. It must be:

$$\omega_i(t) = \sum_{j=1}^{n} \omega_{ij}(t+1) = \sum_{j=1}^{n} \omega_{ji}(t), \qquad \forall\, i. \tag{2}$$

Let $\underline{\omega}(t)$ be the vector with components $\omega_{ij}(t)$, ordered as $\underline{\pi}$. It is well known that:

$$\underline{\omega}(t+1) = \mathbf{P}\,\underline{\omega}(\mathbf{t}).$$

If $\underline{\omega}(t_0)$ is known for some $t_0$, the repeated application of this equation allows us to obtain $\underline{\omega}(t)$ for any $t > t_0$.

Since $\omega_{ij}(t) \to \pi_{ij}$ as $t \to +\infty\, (\forall\, i,\, j)$, we conclude that $\omega_i(t) \to \pi_i$, where:

$$\pi_i = \sum_{j=1}^{n} \pi_{ij} = \sum_{j=1}^{n} \pi_{ji}, \qquad \forall\, i. \tag{3}$$

Since the probability of the certain event is 1, we have:

$$\sum_{h=1}^{n} \pi_{ijh} = 1, \qquad \forall\, i,\, j. \tag{4}$$

These are the usual relations which holds between the components of any transition matrix.

Because of (4), one of the rows (and of the columns) of $\mathbf{P}$ can be expressed as a linear function of the others. Therefore the system (1) is underdetermined, but it is determined if one of his equations is replaced by:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} = 1, \tag{5}$$

which is the well known normalization formula.

The relations (3) are not further conditions on $\underline{\pi}$; indeed from (1) we have:

$$\pi_{ij} = \sum_{h} \pi_{hij} \cdot \pi_{hi}, \qquad \forall\, i,\, j.$$

From this, by summing with respect to $j$ and taking into account (4), we have:

$$\begin{aligned} \sum_{j} \pi_{ij} &= \sum_{j} \sum_{h} \pi_{hij} \cdot \pi_{hi} = \sum_{h} \pi_{hi} \cdot \sum_{j} \pi_{hij} \\ &= \sum_{h} \pi_{hi} = \sum_{j} \pi_{ji}, \qquad \forall\, i. \end{aligned}$$

This shows that (3) are a consequence of (1) and (4), and not further conditions.

## 3. SOME IMPORTANT RELATIONS

If we know $\underline{\omega}(t)$ and the total number $N$ of the items, we easily find the expected number of items in every state $\pi_{ij}$. The expected number of items in every node is obtained by summing on all the states $\pi_{ij}$ which have that node as their second node. If we can admit that between $t$ and $t + \tau$ ($t$ and $\tau$ integers $> 0$) $\mathbf{P}$ is constant, then the state distribution $\underline{\omega}(t + \tau)$ at the time $t + \tau$ is:

$$\underline{\omega}(t + \tau) = \mathbf{P}^{\tau}\, \underline{\omega}(t).$$

We can also obtain the probabilities of the runs (routings) in stationarity conditions for the state density. Let us calculate, for instance, the conditional probability of the run $Z_i \to Z_j \to Z_h \to Z_k$; we have:

$$\begin{aligned}
&P[Z_i \to Z_j \to Z_h \to Z_k | Z(t-3) = Z_i] \\
&\quad = P[S(t) = s_{hk},\ S(t-1) = s_{jh},\ S(t-2) = s_{ij} | Z(t-3) = Z_i] \\
&\quad = P[S(t) = s_{hk} | S(t-1) = s_{jh},\ S(t-2) = s_{ij}] \\
&\qquad \times P[S(t-1) = s_{jh},\ S(t-2) = s_{ij} | Z(t-3) = Z_i] \\
&\quad = P[S(t) = s_{hk} | S(t-1) = s_{jh}] \cdot P[S(t-1) = s_{jh} | S(t-2) = s_{ij}] \\
&\qquad \times P[S(t-2) = s_{ij} | Z(t-3) = Z_i] \\
&\quad = \pi_{jhk} \cdot \pi_{ijh} \cdot \pi_{ij}/\pi_i.
\end{aligned}$$

More generally, we obtain:

$$\begin{aligned}
&P(Z_{i_1} \to Z_{i_2} \to \ldots \to Z_{i_{h-1}} \to Z_{i_h} | Z(t-h+1) = Z_{i_1}) \\
&\quad = \pi_{i_1 i_2 i_3} \cdot \pi_{i_2 i_3 i_4} \cdot \cdots \cdot \pi_{i_{h-2} i_{h-1} i_h} \cdot \pi_{i_1 i_2}/\pi_{i_1}.
\end{aligned}$$

Let $N_{ij}$ be the number of items in the state $s_{ij}$ (namely, the number of transitions from $Z_i$ to $Z_j$ in the process $\Pi^*$, during a unit of time). Let also $\nu_{ij}$ be the expected value of $N_{ij}$ and $\eta_{ij}$ its variance. Since in stationary conditions $N_{ij}$ is a binomial random variable with parameters $N$ and $\pi_{ij}$, we have:

$$\nu_{ij} = N \cdot \pi_{ij}; \qquad \eta_{ij} = N \cdot \pi_{ij} \cdot (1 - \pi_{ij}).$$

If $\Pi^*$ is a Markov Chain, we are in a particular case of the preceding one; therefore the two results found are still valid. Let $\pi_{j|i}$ denote the conditional probability of reaching the node $Z_j$ if the item is in the node $Z_i$ at the preceding time; this probability is defined by:

$$\pi_{j|i} = \frac{\pi_{ij}}{\pi_i},$$

with $\pi_i$ given by (3). Then the preceding formulas can be put in the form:

$$\nu_{ij} = N \cdot \pi_i \cdot \pi_{j|i}; \qquad \eta_{ij} = N \cdot \pi_i \cdot \pi_{j|i} \cdot (1 - \pi_i \cdot \pi_{j|i}).$$

We note that, if $\Pi^*$ admits a stationary density $\underline{\pi}^*$, $\pi_i$ is the $i$-th component of $\underline{\pi}^*$ and $\pi_{j|i}$ is the component $i$, $j$ of the transition matrix.

Finally, let $N_i$ be the number of items in the node $Z_i$; if $\nu_i$ is the mean value of $N_i$ and $\psi_i$ its variance, since $N_i$ is binomial with parameters $N$ and $\pi_i$ we have:

$$\nu_i = N \cdot \pi_i, \qquad \psi_i = N \cdot \pi_i (1 - \pi_i); \qquad \forall i.$$

## 4. FURTHER CONSIDERATIONS ON THE STATIONARITY

If we introduce the conditional densities $\pi_{i|j}$, (3) can be expressed as:

$$\pi_i = \sum_{j=1}^{n} \pi_j \cdot \pi_{i|j} \qquad \forall i,$$

namely:

$$\underline{\pi}^* = P^* \underline{\pi}^*, \tag{6}$$

where by $\underline{\pi}^*$ we mean the vector with components $\pi_i$ $(i = 1, 2, ..., n)$ and by $P^*$ the matrix which has in the entry $k$, $h$ $(h, k = 1, 2, ..., n)$ the transition probability $\pi_{k|h}$.

This is the equation which defines the stationary density in a Markov Chain with $P^*$ as its transition matrix. On the other hand, if we treat $\Pi^*$ as a Markov Chain, when $\Pi$ (hence also $\Pi^*$) has reached the stationarity the transition matrix for $\Pi^*$ is just $P^*$, hence (6) is the equation for the stationary density $\underline{\pi}^*$.

Consequently, **the stationary density $\underline{\pi}^*$ can be obtained by treating the process $\Pi^*$ as a Markov Chain, with transition matrix $P^*$.**

We now will investigate on the speed of convergence to the stationarity. Let $P$ be the transition matrix of a regular Markov Chain; as it is well known, the equation which defines the stationary density is $P\underline{v} = \underline{v}$. Whatever the density $\underline{x}$ can be, for the well known Markov theorem as $n$ diverges $P^n\underline{x}$ converges to $\underline{v}$ and $P^n$ converges to a matrix $Q$ whose columns are all equal to $\underline{v}$. This matrix has rank 1, therefore it has a single eigenvalue, which must be 1 since it is $Q\underline{v} = \underline{v}$.

Let $M$ be the matrix which diagonalizes $P$; then:

$$MPM^{-1} = D, \tag{7}$$

where $D$ is the diagonal matrix which has as diagonal elements the eigenvalues of $P$. We find:

$$\begin{aligned} MP^n M^{-1} &= M(P \cdot P \cdot P \cdot \ldots \cdot P) M^{-1} \\ &= MPM^{-1} MPM^{-1} \ldots MPM^{-1} = D^n. \end{aligned} \tag{8}$$

The matrix $D^n$ is also diagonal, with elements which are the $n$-th power of the corresponding ones of $D$. But they are also the eigenvalues of $P^n$, therefore they must converge to zero excepting one, which is 1. From this we deduce that the matrix $P$ has an eigenvalue 1 associated to the eigenvector $\underline{v}$ and the other eigenvalues must have modulus $< 1$, since their $n$-th powers converge to zero. Therefore the convergence to the stationarity is the same as the convergence to zero of the $n$-th power of the non-unitary eigenvalues of $P$. Obviously the speed of convergence will depend from the initial density vector $\underline{x}$. For calculating the minimal value of $n$ for which the stationarity is practically reached, we must ask for which $n$ we can admit that the difference between the product $P^n \underline{x}$ and $\underline{v}$ is negligible. This aim is reached by multiplying the vector $\underline{x}$ from the left by the matrix $P$, for a number of times sufficient to obtain a vector whose difference from $\underline{v}$ is negligeable. If the convergence is slow, the number of these products is great; therefore, this procedure can be heavily time consuming, especially if the matrix $P$ has great dimension.

We show an alternative procedure, which can be useful in these cases. From (8), by multiplying from the left for $M^{-1}$ and from the right for $M$ we obtain:

$$P^n = M^{-1} D^n M. \tag{9}$$

Therefore we must compare with $\underline{v}$ the vector:

$$\underline{h} = \underline{x} M^{-1} D^n M.$$

But the vector $\underline{b} = \underline{x} M^{-1}$ can be obtained from $M^{-1}$ (a single inversion and a single matrix product) and the vector $\underline{c} = \underline{b} D^n$ is made from the components of $\underline{b}$, each multiplied by the corresponding eigenvalue raised to a power $n$. Therefore, for calculating $\underline{h}$ it is required, besides the determination (once) of $M$, $M^{-1}$, $D$ and of the product $\underline{x} M^{-1}$, the one of the powers of the diagonal elements of $D$ and of the product $\underline{b} D^n M$. This last is rather fast since $\underset{\sim}{c} = \underset{\sim}{b}\, D^n$ is simple to calculate, as shown. In this way the slow operations are executed only once, whereas the iterations (corresponding to the values tried for $n$) are fast. This procedure can be usefully applied to the problem of investigating about the speed of convergence of the $\Pi$ process.

## 5. CONCLUSIONS

In this work we introduced a process, denoted by $\Pi$, for which the states are defined by pairs of nodes (the preceding and the present one). We showed that, if the probability of reaching a node depends on the nodes occupied at the two preceding times, the process $\Pi$ is a Markov Chain. If we can admit that all the states of $\Pi$ are communicating, the process admits a stationary density. In this case the $\Pi^*$ process, for which the state at the time $t$ is the node reached at this time, admits a stationary density as well; this density can be obtained by (6), namely by treating the process $\Pi^*$ as a Markov Chain.

Let us observe that (6) implies a number of equations which is much less than that implied by (1), which gives the stationary density for $\Pi$. Indeed (6) consists of $n$ equations and (1) of a number of equations of the order $n^2$.

Furthermore, the transition probabilities of $\Pi^*$ are much easier and faster to estimate than those of $\Pi$. Therefore, if we are interested only in the stationary density of $\Pi^*$, it is right (and advantageous) to work as it were a Markov Chain.

In Section 4 we have outlined a method for calculating the number of iterations necessary for reaching the stationarity if $\Pi$ is regular.

Moreover, in Section 3 we have quoted formulas which permit to follow the process evolution when the stationarity is not reached, and others which give the probabilities of runs with intermediate nodes. We calculate also the expected values and variances for the numbers of transitions between nodes (fluxes) and for the number of items in each node. These quantities can be interesting if we investigate on the effects of changes in the flux system (changes in the transition probabilities or in the initial density, eliminations of connections between nodes or introductions of not pre-existent ones, etc.).

We conclude with a conjecture, put by F. Giannessi (Dept. of Math., Univ. of Pisa, Italy), in a private communication. Let I be the identity matrix; it is known [5] that the matrix I-P has determinant equal to zero, and its principal minors are all strictly positive and $\leq 1$. From this the question: are these minors the probabilities of meaningful events? Some studies, carried out in [5], seem encouraging but do not offer a definitive answer.

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