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ASYMPTOTIC DIFFERENTIAL APPROXIMATION RATIO: DEFINITIONS, MOTIVATIONS AND APPLICATION TO SOME COMBINATORIAL PROBLEMS (*)

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Abstract. – *We first motivate and define a notion of asymptotic differential approximation ratio. For this, we introduce a new class of problems called radial problems including in particular the hereditary ones. Next, we validate the definition of the asymptotic differential approximation ratio by proving positive, conditional and negative approximation results for some combinatorial problems. We first derive a differential approximation analysis of a classical greedy algorithm for bin packing, the "first fit decreasing". Next we deal with minimum vertex-covering-by-cliques of a graph and the minimum edge-covering-by-complete-bipartite-subgraphs of a bipartite graph and devise a differential-approximation preserving reduction from the former to the latter. Finally, we prove two negative differential approximation results about the ability of minimum vertex-coloring to be approximated by a polynomial time approximation schema.*

Keywords: NP-complete problem, complexity, polynomial time approximation algorithm, bin packing, coloring, covering.

Résumé. – *Nous commençons par définir et motiver une notion de rapport d'approximation différentiel asymptotique. Pour cela, nous introduisons une classe de problèmes que nous appelons radiaux qui comprend en particulier les problèmes héréditaires. Puis nous validons la définition de rapport d'approximation différentiel asymptotique en établissant des résultats positifs, conditionnels et négatifs pour différents problèmes combinatoires. Nous proposons d'abord une analyse, dans le cadre de l'approximation différentielle, d'un algorithme glouton classique (le « first fit decreasing ») pour le problème de bin packing. Nous traitons alors les problèmes de couverture minimum des sommets d'un graphe par des cliques induites et des arêtes d'un graphe biparti par des sous-graphes partiels bipartis complets et donnons une réduction préservant l'approximation différentielle du premier au second. Enfin, nous prouvons deux résultats négatifs concernant la possibilité d'approcher le problème de coloration des sommets d'un graphe par un schéma d'approximation différentiel polynomial.*

Mots clés : Problème NP-complet, complexité, algorithme polynomial approché, *bin packing*, coloration, couverture.

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1. INTRODUCTION

In [5], we have proposed and axiomatized a new approximation theory more compatible with the combinatorial optimization theory. We will try to informally and shortly describe the spirit of this approach.

We first recall that the purpose of the polynomial approximation is to solve optimization NP-complete problems. A common and very complete way for rigorously expressing the most of such problems is *via* the use of mathematical (integer-linear or integer-quadratic) programs. In fact, given a combinatorial optimization problem Π, a generic instance I of Π can be written as follows:

$$I = \begin{cases} \text{opt} & v(\vec{x}) \\ & \vec{x} \in C \\ & x_i \in \{0,1\} \end{cases}$$

where $v(\cdot)$ is a linear (or quadratic) function of $\vec{x}$, called "objective function", and C is a set of linear constraints. A vector $\vec{y}$ verifying the constraints is called "feasible vector" (or "feasible solution") of I and the quantity $v(\vec{y})$ "feasible objective value" (sometimes, for reasons of simplicity, the term feasible is omitted). The best between the objective values of I is called "optimal value" of I.

Key-requirement of the differential approximation framework is the stability of any adopted approximation ratio with respect to the affine transformation of the objective function. In what follows, we call *affine-equivalent* problems for which the objective function of the former is an affine transformation of the objective function of the latter. Affine transformation is very natural and frequent in combinatorial optimization and the stability of the approximation ratio under this type of transformation is not taken into account by the classical theory as it is shown in the following example.

Example 1: One of the most famous cases, in complexity theory, of affine transformation is the relationship between maximum independent set[1] (IS) and minimum vertex covering[2] (VC).

[1] Let $G = (V, E)$ be a graph of order n; an *independent set* is a subset $V' \subseteq V$ such that whenever $\{v_i, v_j\} \subseteq V'$, $v_i v_j \notin E$, and the *maximum independent set problem (IS)* is to find an independent set of maximum size.

[2] In a graph $G = (V, E)$, a *vertex cover* is a subset $V' \subseteq V$ such that, for each edge $uv \in E$, at least one of u and v belongs to V' and the *minimum vertex cover problem (VC)* is to find a minimum size vertex cover.

Given a graph, *a maximal (resp., maximum) independent set is the complement, with respect to the vertex set of the graph, of a minimal (resp., minimum) vertex covering*. So, every algorithm solving the former solves, *via* a a set-difference with respect to the vertex-set of the graph, the latter and *vice-versa*.

These problems are affine-equivalent following the problematic of [5] (*i.e.*, every algorithm supposed to solve them has the same approximation ratio regarding both problems), while they have conflicting behaviour in the usual approximation framework. In fact, the maximal matching algorithm guarantees an approximation ratio 2 for VC, while its approximation ratio tends to 0 for IS. This asymmetry becomes even more embarrassing given that, as it has been proved in 1992, *no polynomial time approximation algorithm can guarantee constant ratio for IS, unless* $P = \text{NP}$ [1].

The theory of differential approximation is based upon the equivalence of problems under affine transformation, since the main axiom we have imposed for a relevant approximation measure is to respect it. The problematic outlined above has led us, in [5], by an axiomatic thought process, to the definition of a notion of approximation measure quite different from the usual one (presented in [8]), the *differential approximation ratio* defined by

$$\delta_{\mathbf{A}}(I) = \frac{\omega(I) - \lambda(I)}{\omega(I) - \beta(I)}$$

$$\delta_{\mathbf{A}}(\Pi) = \inf_{I} \left\{ \frac{\omega(I) - \lambda(I)}{\omega(I) - \beta(I)} \right\}$$

where, given an instance I of a combinatorial problem, $\omega(I)$, $\lambda(I)$ and $\beta(I)$ are the values of the worst-case solution (notion discussed in [5]), the approximated one (provided by algorithm $\mathbf{A}$, supposed to solve problem Π), and the optimal one, respectively. It is easy to see that $\delta_{\mathbf{A}}(\Pi) = 1$ corresponds to an exact algorithm $\mathbf{A}$ for Π; otherwise, $\delta_{\mathbf{A}}(\Pi) \leq 1$.

Loosely speaking, a differential ratio $1 - \epsilon$ means that $\mathcal{A}$ is to an extent of ϵ like the most stupid algorithm, and to an extent of $(1 - \epsilon)$ like the most intelligent algorithm. If $\lambda(I) = \beta(I)$, then the ratio is equal to 1, and if $\lambda(I) = \omega(I)$, then it is equal to 0. If, on the other hand, $\beta(I) = \omega(I)$, then the ratio is undefined, but this case is trivial since the optimal solution can be found in polynomial time. Finally, the better the value of the approximation solution, the better (closer to 1) the differential ratio. Remark that a possible interpretation of differential approximation ratio is that it expresses the position of $\lambda(I)$ into the interval of the possible objective function's values.

As it is shown in [4, 5, 9, 10], many combinatorial optimization problems have, in the differential approximation framework, a behaviour completely different from the one in the usual approximation framework, where the approximation ratio for minimization (maximization) problems is defined as

$$\rho_{\mathbf{A}}(I) = \frac{\lambda(I)}{\beta(I)} \quad \left(\frac{\beta(I)}{\lambda(I)}\right).$$

In this note, we extend the work of [5] by introducing a notion of *asymptotic* differential approximation ratio (the asymptotic approximation ratio is well-known in the classical approximation framework too). Next, we use this notion to derive several approximation results.

More precisely, we first evaluate the asymptotic differential approximation behaviour of the *first fit decreasing* algorithm (FFD) used to solve a very well-known NP-complete problem, the *bin packing*.

Next we consider two other well-known combinatorial optimization problems, the *minimum vertex-covering-by-cliques of a graph* and the *minimum edge-covering-by-complete-bipartite-sub-graphs of a bipartite graph*. For these problems, we show how, for every $\delta \leq 1$, the existence of a δ-differential approximation algorithm for the former implies the existence of a polynomial time approximation algorithm for the latter, guaranteeing a constant *asymptotic* differential ratio of value δ.

Finally, we prove that, unless $P = \mathrm{NP}$, the *minimum vertex-coloring of a graph* cannot be approximated neither by a fully polynomial time approximation schema, nor by an asymptotic one.

The problems we mainly deal with in this paper are defined as follows.

Bin packing (BP)

We are given a finite set $L = \{x_1, \ldots, x_n\}$ of n rational numbers and an unbounded number of bins, each bin having capacity equal to 1; we wish to arrange all these numbers in the least possible bins in such a way that the sum of the numbers in each bin does not violate its capacity.

Vertex-covering-by-cliques of a graph (VCC)

Given a graph $G = (V, E_G)$, we wish to cover the vertices of V by a minimum-cardinality system of cliques of G.

Edge-covering-by-complete-bipartite-subgraphs of a bipartite graph (CCB)

Given a bipartite graph $B = (V_1 \cup V_2, E_B)$, we wish to cover the edges of E_B by a minimum-cardinality system of complete bipartite subgraphs of B.

Vertex-coloring of a graph (C)

We are given a graph of order n and wish to color its vertices with as few colors as possible, so that no two adjacent vertices receive the same color.

We now give a definition-panorama of the several approximation-theory terms which will be used in the sequel.

DEFINITION 1: Consider a generic instance I of size n of an NP-complete combinatorial optimization problem Π.

- A polynomial time algorithm **A** sub-optimally solving Π is called *polynomial differential δ-approximation algorithm (δ-DPTAA)* if **A** guarantees differential approximation ratio δ for every instance of Π.
- A *differential fully polynomial time approximation schema (DFPTAS)* is a sequence of δ-DPTAAs $\mathbf{A}_\epsilon$ guaranteeing, for every $\epsilon > 0$, approximation ratio $\delta \geq 1-\epsilon$ with complexity $O(p(1/\epsilon)O(n^k))$, where p is a polynomial not depending on n, and k a constant not depending neither on n, nor on ϵ.

2. ASYMPTOTIC DIFFERENTIAL APPROXIMATION RATIO

In the classical approximation theory, the notion of asymptotic approximation ratio deals, informally, with instances for which the optimal solution value is unbounded. This notion appears to be very meaningful given that a great number of NP-complete optimization problems become polynomial if we consider that their optimal solution value is bounded by a fixed constant.

These problems are called *simple* in the literature [13]. For example, IS, VC, BP, etc., are simple since, for instances with $\beta(I) \leq \kappa$, for a fixed

constant κ, an exhaustive search, where at most all the κ-element subsets of the input-data are examined, suffices to produce the optimal solutions. Notice that in such a case, the complexity of an exhaustive search is at most $O(n^{\kappa+1})$, hence polynomial in n where n denotes the size of problem's instance. On the other hand, since C is NP-complete for any minimum number of colors greater than, or equal to, 3, it is not simple.

Moreover, we think that in any approximation framework, the notion of asymptotic approximation (dealing with a class of "interesting" instances) is pertinent. In the usual one, there exist numerous "asymptotic" approximation results, and in the differential one, such results can be produced as we show in the sequel. But first, we define and discuss what we call "differential asymptotic approximation ratio" (denoted by $\delta_{\mathbf{A}}^{\infty}$). As we will see, the hypotheses upon which the definition of $\delta_{\mathbf{A}}^{\infty}$ is based are different from the ones of the usual approximation framework.

In fact, the asymptotic approximation ratio in the usual framework is defined on the hypothesis that the interesting (from an approximation point of view) instances of the simple problems are the ones whose values of the optimal solutions tend to ∞ (because, in the opposite case[3], the problems are polynomial). So by ruling out a few of "non-interesting" instances, people can restrain themselves to the interesting ones and they produce approximation results holding for the quasi-totality of the instances of a given problem (asymptotic results). This point of view justifies the definition of the usual asymptotic approximation ratio as it is given in [8]:

$$\rho_{\mathbf{A}}^{\infty} = \inf\{\rho \geq 1 : \text{for some } \kappa \in \mathbf{Z}^{+}, \rho_{\mathbf{A}}(I) \leq \rho, \forall I \text{ with } \beta(I) \geq \kappa\}.$$

The thought process that has led us to axiomatize the differential approximation framework allows us to bring to the fore cases where the size (or the value) of the optimal solution is not always a pertinent hardness criterion for the instances of a problem. Let us explain for a while this point of view by means of the three following examples.

Example 2: **Vertex covering and independent set**

We revisit the two well-known combinatorial optimization problems VC and IS. As we have already mentioned, these two problems are affine-equivalent and it is exactly on such a kind of equivalence that the differential

[3] The case where optimal values are bounded by fixed constants.

approximation theory is built. Consequently, these two problems are affine-equivalent and one of the main impacts of this equivalence is that the two problems have the same sets of "interesting" (hard) instances. Consider now a complete graph K_n, for an n arbitrarily large. Then $\beta_{\mathrm{VC}}(K_n) = n - 1$ and $\beta_{\mathrm{IS}}(K_n) = 1$. Therefore, if a hardness criterion for the instances of a problem is the value of their optimal solutions, then K_n should be considered hard for VC but easy for IS; this contradicts the equivalence of the two problems. In fact, it is commonly accepted (even in the usual approximation framework) that K_n is an easy instance for both the problems under discussion.

Example 3: **Bin packing**

Consider an instance L of BP and another instance L_κ obtained by appending, in the list of items of L, a list of κ numbers, all equal to 1. It is obvious that every solution for L_κ will be a solution for L followed by κ bins, each one of these bins containing one of the κ 1s appended. In this sense, solving BP in L is exactly as hard as solving it in L_κ and *vice-versa*. If we adopt the size of the optimal solution as a hardness criterion, then L should be easier than L_κ and this does not seem pertinent.

Example 4: **Coloring**

As a last example, consider C. In [4], we have devised a DPTAA guaranteeing an approximation ratio of 1/2 for C. In order to do this, we transformed C into an affine-equivalent maximization problem $\overline{\mathrm{C}}$ and then, by solving the latter, we constructed a solution for the former with the same approximation ratio.

The problem $\overline{\mathrm{C}}$ is the following: "given a graph $G = (V, E)$, find a partial sub-graph H of $\bar{G}$ (the complement of G) having a maximum number of edges and such that (i) H is acyclic (or, equivalently, H is a forest), and (ii) every connected component (tree) of H is included in a clique of $\bar{G}$".

We show now that $\overline{\mathrm{C}}$ is affine-equivalent to C.

With every instance G of C, we associate the instance $\bar{G}$ of $\overline{\mathrm{C}}$ and *vice-versa*.

With every feasible solution $H = (T_1, \ldots, T_p)$ of $\overline{\mathrm{C}}$, we associate the following vertex coloring X of G:

- the vertices of a component-tree T_i, $i = 1, \ldots, p$, of H are colored by a new color;

- we color the vertices of $(V(G) \setminus \cup_{i=1,\ldots,p} V(T_i))$ by using a new color per vertex[4].

The set of used colors constitutes the solution X for C.

The so-obtained set X is feasible for C since tree T_i of H is the spanning tree of a clique of $\bar{G}$, hence $V((T_i)$ is an independent set in G. Moreover, the second of the above items guarantees that all the vertices of G have been colored.

So, given a feasible solution H of $\overline{\mathrm{C}}$, a solution $X = (S_1, \ldots, S_p, S_{p+1}, \ldots, S_{\chi'(G)})$ is constructed (where S_j is the subset of $V(G)$ colored by color j and sets $S_{p+1}, \ldots, S_{\chi'}$ are the singletons of the second of the above items) of cardinality $\chi'(G)$. The objective value of the solution H (the number $|E(H)|$ of the edges of H) is then $|E(H)| = \sum_{i=1}^{p}(|S_i| - 1) = \sum_{i=1}^{\chi'(G)}(|S_i| - 1) = n - \chi'(G)$.

Conversely, given a solution $X = (S_1, \ldots, S_{\chi'(G)})$ for G (of objective value $\chi'(G)$; recall that a vertex coloring is a partition of the vertices of the input-graph into independent sets), one can easily construct a solution H for $\overline{\mathrm{C}}$ in $\bar{G}$ by simply taking the subgraph of $\bar{G}$ induced by S_i, $i = 1, \ldots, \chi'(G)$ (this sub-graph is obviously a clique) and by extracting (using, for example, depth-first-search) a spanning tree T_i. Such a solution is feasible for $\overline{\mathrm{C}}$ (by its definition) and constitutes a forest H on $\chi'(G)$ trees. It is well-known that the number of edges of a p-trees forest of order n is $n - p$; so, $E(H) = n - \chi'(G)$.

We have shown that given a coloring of value x, one can polynomially construct a feasible solution for $\overline{\mathrm{C}}$ of value $n - x$ and *vice-versa*. So the affine-equivalence between C and $\overline{\mathrm{C}}$ is proved.

But even if C and $\overline{\mathrm{C}}$ are affine-equivalent the latter is simple while the former is not. Since affine-equivalent problems have the same differential approximation ratio, it seems natural that they also have *the same asymptotic differential approximation ratio*. This is not the case for the two problems under discussion (if we adopt as hardness criterion the size of the optimal solution), since for C we cannot rule out instances with bounded optimal solutions while, for $\overline{\mathrm{C}}$, such instances, being polynomially solved, can be neglected.

[4] Given a graph G, we denote by $V(G)$ its vertex-set.

Considerations like the ones exposed in Examples 2, 3 and 4 have led us to look for other hardness criteria better adapted to the differential problematic. As we have already noticed, a possible interpretation of the differential approximation ratio is that it expresses the position of $\lambda(I)$ into the interval of the possible objective function's values (as we have restricted ourselves to discrete problems, there exists a finite number of such values). Moreover, for the most of the maximization problems, since the value of the worst solution is equal to 0, the optimal value is an upper bound for the number of the objective function's values. Observations like the ones just mentioned, give us the terms of another hardness criterion, *the number $\sigma(I)$ of the feasible values of I*, which we adopt as hardness criterion for the instances of a problem. Consequently, we propose the following definition.

DEFINITION 2: **Asymptotic differential approximation ratio**

The *asymptotic approximation ratio* of a DPTAA **A** is defined as

$$\delta_{\mathbf{A}}^{\infty} = \lim_{k \to \infty} \inf_{\substack{I \\ \sigma(I) \geq k}} \left\{ \frac{\omega(I) - \lambda(I)}{\omega(I) - \beta(I)} \right\}.$$

In other words, $\delta_{\mathbf{A}}^{\infty}$ is the lower limit of the sequence of ratios indexed by the value of $\sigma(I)$.

Let us note that the condition $\sigma(I) \geq k$ used in the definition of the asymptotic approximation ratio for characterizing "the sequence of unbounded instances" (or "limit instances") cannot be polynomially verified[5]. But in practice, for a given problem, it is possible to directly interpret condition $\sigma(I) \geq k$ by means of $\omega(I)$ and $\beta(I)$ (note that $\sigma(I)$ is not a function of these values). For example, for numerous cases of discrete problems, we are able to determine, for each instance, a step π defined as the least variation between two feasible values of the instance (for example, for BP $\pi = 1$); then, $\sigma(I) \leq [\omega(I) - \beta(I)]/\pi$ and consequently,

$$\delta_{\mathbf{A}}^{\infty} \geq \lim_{k \to \infty} \inf_{\substack{I \\ \frac{\omega(I) - \beta(I)}{\pi} \geq k}} \left\{ \frac{\omega(I) - \lambda(I)}{\omega(I) - \beta(I)} \right\}.$$

Whenever π can be determined, condition $[\omega(I) - \beta(I)]/\pi \geq k$ can be easier to evaluate than $\sigma(I) \geq k$, and in this case, the former is used (this is not senseless since we try to bound below the ratio).

[5] Of course, the same holds for the condition $\beta(I) \geq k$ induced by the hardness criterion in the usual approximation framework.

3. RADIAL PROBLEMS

In order to justify the adoption of $\sigma(I)$ as hardness criterion in the definition of the asymptotic approximation ratio, let us introduce a class of problems, called *radial problems*, large enough and including a lot of well-known combinatorial optimization problems.

We prove in this section that, for a radial problem Π, if $\sigma(I)$ is bounded above, then Π is polynomially solvable (see Prop. 1). Roughly speaking, instances verifying $\sigma(I) < k$ are not very interesting, since there is a bounded number of elementary stages from any feasible solution to the best one. So, the test $\sigma(I) \geq k$ rules the less interesting instances (that are polynomial for radial problems) out, which is just the aim of the asymptotic notions. This thought process is largely inspired by the spirit of the notion of *simplicity* [13] but seems more pertinent in the differential approximation framework.

Before formally introducing radial problems (Def. 3), let us give an informal description for these problems.

Informally, a problem Π is radial if, given a generic instance I of Π and a feasible solution for I, one can, in polynomial time, on the one hand, deteriorate it as much as one wants (until to finally obtain a worst-value solution) and, on the other hand, greedily improve it in order to obtain (always in polynomial time) a sub-optimal solution (eventually the optimal one). The part of the informal description concerning deterioration is formally expressed in definition 3 by conditions **(i)**, **(ii)** and **(iii)** while improvement is expressed by condition **(iv)**.

DEFINITION 3: **Radial problems**

Consider a combinatorial problem Π and a generic instance I of Π of size n. If there exist three polynomial (in n) algorithms ξ, ψ and φ such that, for all I:

(i) ξ computes a feasible solution $\vec{x}^{(0)}$;

(ii) for every feasible vector $\vec{x}$ of I strictly better (in the sense of the objective) than $\vec{x}^{(0)}$, algorithm φ computes a feasible solution $\varphi(\vec{x})$ (if any exists) with $v(\varphi(\vec{x}))$ strictly worse than $v(\vec{x})$, *i.e.*, $v(\varphi(\vec{x})) > v(\vec{x})$, if Π is a minimization problem and $v(\varphi(\vec{x})) < v(\vec{x})$, if Π is a maximization one;

(iii) for every feasible vector $\vec{x}$ of I with objective value strictly better than $v(\vec{x}^{(0)})$, there exists an integer k such that $\varphi^k(\vec{x}) = \vec{x}^{(0)}$ (where we denote by φ^k the k-times iteration of φ);

(iv) for a vector $\vec{y}$ such that, either $\vec{y} = \vec{x}^{(0)}$, or $\vec{y}$ is any feasible vector of I with objective value strictly better than $v(\vec{x}^{(0)})$, $\psi(\vec{y})$ computes the set of ancestors of $\vec{y}$, defined by $\psi(\vec{y}) = \varphi^{-1}(\{\vec{y}\}) = \{\vec{z} : \varphi(\vec{z}) = \vec{y}\}$ (this set is eventually empty),

then Π is *radial*.

In what follows in this section, we consider maximization problems, the case of minimization ones being completely analogous.

The notion of radial problems includes in particular the usual notion of *hereditary* problems for which every subset of a feasible solution remains feasible (for example, IS is a hereditary problem). In fact, for hereditary problems,

(i) feasible solutions $\vec{x}$ are characteristic vectors of sets;

(ii) for all instances I, $\vec{x}^{(0)} = \vec{0}$;

(iii) for every other feasible vector $\vec{x}$, $\varphi(\vec{x})$ is just obtained from $\vec{x}$ by setting to 0 a component which equals 1 in $\vec{x}$.

Example 5: **The bin packing is radial**

Let us revisit BP. First, observe that it is not hereditary, since given a feasible solution B consisting of a certain number of bins correctly[6] including all the elements of the initial list, if we remove some bins of B, the remaining solution is not feasible since it does not contain all the members of L.

Let us now show that BP satisfies conditions **(i)** to **(iv)** of Definition 3. Consider for this an instance L of BP:

- one can easily compute a solution $B^{(0)}$ for BP consisting of putting an item per bin (so $B^{(0)}$ consists of n one-item bins); so, condition **(i)** is satisfied;
- given a feasible solution B, one can deteriorate it by removing an element from a bin of B containing at least two items and by putting this removed element in a new (unused) bin; so, condition **(ii)** is satisfied;

[6] In the sense that the sum of the sizes of the elements in each bin does not exceed 1.

- one can continue this deterioration of B by repeatedly executing the above item until solution $B^{(0)}$ is obtained; so, condition **(iii)** is satisfied;
- finally, given either solution $B^{(0)}$ or any other feasible solution B
  - one can obtain better solutions by iteratively trying to empty some bins of $B^{(0)}$ (or B), *i.e.*, by considering an single-item bin and by trying to put its item in another non-empty bin of $B^{(0)}$ (or B);
  - one can continue this procedure as far as it leads to smaller feasible BP-solutions;
  - moreover, if one guesses successfully the single items one tries to move, then one could obtain even an optimal BP-solution;

  hence condition **(iv)** of definition 3 is also satisfied.

Consequently, BP is radial.

Example 5 illustrates also that the class of radial problems is significantly more general than the one of the hereditary problems[7]. By a similar thought process to the one of Example 5, one can prove that the radial property described in Definition 3 is verified by a very large number of other combinatorial problems (in particular, all the problems of this paper are radial) such as VC, BP or, even, C and CCB.

On the contrary, the Proposition 1 below brings to the fore some problems which do not belong to this class. For example, the problem of coloring with as few colours as possible a 4-colorable graph (*i.e.*, we suppose that a feasible solution does not contain more than four colours) is not radial.

Let us now introduce Proposition 1 that illustrates why we consider as the most interesting, under the adopted asymptotic ratio's definition, the instances with large σ.

PROPOSITION 1: *Let κ be a fixed constant and consider a problem Π satisfying the following properties:*

(i) Π is radial, and,

[7] In fact, the hereditary notion deals with problems for which a feasible solution is a subset of the input-data, while the radial notion includes problems for which a feasible solution is not only a subset but a second-order sub-structure of these data.

(ii) for every instance I of Π, $\sigma(I) \leq \kappa$.

Then, Π is polynomial.

Proof: Let I be an instance of Π and consider the feasible vector $\vec{x}^{(0)}$ computed by algorithm ξ. Let us now consider an optimal solution $\vec{\hat{x}}$ of I; if $\vec{x}^{(0)}$ is not optimal, then, by property **(iii)** of Definition 3, $\exists k$ such that $\phi^k(\vec{\hat{x}}) = \vec{x}^{(0)}$. But from hypothesis **(ii)** of Proposition 1 and property **(ii)** of Definition 3, $k \leq \kappa$.

Then, in order to compute $\vec{\hat{x}}$, one has to perform an exhaustive search of the at most $p(n)^\kappa$ feasible solutions of I, where $p(n)$ is an upper bound of $\psi(\vec{y})$. □

4. ASYMPTOTIC DIFFERENTIAL-APPROXIMATION RESULTS

4.1. Bin packing and the asymptotic differential-approximation behaviour of "first fit decreasing"

A broad discussion of approximation strategies for BP is performed in [8, 11, 12]. In the usual approximation framework, the strongest approximation result for BP is the one of Fernandez de la Vega and Lueker [7] who prove that it can be solved in linear time by an asymptotic fully polynomial time approximation schema.

One of the most classical and efficient algorithms computing good solutions for BP is the following $O(n \log n)$ first fit decreasing (Algorithm 1).

Algorithm 1. Algorithm FFD

BEGIN

sort the numbers in L in decreasing order;
let $L = \{x_1, \ldots, x_n\}$ be the resulting list;
FOR all p DO $B_p \leftarrow \emptyset$ OD
FOR $i \leftarrow 1$ TO n DO
$j \leftarrow min\{k : \sum_{x \in B_k} x \leq 1 - x_i\}$;
$B_j \leftarrow B_j \cup \{x_i\}$;
OD
remove the empty bins;

END.

The best (classical) approximation result for FFD (obtained by a rather complicated and long proof) is the one cited in [8], *i.e.*, for all instances L

of BP, $\lambda(L) \leq (11/9)\beta(L)+4$, in other words, the asymptotic approximation ratio of FFD (in the usual approximation framework) is 11/9.

We show in this section how the notion of asymptotic differential approximation ratio provides a framework in which Algorithm 1 can be also analyzed with good results. We first recall the following lemma proved in [11, 12].

LEMMA 1 [11, 12]: *Let* $L = \{x_1, \ldots, x_n\}$ *be an instance of BP, let* $\bar{L} = \{x_i \in L : x_i \leq 1/3\}$ *and* $\bar{m} = |\bar{L}|$. *Algorithm 1 optimally solves instances* L *of BP for which* $\bar{L} = \emptyset$.

THEOREM 1:

- $\delta^{\infty}_{\mathbf{FFD}} \geq 2/3$
- $\delta^{\infty}_{\mathbf{FFD}} \leq 11/12$
- $\delta_{\mathbf{FFD}} \leq 3/4$.

Proof: Let us consider an instance $L = \{x_i : i = 1, \ldots, n\}$ of BP. It is easy to see that $\omega(L) = n$. List L is divided into two sub-lists $\bar{L}$ of cardinality $\bar{m}$, and L' of cardinality m', with

$$\bar{L} = \left\{x_i \in L : x_i \leq \frac{1}{3}\right\}$$
$$L' = L \setminus \bar{L}.$$

Let us consider two distinct applications of FFD, first in L', then in $\bar{L}$ and let B' and $\bar{B}$ be the corresponding BP-solutions; denote by $\lambda(L')$ and $\lambda(\bar{L})$ the values of B' and $\bar{B}$, respectively. Obviously, $B' \cup \bar{B}$ is a feasible BP-solution for L of value $\lambda(L') + \lambda(\bar{L})$.

In Algorithm 1, since L is sorted in increasing order, L' is firstly treated. Next, in order to treat $\bar{L}$, the algorithm will firstly search to put some of its elements in bins already used for L' and next, for the rest of elements of $\bar{L}$ empty (still unused) bins will be used. Consequently, the value $\lambda(L)$ of the obtained solution is *a priori* smaller (at most equal) to the value $\lambda(L') + \lambda(\bar{L})$ of the solution $B' \cup \bar{B}$ (thanks to the fact that some elements of $\bar{L}$ are, eventually, already arranged in bins already used for the items of L'); so,

$$\lambda(L) \leq \lambda(L') + \lambda(\bar{L})$$

with equality in the case where no item of $\bar{L}$ can be introduced by algorithm FFD in a bin already used to arrange some elements of L'.

The following holds for $\lambda(L')$, $\beta(L')$, $\lambda(\bar{L})$ and $\beta(L)$:

$$\begin{aligned}
&\lambda(L') = \beta(L') \text{ (according to Lem. 1)}\\
&\lambda(\bar{L}) \leq \left\lceil \frac{\bar{m}}{3} \right\rceil \leq \frac{\bar{m}}{3} + \frac{2}{3}\\
&\beta(L) \geq \beta(L');
\end{aligned}$$

so,

$$\begin{aligned}
\frac{\omega(L) - \lambda(L)}{\omega(L) - \beta(L)} &\geq \frac{m' - \beta(L') + \frac{2\bar{m}}{3} - \frac{2}{3}}{\omega(L) - \beta(L)}\\
&\geq \frac{m' - \beta(L') + \frac{2\bar{m}}{3}}{m' - \beta(L') + \bar{m}} - \frac{2}{3[\omega(L) - \beta(L)]}.
\end{aligned}$$

As $m' - \beta(L') \geq 0$, we get

$$\frac{\omega(L) - \lambda(L)}{\omega(L) - \beta(L)} \geq \frac{2}{3} - \frac{2}{3[\omega(L) - \beta(L)]}.$$

Given that for BP $\pi = 1$, we have

$$\delta^{\infty}_{\mathbf{FFD}} \geq \frac{2}{3}$$

which concludes the proof of first item.

Of course, it is natural to wonder what is the best value for $\delta^{\infty}_{\mathbf{FFD}}$. The second item of the theorem affirms that it cannot be greater than 11/12, while, in the third item we prove that $\delta_{\mathbf{FFD}}$ cannot be larger than 3/4.

Consider a 18m-element list $L = \{x_1, \ldots, x_{18\mathrm{m}}\}$ with

$$x_i = \begin{cases} \frac{1}{3} + 2\epsilon & 1 \leq i \leq 6\mathrm{m} \\ \frac{1}{3} - \epsilon & 6\mathrm{m} + 1 \leq i \leq 18\mathrm{m}. \end{cases}$$

Then,

$$\beta(L) = 6\mathrm{m} \tag{1}$$

$$\lambda_{\mathbf{FFD}}(L) = 7\mathrm{m} \tag{2}$$

$$\omega(L) = 18\mathrm{m}$$

where (1) holds because, in the optimal solution for L, each bin contains the triple $(1/3+2\epsilon, 1/3-\epsilon, 1/3-\epsilon)$ per bin, while (2) holds because, in the solution provided by FFD, 3m bins will contain the pair $(1/3+2\epsilon, 1/3+2\epsilon)$ per bin and 4m bins will contain the triple $(1/3-\epsilon, 1/3-\epsilon, 1/3-\epsilon)$ per bin.

It is easy to see that $\sigma(L) = 12\text{m}$ can be unbounded. So, the ratio $(18\text{m}-7\text{m})/(18\text{m}-6\text{m}) = 11/12$ is an upper bound for $\delta^{\infty}_{\mathbf{FFD}}$.

In order to prove the third item, consider a 6-element list

$$L = \left\{\frac{1}{3}+\epsilon, \frac{1}{3}+\epsilon, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}-\epsilon, \frac{1}{3}-\epsilon\right\}.$$

It is easy to see that the value of the optimal BP-solution for L equals 2, while FFD provides solution of value 3 and the worst value is 6. So, for this list, the approximation ratio of FFD is $(6-3)/(6-2) = 3/4$ and this completes the proof of the third item and of the theorem. □

4.2. Covering problems on graphs

In what follows, *via* an approximation-preserving reduction, we show how the existence of a DPTAA for CCB with a certain ratio, would lead to the devising of a DPTAA for VCC with asymptotically the same ratio.

Let B and G be two generic instances of CCB and VCC, respectively.

For $\omega(G)$, it suffices to remark that every vertex of G can be considered as a clique K_1.

For $\omega(B)$, two immediate alternatives exist for the worst-case solution. We can either consider each edge of E_B as a complete bipartite sub-graph of B, or consider the complete bipartite subgraph of B $(v \cup \Gamma(v), \{vu : u \in \Gamma(v)\})$, with $v \in V_i$, $i = 1, 2$ and where $\Gamma(v)$ is the set of neighbours of v. It is easy to see that such a graph is a complete bipartite graph and that $\cup_{v \in V_i}\{vu : u \in \Gamma(v)\}) = E_B$. The two candidate solutions are both unwarranted and either the former, or the latter can be effectively considered as worst-case ones. But since the latter, being smaller than the former, is as unwarranted and natural as the former, it is natural to require from an efficient approximation algorithm for CCB to provide solutions better than the one $\{(v \cup \Gamma(v), \{vu : u \in \Gamma(v)\}) : v \in \text{argmin}\{|V_1|, |V_2|\}\}$. Moreover, from a mathematical point of view, the greater the worst-case value considered, the

easier the achievement of positive differential approximation results[8] (a long discussion on the notion of the worst-case solution in complexity theory and, also on how its value can be choosen, is performed in [6]). Consequently, in order to render the differential-approximation framework as restrictive as possible and, therefore, the obtained results to be as non-trivial as possible, we systematically consider the smallest between the unwarranted solutions as the worst case one.

So, the following holds for the sizes of the worst-case solutions for CCB and VCC, respectively:

$$\omega(B) = \min\{|V_1|, |V_2|\}$$
$$\omega(G) = |V|.$$

A very interesting problem in complexity theory is the devising of approximation preserving reductions between NP-complete problems, *i.e.*, reductions establishing that an approximation algorithm for an NP-complete problem can be transformed into an approximation algorithm with the same approximation ratio (up to a multiplicative factor) for another NP-complete problem. Although this aspect of theoretical computer science is relatively well-studied in the framework of the conventional approximation theory, in the differential one (due also to the youth of this framework), it is not yet satisfactorily developed.

Moreover, let us notice that approximation-preserving reductions for the conventional framework does not always remain approximation-preserving for the differential one, since the approximation ratio adopted plays a key-role to the proofs of the preservation property, strongly conditions its existence and requires mathematical arguments proper to the adopted framework.

In our case, a slightly different result (where no asymptotic ratios intervene) happens to exist also in the conventional framework [14]; the reduction proving the result of this section (Th. 2) is an adaptation of the one of [14].

[8] Let us notice that another "possible" worst-case solution for CCB could be the family of all the existing complete bipartite sub-graphs of B; but since the cardinality of such a family is exponential in $|V_1| + |V_2|$, then every DPTAA for CCB (providing, naturally, a feasible solution of size polynomial in $|V_1| + |V_2|$) would be considered as asymptotically optimal for this problem (since it would always achieve a differential approximation ratio $1 - \epsilon$ for an ϵ tending to 0); the same remarks hold also for $\omega(B) = |V_1||V_2|$; finally, it is also a matter of intellectual honnesty for a researcher to render the achievement of mathematical results as least trivial as possible.

THEOREM 2: *Let $\delta \leq 1$ be a fixed positive constant. If there exists a δ-DPTAA for CCB, then there exists a DPTAA for VCC guaranteeing asymptotic differential approximation ratio δ.*

Proof: Let $G = (V, E)$ be an instance of VCC; let us suppose that $V = \{1, \ldots, n\}$ and $m = |E|$.

Starting from G, we first construct the following bipartite graph $B = (V_1 \cup V_2, E_B)$:

$$\begin{aligned} V_1 &= \{x_1, \ldots, x_n\} \\ V_2 &= \{y_1, \ldots, y_n\} \\ E_B &= E_B^1 \cup E_B^2 \\ E_B^1 &= \{x_i y_i : i = 1, \ldots, n\} \\ E_B^2 &= \{x_i y_j : ij \in E\}; \end{aligned}$$

and then, we construct an instance $\bar{B} = (\bar{V}_1 \cup \bar{V}_2, E_{\bar{B}})$ of CCB as follows:

$$\begin{aligned} \bar{V}_i &= V_i \times \{1, \ldots, m\} \times \{1, \ldots, m\} \quad i = 1, 2 \\ E_{\bar{B}} &= E_{\bar{B}}^1 \cup E_{\bar{B}}^2 \cup E_{\bar{B}}^3 \\ E_{\bar{B}}^i &= E_B^i \times \{1, \ldots, m\} \times \{1, \ldots, m\} \quad i = 1, 2 \\ E_{\bar{B}}^3 &= \{(x_i, p, q)(y_j, k, l) : p = k, q \neq l, i < j, ij \in E\} \\ &\quad \cup \{(x_i, p, q)(y_j, k, l) : p \neq k, q = l, i > j, ij \in E\}. \end{aligned}$$

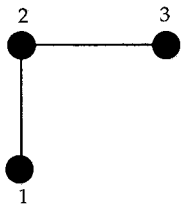
In Figure 1, an example of the reduction just described is shown.

Notice that the above construction has the following properties:

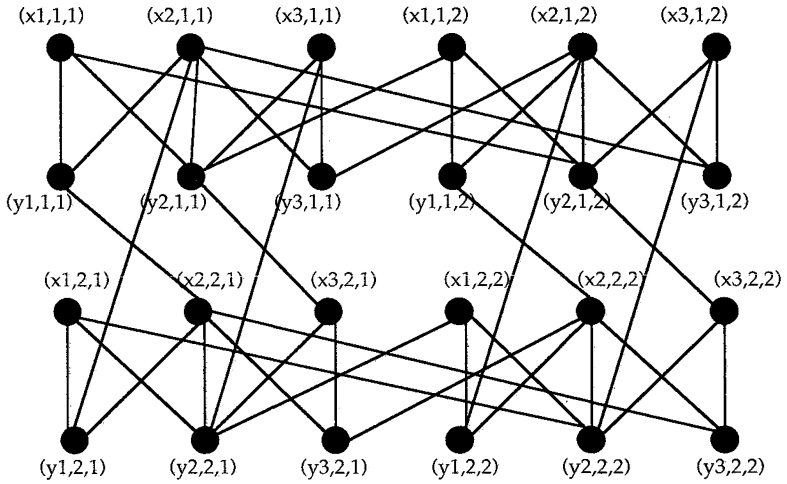
(i) every clique of G covering a set $V' \subset V$ corresponds, in B, to a complete bipartite graph covering, in particular, the edges of the form $x_i y_i$, $i \in V'$; conversely, every complete bipartite graph covering edges of the form $x_i y_i$, $i \in V'$, corresponds to a clique of G covering V'; so,

- the members of an edge-covering by complete bipartite graphs of B are all balanced[9]; moreover, if $\{B_r = (V_1^r, V_2^r, E_r) : r = 1, \ldots\}$, $V_1^r \subseteq V_1$, $V_2^r \subseteq V_2$, $r = 1, \ldots$, is such an edge-covering, then the members of V_1^r and V_2^r have the same set of indices;

[9] A bipartite graph $B = (V, V', E)$ is called balanced if $|V| = |V'|$.



G



$\overline{B}$

Figure 1. – A graph G, instance of VCC, and the derived bipartite graph $\bar{B}$, instance of CCB.

- every vertex-covering by ν cliques in G corresponds to a family of ν complete bipartite graphs of B covering E^1_B and *vice-versa* (but it does not necessary cover E^2_B, which explains the usefulness of introducing $\bar{B}$);

(ii) for every $p = 1, \ldots, m$ and for every edge $ij \in E$, $i < j$, the vertices $\{(x_i, p, q), (y_j, p, l) : q, l = 1, \ldots, m\}$ induce in $\bar{B}$ a complete bipartite graph; the same holds for the vertices $\{(x_j, q, p), (y_i, l, p) : q, l = 1, \ldots, m\}$; so, the $2m^2$ corresponding complete bipartite graphs cover the edges of the set $E^2_{\bar{B}} \cup E^3_{\bar{B}}$;

(iii) for every $i \in V$ and every $(p,q) \neq (k,l)$, edges $(x_i,p,q)(y_i,p,q)$ and $(x_i,k,l)(y_i,k,l)$ cannot belong to the same complete bipartite subgraph of $\bar{B}$; consequently, every edge-covering of $\bar{B}$ by μ complete bipartite graphs is partitioned into m^2 edge-coverings of B, each covering containing μ_i complete bipartite graphs; since $\sum_{i=1}^{m^2} \mu_i = \mu$, it is immediately deduced that $\min_i\{\mu_i\} \leq \mu/m^2$.

Let us now suppose that we are given a δ-PTAA $\mathbf{A_{CCB}}$ for CCB. Then, we consider the following algorithm for VCC.

Algorithm 2. A VCC-algorithm $\mathbf{A_{VCC}}$

BEGIN

starting from G construct B and $\bar{B}$;
Apply $\mathbf{A_{CCB}}$ on $\bar{B}$;
let $\mathcal{B}_0 = \text{argmin}_i\{\mu_i\}$;
construct, following (i), the corresponding clique-covering of V;

END.

We then have the following:

$$\begin{aligned}
&\lambda(G) \leq \frac{\lambda(\bar{B})}{m^2} \quad \text{following (iii)} \\
&\beta(\bar{B}) \leq m^2\beta(G) + 2m^2 \quad \text{following (i) and (ii)} \\
&\omega(G) = n \\
&\omega(\bar{B}) = nm^2.
\end{aligned} \tag{3}$$

Notice that the righthand side of expression (3) represents the size of the feasible solution deduced from items **(i)** and **(ii)** above. The size of this solution is, *a priori*, greater than, or equal to, the size of the optimal solution for $\bar{B}$.

Based upon the above expressions and the hypothesis that $\mathbf{A_{CCB}}$ guarantees differential approximation ratio δ, we get:

$$\begin{aligned}
\delta_{\mathbf{A_{VCC}}} &= \frac{\omega(G) - \lambda(G)}{\omega(G) - \beta(G)} = \frac{\omega(G) - \lambda(G)}{\omega(G) - \beta(G)} + \frac{2}{\omega(G) - \beta(G)} - \frac{2}{\omega(G) - \beta(G)} \\
&\geq \frac{\omega(\bar{B}) - \lambda(\bar{B}) + 2m^2}{\omega(\bar{B}) - \beta(\bar{B}) + 2m^2} - \frac{2}{\omega(G) - \beta(G)} \geq \delta - \frac{2}{\omega(\bar{B}) - \beta(\bar{B})}.
\end{aligned}$$

To complete the proof of the theorem, it suffices to remark that VCC is radial. So $\delta^{\infty}_{\mathbf{A_{VCC}}} \geq \delta$, and this completes the proof of the theorem. □

4.2.1. *Graph coloring*

We prove in this section that, unless $P = \mathrm{NP}$, C cannot be approximated by neither a DFPTAS, nor by an asymptotic DFPTAS.

Let us notice that the non-existence of a DFPTAS for C was already announced in [4] but without proof because of the imperatives on the length of the paper. So, here we give the full proof of this result and, moreover we somewhat strengthen it by proving that even the existence of an asymptotic DFPTAS of a certain form for C is also to be excluded, unless $P = \mathrm{NP}$.

Before stating and proving our result, we need the following definition (already given in [3] and also used in [8], Chap. 6, Sect. 6.2, pp. 142-143).

DEFINITION 4: Consider two graphs $H = (U, E_H)$ and $I = (W, E_I)$. The *composition* (product in [3]) $G = H[I]$ of H and I, is a graph $G = (V, E)$ with

$$V = U \times W$$
$$E = \{(u,w)(u',w') : (uu' \in E_H) \vee [(u = u') \wedge (ww' \in E_I)]\}.$$

In other words, G is constructed by replacing each vertex of H by a copy of I and then replacing each edge uu' of H by a complete bipartite graph joining the copy corresponding to u to the one corresponding to u'.

Figure 2 shows an example of the composition described in Definition 4.

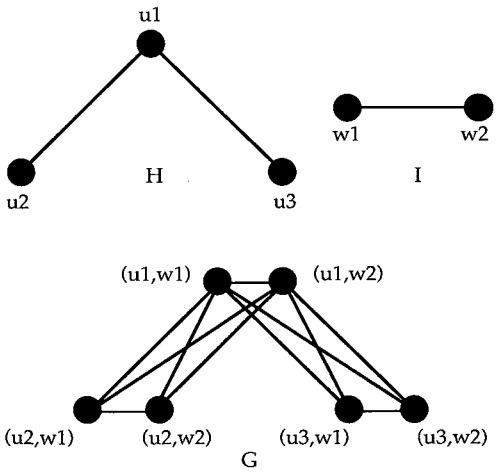


Figure 2. – The graphs H and I and their composition graph $G = H[I]$.

THEOREM 3: *Consider a graph G of order n, generic instance of C. Then, unless $P = \mathrm{NP}$, C cannot be approximated*

- *neither by a DFPTAS;*
- *nor by an asymptotic DFPTAS guaranteeing, for every $\epsilon > 0$, differential ratio of the form*

$$\delta = 1 - \epsilon - \left(\frac{1}{\omega(G) - \beta(G)}\right)^{\alpha}$$

where α is a fixed positive constant.

Proof: In order to prove the first item, let us suppose the existence of a DFPTAS $\mathbf{A}_\epsilon$ for C. We will show how we can use it to polynomially solve the 3-coloring problem[10].

Consider a graph $G = (V, E)$ with $|V| = n$. Apply $\mathbf{A}_{1/n}$ to G ($\mathbf{A}_\epsilon$ being a DFPTAS, its complexity is not exponential in ϵ; hence $\mathbf{A}_{1/n}$ is polynomial in n) and recall that that $\omega(G) = n$. So,

$$\frac{\omega(G) - \lambda_{\mathbf{A}_{1/n}}(G)}{\omega(G) - \beta(G)} \geq 1 - \frac{1}{n}$$

which, after some easy and short algebra, gives

$$\lambda_{\mathbf{A}_{1/n}}(G) \leq 1 + \left(1 - \frac{1}{n}\right)\beta(G) = \frac{1}{n}n + \left(1 - \frac{1}{n}\right)\beta(G).$$

It is easy to see that if G is 3-colorable, then $\beta(G) = 3$ and, consequently, $\lambda_{\mathbf{A}_{1/n}}(G) < 4$; on the contrary, if G is not 3-colorable, then obviously, $\lambda_{\mathbf{A}_{1/n}}(G) > \beta(G) > 4$. Therefore, following the value of $\mathbf{A}_\epsilon$ (obtained in polynomial time thanks to the fact that $\mathbf{A}_\epsilon$ is polynomial for every ϵ), one can decide in polynomial time if G is 3-colorable[11] or not ($\lambda_{\mathbf{A}_{1/n}}(G) > 4$). The fact that 3-colorability is NP-complete, suffices to deduce that a DFPTAS for C cannot exist unless $P = \mathrm{NP}$.

Let us now prove the second item of the theorem. Let us notice that the ratio claimed in this item is a restricted form of asymptotic ratio.

[10] Given a graph, we wish to decide if its vertices can be colored by using only 3 distinct colors; this version of C is another famous NP-complete problem [8].

[11] $\lambda_{\mathbf{A}_{1/n}}(G) < 4$ means that G is 3-, or 2-colorable; on the other hand, deciding if a graph is 2-colorable is polynomial.

Consider a graph $G = (V, E)$ of order n, instance of C and the composition graph $K_\ell[G]$, where K_ℓ is a clique on ℓ vertices.

It is easy to see that in $K_\ell[G]$ a vertex of a copy corresponding to a vertex of G is linked to every vertex of any other copy; so for the coloring of $K_\ell[G]$, the following holds:

$$\begin{aligned} \beta(K_\ell[G]) &= \ell\beta(G) \\ \omega(K_\ell[G]) &= \ell\omega(G) \\ \lim_{\ell\to\infty} [\omega(K_\ell[G]) - \beta(K_\ell[G])] &= \infty. \end{aligned}$$

Moreover, at every vertex-coloring of G by $\lambda(G)$ colors, a vertex-coloring of $K_\ell[G]$ by $\ell\lambda(G)$ colors is easily constructed by considering λ new colors proper to every copy of G in $K_\ell[G]$. Conversely, given a coloring of $K_\ell[G]$ by $\lambda(K_\ell[G])$ colors, a vertex-coloring of G by at most $\lambda(K_\ell[G])/\ell$ colors can easily determined. This can be done by projecting the coloring of $K_\ell[G]$ onto each copy of G; we so obtain ℓ (not necessarily distinct) vertex-colorings of G and by choosing as final coloring-solution of G the projection inducing the best (largest) between the ℓ differential approximation ratios, we get the coloring claimed. So, every coloring-algorithm has on G approximation ratio greater than its approximation ratio on $K_\ell[G]$.

Suppose now that C can be approximately solved by an asymptotic DFPTAS with ratio as the one claimed and consider an instance G of C. Fix a positive integer ϵ and construct $K_{\lceil\epsilon^{-(1/\alpha)}\rceil}[G]$. Since the quantity $\lceil\epsilon^{-(1/\alpha)}\rceil$ is polynomial in n, the whole construction can be performed in polynomial time.

We run $\mathbf{A}_\epsilon$ on $K_{\lceil\epsilon^{-(1/\alpha)}\rceil}[G]$ and starting from the coloring solution obtained, we construct, as discussed just above, a coloring solution for G. For this latter coloring, the following holds:

$$\begin{aligned} \delta_{\mathbf{A}_\epsilon}(G) &\geq \delta_{\mathbf{A}_\epsilon}\left(K_{\lceil\epsilon^{-(1/\alpha)}\rceil}[G]\right) \\ &\geq 1 - \epsilon - \left(\frac{1}{\left\lceil\left(\frac{1}{\epsilon}\right)^{\left(\frac{1}{\alpha}\right)}\right\rceil[\omega(G) - \beta(G)]}\right)^{\alpha} \geq 1 - 2\epsilon. \end{aligned}$$

But $\delta_{\mathbf{A}_\epsilon}(G) \geq 1 - 2\epsilon$ means that C admits a real (non-asymptotic) DFPTAS, contradicting so the first item of the theorem and completing the proof of the second item. □

5. DISCUSSION

The purpose of this paper was less to produce particular results for combinatorial problems than to show that the differential approximation theory is as rich as the usual one. We have tried to enrich the differential polynomial approximation theory by adding new concepts (already existing in the classical approximation theory) and by justifying the working hypotheses associated with these concepts. In this spirit we have defined the class of radial problems which justifies why one can consider as "interesting instances", the ones for which the number of feasible solution-values is unbounded; next, based upon this characterization of the hard instances, we have introduced the concept of the differential asymptotic approximation ratio.

Next, we have proved that the definition of δ^{∞} is scientifically non-empty, since it provides a working-framework for the asymptotic analysis of the behaviour of approximation algorithms.

We now informally describe under which problematic we have adopted the quantity $\sigma(I)$ as threshold between easy and hard instances. First, we must notice that the concept itself of the instance has some structural characteristics which have to be explored in order to devise algorithms that both come up to the optimal solution and move away from the worst one (this is the spirit of the differential approximation measure).

In the exciting work of [2], the authors propose the following notion of the structure of an instance.

DEFINITION 5: **Structure of the instance of a combinatorial problem [2]**

The *structure* $S(I)$ of an instance I of a combinatorial problem is the list $(a_1, \cdots, a_c)$ with $c = |\beta(I) - \omega(I)|$ and $a_i = |\{\vec{x} \in C_I : |v(\vec{x}) - \omega(I)| = i\}|$, where C_I is the constraint-set of the instance I and $v(\vec{x})$ is the value of the solution $\vec{x}$.

This notion of instance-structure is quite adequate for unweighted (cardinality) combinatorial problems but somewhat missleading when weighted problems are treated[12]. So, in a first time, we define the generalized structure $S^*(I)$ of an instance I as follows.

[12] For example, the weighted version of IS is the one where weights are associated with the vertices of the graph and the objective becomes to maximize the sum of the weights of an independent set.

DEFINITION 6: **Generalized structure of an instance**

Given a combinatorial optimization problem Π, the *generalized structure* $S^*(I)$ of an instance I of Π is the list $((v_i, a_i))_{i=1,\ldots,\sigma}$, ordered in increasing order with respect to v_i, where σ is the number of feasible values of I and $a_i = |\{\vec{x} \in C_I : |v(\vec{x}) - \omega(I)| = v_i\}|$; let us note that $v_1 = 0$ and $v_\sigma = |\beta(I) - \omega(I)|$.

But even the concept of $S^*(I)$ has the drawback to be incompatible with the affine (and homothetic) transformations (and such transformations play, as we have seen, a crucial role to the consideration of the differential framework). Let us explain it by means of the following example.

Example 6: We revisit our old friend IS and express it as a linear-integer program:

$$\text{IS} = \begin{cases} \max & \vec{1}_{|V|} \cdot \vec{x} \\ & A \cdot \vec{x} \leq \vec{1}_{|E|} \\ & \vec{x} \in \{0,1\}^{|V|} \end{cases}$$

where A is the edge-vertex incidence matrix of the input-graph and, by $\vec{1}_D$, we denote the one-column vector of $\mathbb{R}^D$ ($D \in \mathbb{N}$), all the coordinates of which are equal to 1. For this problem, it can be seen that all the integer values in $[\beta(I), \omega(I)]$ can be attained.

Consider now the following problem:

$$\text{HIS} = \begin{cases} \max & \vec{2}_{|V|} \cdot \vec{x} \\ & A \cdot \vec{x} \leq \vec{1}_{|E|} \\ & \vec{x} \in \{0,1\}^{|V|} \end{cases}$$

where $\vec{2}_{|V|}$ is the one column vector of $\mathbb{R}^{|V|}$ with all its components equal to 2. For HIS, which is an IS where we simply search to maximize the double of an independent set of the input-graph (we apply a simple homothetic transformation of the objective function of IS), all integer values in $[\beta(I), \omega(I)]$ cannot be attained since the odd ones are never attained; so, HIS has not the same generalized structure as IS.

But homothetic transformations of the objective function (which are very natural, in particular when dealing with weighted problems) do not change the nature of the resulting problems (by making them, for instance, more or less significant regarding their asymptotic approximation behaviour). They

generate problems strictly identical (from optimization point of view) to the initial one, and in the differential approximation context, we consider that not only homothetic but also affine transformations generate problems of the same structure, since such problems are affine-equivalent.

So we replace the notion of the (generalized) structure by the one of the distribution $\Sigma(I)$ of the values of an instance defined as follows.

DEFINITION 7: **Distribution of the feasible values of an instance**

Given an instance I of a combinatorial optimization problem and the list $S^*(I)$, the *distribution* $\Sigma(I)$ of the feasible values of I is the equivalence class of $S^*(I)$ defined by the following equivalence relation:

$$\begin{aligned}&\{(v_i, a_i)\}_{i\in\{1,\ldots,p\}} \equiv \{(w_i, b_i)\}_{i\in\{1,\ldots,q\}}\\ &\quad\Longleftrightarrow (p = q) \wedge [\exists \lambda \in \mathbb{R}^*, \exists \mu \in \mathbb{R},\\ &\forall i \in \{1, \ldots, p\} : [(a_i = b_i) \wedge (\ v_i = \lambda w_i + \mu)]].\end{aligned}$$

The quantity P is exactly the threshold $\sigma(I)$ discussed in the previous sections.

We conclude this paper by noticing that the affine-equivalence preserves distribution. Moreover, all the "structure-preserving-reductions" of [2] preserve also distribution.

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