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THE LORENZ TRANSFORM APPROACH TO THE OPTIMAL REPAIR-COST LIMIT REPLACEMENT POLICY WITH IMPERFECT REPAIR

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Abstract. In this paper, we consider a repair-cost limit replacement problem with imperfect repair and develop a graphical method to determine the optimal repair-cost limit which minimizes the expected cost per unit time in the steady-state, using the Lorenz transform of the underlying repair-cost distribution function. The method proposed can be applied to an estimation problem of the optimal repair-cost limit from empirical repair-cost data. Numerical examples are devoted to examine asymptotic properties of the non-parametric estimator for the optimal repair-cost limit.

Keywords: Repair-cost limit, imperfect repair, Lorenz transform, nonparametric estimation, maintenance policy.

Mathematics Subject Classification. 90B25, 62N05, 62N02.

1. Introduction

The repair-cost limit replacement policies can provide how to design the recovery mechanism of a system using two maintenance options; repair and replacement, in terms of cost minimization. That is, if the repair cost of a failed unit is greater than the replacement cost, one should replace a failed unit, otherwise one should repair it. First this problem was considered by Drinkwater and Hastings [6] and Hastings [9] for army vehicles. Especially, Hastings [9] proposed

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three methods of optimizing the repair-cost limit replacement policies by simulation, hill-climbing and dynamic programming. Since the seminal contributions above, a number of authors dealt with a variety of repair-cost limit replacement problems. For instance, Nakagawa and Osaki [16], Kaio and Osaki [10] and Nguyen and Murthy [17] reformulated the Hastings' original problem from the viewpoint of renewal reward argument and discussed both continuous and discrete models. Also, Nguyen and Murthy [18] took account of the imperfection of repair and analyzed the repair-limit replacement model with imperfect repair. Love *et al*. [14] examined the similar problem for vehicle replacement using postal Canada data which was constructed by dividing the life of the vehicle into discrete ages. Park [19] considered a simple but interesting cost limit replacement policy under minimal repair. Love and Guo [15] extended the repair-limit analysis by incorporating a changing force of mortality as the unit ages in the framework of a Markov or semi-Markov decision process.

As Love and Guo [15] pointed out implicitly, it is often assumed that the repair-cost distribution function is arbitrary but known. Of course, this seems to be rather restrictive in many practical situations. In other words, practitioners have to determine the repair-cost limit under incomplete information on the repair-cost distribution in most cases. Dohi *et al*. [4, 5] proposed non-parametric estimators of the optimal repair-time limit and repair-cost limit from the empirical repair time and cost data, respectively. More precisely, they applied the total time on test (TTT) statistics to those estimation problems in accordance with the graphical idea by Bergman [1] and Bergman and Klefsjö [2]. If the optimal repair-time or cost limit has to be estimated from the sample data with unknown repair-time or cost distribution, their method will be useful in practice, since one needs not specify the repair-time or cost distribution in advance.

However, it should be noted that the repair-cost limit replacement problem in Dohi *et al*. [5] was very interesting but somewhat different from existing ones. More specifically, their main objective was to derive the optimal repair-cost limit to retire the repair action, *i.e.* if the repair is not completed within a cost limit, the failed unit is scrapped and then a new spare is ordered. Such a policy seems to be plausible in some practical situations, but should be distinguished from the original repair-cost limit problem. In this paper, we consider a repair-cost limit replacement problem with imperfect repair in the framework of renewal reward processes and propose a statistical estimation method based on the Lorenz curve. Notice that the basic idea in this paper is similar to the graphical one used in Dohi *et al*. [5] but the statistical device employed here is different from the TTT statistics. The Lorenz curve was first introduced by Lorenz [13] into economics to describe income distributions. Since the Lorenz curve is essentially equivalent to the Pareto curve used in the quality control, it will be one of the most important statistics in every social sciences.

The more general and tractable definition of the Lorenz curve was made by Gastwirth [7]. Goldie [8] proved the strong consistency of the empirical Lorenz curve and discovered its several convergence properties. Chandra and Singpurwalla [3] and Klefsjö [11] investigated the relationship between the TTT statistics and the Lorenz statistics, and derived a few aging and partial ordering properties. Recently, the further results on two statistics were examined by Pham and Turkkan [21] and Perez–Ocon *et al*. [20]. It is shown that the estimator of the optimal repair-cost limit derived in this paper has also several powerful properties proved in earlier contributions above.

The paper is organized as follows. In Section 2, we describe the repair-cost limit replacement problem with imperfect repair under consideration. In Section 3 we develop a graphical method to calculate the optimal repair-cost limit which minimizes the expected cost per unit time in the steady-state. Then, it is seen that the Lorenz curve plays an important role to derive the optimal solution on the graph. In Section 4, the statistical estimation problem is discussed. We show that the estimator of the optimal repair-cost limit has a strong consistency, and examine its convergence properties. Numerical examples are presented for illustration of the graphical method throughout the paper.

2. Model description

Notation

The repair cost V for each unit is a non-negative i.i.d. random variable and unknown. The decision maker has a *subjective* probability distribution function $\Pr\{V \leq v\} = H(v)$ on the repair cost, with density $h(v)(> 0)$ and finite mean m_m (> 0). We suppose that the distribution function $H(v) \in [0, 1]$ is arbitrary, continuous and strictly increasing in $v \in [0, \infty)$, and has an inverse function, *i.e.* $H^{-1}(\cdot)$. Further, we define:

$v_0 \in [0, \infty)$**:** repair cost limit;
$m_l(> 0)$**:** mean time to failure for a new unit;
$m_s(> 0)$**:** mean time to failure for a repaired unit;
$m_a(> 0)$**:** mean repair time (for units repaired);
$k_f(> 0)$**:** penalty cost per unit time when the system is in down state;
$c(> 0)$**:** cost associated with the ordering of a new unit;
$L(> 0)$**:** lead time for delivery of a new unit.

Model description

Consider a single-unit repairable system, where each spare is provided only by an order after a lead time L and each failed unit is repairable. The original unit begins operating at time 0. When the unit has failed, the decision maker wishes to determine whether he or she should repair it or order a new spare. If the decision maker estimates that the repair is completed within a prespecified cost limit $v_0 \in [0, \infty)$, then the repair is started immediately at the failure time. In this case, the mean time to complete the repair is m_a. Since the repair is imperfect, the unit fails again for a finite time span. Then, the mean failure time when the repair is completed is m_s.

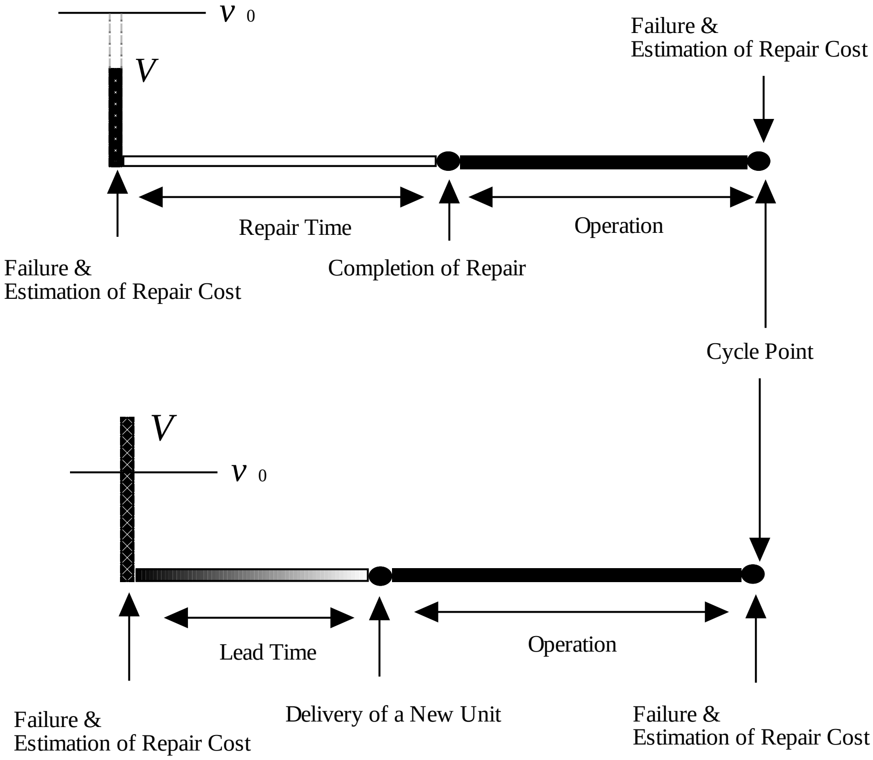

FIGURE 1. Configuration of repair-cost limit replacement problem with imperfect repair.

On the other hand, if the decision maker estimates that the repair cost exceeds the cost limit v_0, then the failed unit is scrapped immediately and a new spare unit is ordered. After the lead time L, a new unit is delivered. Further, the new unit also fails for a finite time span and then the mean failure time is m_l. Without any loss of generality, it is assumed that the time required for replacement is negligible. Under these model assumptions, we define the interval from the failure point of time to the following failure time as one cycle. Figure 1 depicts the configuration of the model under consideration.

We make the following additional assumptions:

(A-1) $m_a + m_s > L + m_l$.
(A-2) $k_f m_a < k_f L + c$.

The assumption (A-1) implies that the mean time length of one cycle when the repair cost does not arrive at the repair-cost limit is greater than that when the cost is beyond it with probability one. In the assumption (A-2), it is meant that the mean shortage cost to finish the repair when the repair is executed is smaller than the mean shortage cost plus the ordering cost in another case. It is noticed that these assumptions motivate the underlying problem to determine the optimal repair-cost limit.

Analysis

Let us formulate the expected cost during one cycle. If the decision maker judges that a new spare unit should be ordered, then the ordering cost for one cycle is $c\overline{H}(v_0)$, where $\overline{H}(\cdot) = 1 - H(\cdot)$. In this case, the expected shortage cost for one cycle is $k_f L\overline{H}(v_0)$. On the other hand, if he or she selects the repair option, the expected repair cost for one cycle is $\int_0^{v_0} v\mathrm{d}H(v)$ and the expected shortage cost for one cycle is $k_f m_a H(v_0)$. Thus the total expected cost during one cycle is

$$E_C(v_0) = \int_0^{v_0} v\mathrm{d}H(v) + (k_f m_a - c - k_f L)H(v_0) + c + k_f L. \tag{1}$$

Also, the mean time length of one cycle is

$$E_T(v_0) = (m_a + m_s - L - m_l)H(v_0) + L + m_l. \tag{2}$$

It may be appropriate to adopt an expected cost per unit time in the steady-state over an infinite planning horizon, as a criterion of optimality. The expected cost per unit time in the steady-state is, from the renewal reward argument,

$$C(v_0) = \lim_{t\to\infty} \frac{E[\text{the total cost on } (0,t]]}{t} = E_C(v_0)/E_T(v_0). \tag{3}$$

The problem is to derive the optimal repair-cost limit ${v_0}^*$ such as

$$C(v_0^*) = \min_{0\le v_0<\infty} C(v_0). \tag{4}$$

Then, we have the following result on the optimal repair-cost limit.

Theorem 2.1. *Define the numerator of the derivative of equation (3) with respect to v_0, divided by $h(v_0)$, as $q_0(v_0)$, i.e.*

$$q_0(v_0) \equiv (k_f m_a - c - k_f L + v_0)E_T(v_0) - (m_a + m_s - L - m_l)E_C(v_0). \tag{5}$$

Suppose that both assumptions (A-1) and (A-2) hold. Then there exists a unique optimal repair-cost limit v_0^* $(0 < v_0^* < \infty)$ satisfying $q_0(v_0^*) = 0$ and the minimum expected cost is

$$C(v_0^*) = \frac{k_f m_a - c - k_f L + v_0^*}{m_a + m_s - L - m_l}. \tag{6}$$

Proof. Differentiating $C(v_0)$ with respect to v_0 and setting it equal to zero implies $q_0(v_0) = 0$. This leads to $\mathrm{d}q_0(v_0)/\mathrm{d}v_0 = E_T(v_0) > 0$ and the fact that the function $C(v_0)$ is strictly convex in v_0. Since $\lim_{v_0\to\infty} q(v_0) \to \infty$ and $q(0) < 0$ under

(A-1) and (A-2), there exists a unique optimal repair-cost limit v_0^* $(0 < v_0^* < \infty)$ satisfying $q_0(v_0^*) = 0$. The proof is completed. □

From Theorem 2.1, one sees that the optimal repair-cost limit can be calculated easily, by solving the nonlinear equation $q_0(v_0) = 0$, if the repair-cost distribution is completely known. In the following section, the minimization problem in equation (4) is transformed to a simple graphical one on the Lorenz curve.

3. Graphical method

In stead of differentiating $C(v_0)$ with respect to v_0 directly, we here employ an interesting graphical method. Define the Lorenz transform of the repair-cost distribution $p \equiv H(v)$ by

$$\phi(p) = \frac{1}{m_m} \int_0^p H^{-1}(v)\mathrm{d}v, \; 0 \leq p \leq 1. \tag{7}$$

Then the curve $\mathcal{L} = (p, \phi(p)) \in [0,1] \times [0,1]$ is called the *Lorenz curve* (see Gastwirth [7] and Goldie [8]). It should be noted that the curve $\mathcal{L}$ is absolutely continuous from the continuity of $H(v)$. From a few simple algebraic manipulations, we have the following useful result:

Theorem 3.1. *Suppose that the assumption (A-1) holds. The minimization problem in equation (4) is equivalent to*

$$\min_{0 \leq p \leq 1} : M(p, \phi(p)) \equiv \frac{\phi(p) + \xi}{p + \eta}, \tag{8}$$

where

$$\xi = \frac{1}{m_m} \left\{ c + k_f L - \frac{(L + m_l)(k_f m_a - c - k_f L)}{m_a + m_s - L - m_l} \right\} \tag{9}$$

and

$$\eta = \frac{L + m_l}{m_a + m_s - L - m_l}. \tag{10}$$

The proof is omitted for brevity. From Theorem 3.1, the optimal repair-cost limit is determined by $p^* = H(v_0^*)$ which minimizes the slope from the point $B = (-\eta, -\xi) \in (-\infty, 0) \times (-\infty, 0)$ to the curve $\mathcal{L}$ in the plane $(x, y) \in (-\infty, 1)$ $\times(-\infty, 1)$ under the assumption (A-2).

More precisely, we prove the uniqueness of the optimal repair-cost limit.

Theorem 3.2. *Suppose that both assumptions (A-1) and (A-2) hold. Then there exists a unique optimal solution $p^* = H(v_0^*)$ $(0 < v_0^* < \infty)$ minimizing $M(p, \phi(p))$, where p^* is given by the x-coordinate at the point of contact for the curve $\mathcal{L}$ and the straightline from the point B.*

Proof. From (A-1) and (A-2), it can be seen that $\xi > 0$ and $\eta > 0$. Differentiating $M(p, \phi(p))$ with respect to p and setting it equal to zero implies

$$q(p) \equiv (\mathrm{d}\phi(p)/\mathrm{d}p)(p+\eta) - (\phi(p)+\xi), \tag{11}$$

where $\mathrm{d}\phi(p)/\mathrm{d}p = H^{-1}(p)/m_m$. Further, we have

$$\mathrm{d}q(p)/\mathrm{d}p = \mathrm{d}^2\phi(p)/\mathrm{d}p^2(p+\eta) > 0 \tag{12}$$

and the function $M(p, \phi(p))$ is strictly convex in p, since $\mathrm{d}^2\phi(p)/\mathrm{d}p^2 = 1/\{m_m h(H^{-1}(p))\} > 0$. From $q(0) = -\xi < 0$ and $q(1) \to \infty$, the proof is completed. □

The result above is a dual theorem and is essentially same as Theorem 2.1. The interesting point of Theorem 3.2 is to determine the optimal solution on the graph instead of solving the nonlinear equation.

Example 3.3. We give an example for the graphical method proposed above. Suppose that the repair-cost distribution $H(v)$ is known and obeys the Weibull distribution;

$$H(v) = \exp\left\{-\left(\frac{v}{\theta}\right)^{\beta}\right\} \tag{13}$$

with the shape parameter $\beta = 4.0$ and the scale parameter $\theta = 0.9$. The other model parameters are $c = 0.4000$ (\$), $L = 0.3500$ (day), $k_f = 0.3500$ (\$/day) $m_a = 0.5500$ (day), $m_s = 1.2000$ (day), $m_l = 0.4500$ (day) and $m_m = 0.8862$ (\$). The determination of the optimal repair-cost limit is presented in Figure 2. In this case, we have $B = (-0.8421, -0.9032)$ and the optimal point with minimum slope from B is $(p^*, \phi(p^*)) = (0.4630, 0.2574)$. Thus, the optimal repair-cost limit is $v_0^* = H^{-1}(0.4630) = 0.7885$.

The basic idea for the graphical method proposed in this section can be applied to an estimation problem of the optimal repair-cost limit when the empirical repair-cost data are available. In the following section, the statistical optimization technique is developed for the empirical counterparts of the Lorenz curve.

4. Statistical estimation method

Based on the graphical idea in Section 3, we propose a non-parametric method to estimate the optimal repair-cost limit. Suppose that the optimal repair-cost limit has to be estimated from an ordered complete sample $0 = x_0 \le x_1 \le x_2 \le \cdots \le x_n$ of repair cost data from an absolutely continuous repair-cost distribution H, which is unknown. The estimator of $H(v) = p$ is the empirical distribution

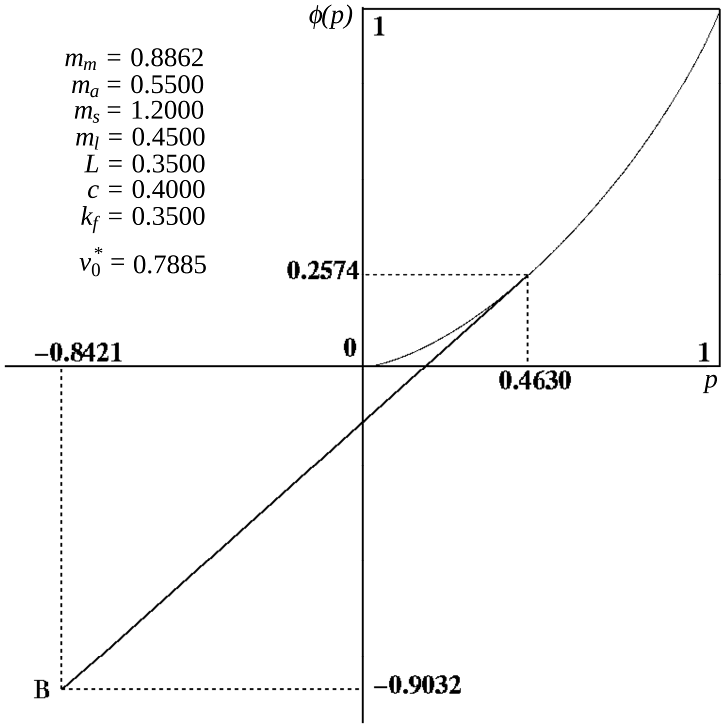

FIGURE 2. Determination of optimal repair-cost limit on the Lorenz curve.

given by

$$H_{in}(x) = \begin{cases} i/n & \text{for} \quad x_i \le x < x_{i+1}, i = 0, 1, 2, \cdots, n-1 \\ 1 & \text{for} \quad x_n \le x, i = n. \end{cases} \tag{14}$$

Then the *Lorenz statistics* (see Goldie [8]) is defined as

$$\phi_{in} \equiv \sum_{k=1}^{[np]} x_k / \sum_{k=1}^{n} x_k, \ \phi_{0n} = 0, i = 1, 2, 3, \cdots, n, \tag{15}$$

where $[a]$ is the greatest integer in a. Plotting the point $(i/n, \phi_{in})$, $i = 0, 1, 2, \cdots, n$, and connecting them by line segments, we obtain the empirical Lorenz curve $\mathcal{L}_n \in [0, 1] \times [0, 1]$.

As an empirical counterpart of Theorem 3.1, we obtain a non-parametric estimator of the optimal repair-cost limit in the following theorem:

Theorem 4.1. *(i) The optimal repair-cost limit can be estimated by* $\hat{v}_{0n}^* = x_{i^*}$*, where*

$$\left\{ i^* \mid \min_{0 \le i \le n} \frac{\phi_{in} + \xi}{i/n + \eta} \right\}. \tag{16}$$

(ii) The estimator $\hat{v}_{0n}^* = x_{i^*}$ *in equation (16) is strongly consistent, i.e.* $\hat{v}_{0n}^* = x_{i^*} \to v_0^*$ *as* $n \to \infty$.

The result in (i) is trivial. The proof of (ii) is based on the asymptotic property $\phi_{in} \to \phi(p)$ as $n \to \infty$, which is due to Goldie [8].

Example 4.2. The repair-cost data are made by the random number following the Weibull distribution with shape parameter $\beta = 4.0$ and scale parameter $\theta = 0.9$. The other model parameters are same as Example 3.3 except that $m_m = 0.8076$ (\$). The empirical Lorenz curve based on the 30 sample data is shown in Figure 3. Since $B = (-0.8421, -0.9911)$, the optimal point with minimum slope from B becomes $(i^*/n, \phi_{i^*n}) = (14/30, \phi_{14,30}) = (0.5000, 0.3061)$. Thus, the estimate of the optimal repair-cost limit is $\hat{v}_{0n}^* = 0.7618$ (\$).

If the estimator $\hat{v}_{0n}^* = x_{i^*}$ is obtained, it is easy to calculate the estimate of the minimum expected cost. That is, from equation (6),

$$C(\hat{v}_{0n}^*) = \frac{k_f m_a - c - k_f L + \hat{v}_{0n}^*}{m_a + m_s - L - m_l}, \tag{17}$$

which may be strongly consistent.

Of our next interest is the convergence speed of the estimators $\hat{v}_{0n}^*$ and $TC(\hat{v}_{0n}^*)$. We examine numerically the strong consistency of the estimator derived in Theorem 4.1.

Example 4.3. Suppose that the repair-cost distribution and model parameters are similar to those in Example 4.2. Then the real optimal repair-cost limit and the minimum expected cost become $v_0^* = 0.7885$ (\$) and $C(v_0^*) = 0.4826$ (\$), respectively. On the other hand, the asymptotic behaviours of estimates for the optimal repair-cost limit and its associated minimum expected cost are depicted in Figures 4 and 5, respectively. From these figures, we observe that the estimates converge to the corresponding real values around where the number of data is 30. In other words, without specifying the repair-cost distribution, the proposed non-parametric method may function to estimate the optimal repair-cost limit precisely.

Finally, utilizing the results above, we determine the asymptotic valid confidence interval for the optimal repair-cost limit approximately. Recall that determining asymptotically valid confidence intervals for the probability distribution function is based on the normal approximation to the binomial distribution, when there are no

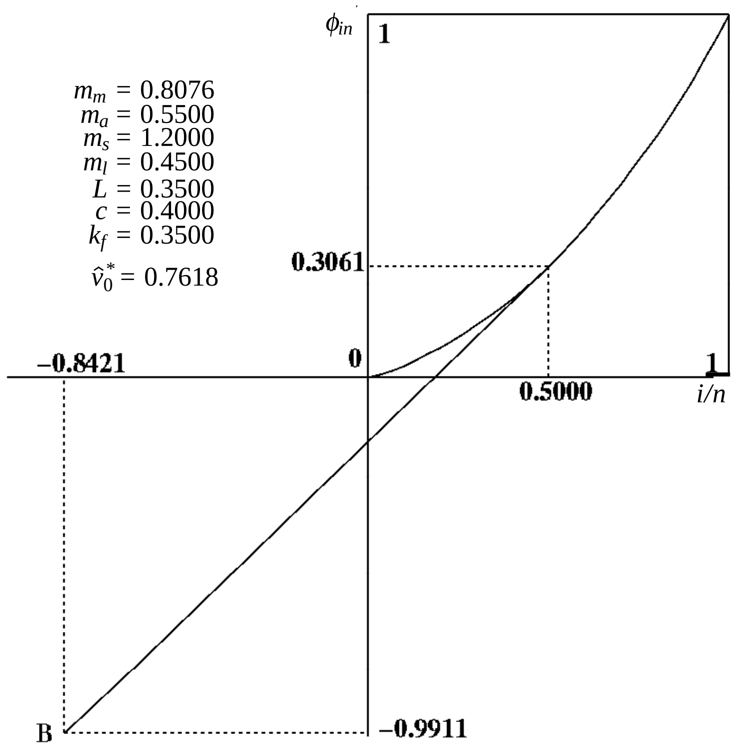

FIGURE 3. Estimation of optimal repair-cost limit on the empirical Lorenz curve.

ties in the data set. Notice that the empirical distribution defined in equation (14) can be regarded as the binomial random variable having mean $E[H_{in}(x)] = H(x)$ and variance $\mathrm{Var}[H_{in}(x)] = H(x)\overline{H}(x)/n$. Furthermore, when the sample size n is large and $H(x)$ is not too close to 0 or 1, the binomial distribution may have a shape that is closely approximated by a normal distribution, and can be used to find interval estimates for $H(x)$. It should be noted that these interval estimates are most accurate, in terms of coverage, around the median of the distribution, since the normal approximation to the binomial distribution works best when the probability of success is about 0.5, where the binomial distribution is symmetric.

Replacing $H(v)$ by $H_{in}(v)$ in the variance formula, an asymptotically valid $100(1-\alpha)\%$ confidence interval for the repair-cost distribition is approximately

$$H_{in}^L < H(v) < H_{in}^U, \tag{18}$$

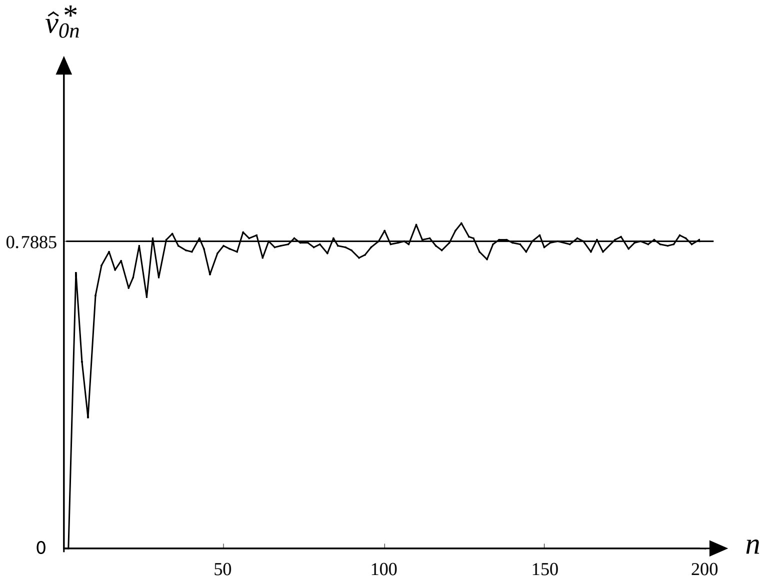

FIGURE 4. Asymptotic property of estimates for the optimal repair-cost limit.

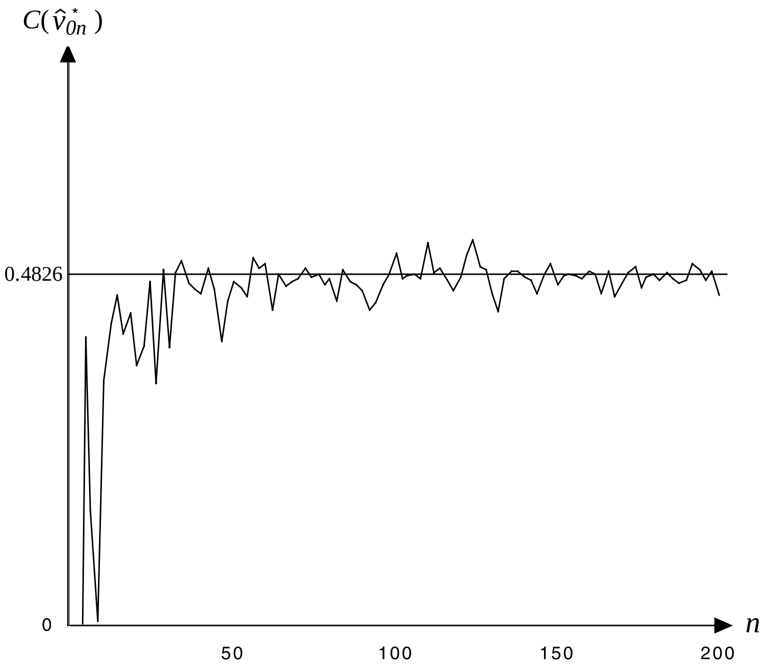

FIGURE 5. Asymptotic property of estimates for the associated minimum expected cost.

where $z_{\alpha/2}$ is the $1-\alpha/2$ fractile of the standard normal distribution, and the lower and upper bounds are

$$H_{in}^{L} = H_{in}(v) - z_{\alpha/2}\sqrt{\frac{H_{in}(v)\overline{H}_{in}(v)}{n}} \tag{19}$$

and

$$H_{in}^{U} = H_{in}(v) + z_{\alpha/2}\sqrt{\frac{H_{in}(v)\overline{H}_{in}(v)}{n}}, \tag{20}$$

respectively. This confidence interval is appropriate when there are tied observations, although it becomes more approximate as the number of ties increases (see Lee [12]). The following theorem is the direct application of the asymptotically valid $100(1-\alpha)\%$ confidence interval for the repair-cost distribution.

Theorem 4.4. *(i) The lower and upper bounds of the Lorenz transform are approximately given by*

$$\phi_{in}^{L} < \phi(H^{-1}(v)) < \phi_{in}^{U}, \tag{21}$$

where

$$\phi_{in}^{L} = \sum_{k=1}^{[nH_{in}^{L}]} x_k \Big/ \sum_{k=1}^{n} x_k \tag{22}$$

and

$$\phi_{in}^{U} = \sum_{k=1}^{[nH_{in}^{U}]} x_k \Big/ \sum_{k=1}^{n} x_k, \tag{23}$$

respectively.

(ii) The asymptotically valid $100(1-\alpha)\%$ confidence interval for the optimal repair-cost limit is approximately given by

$$v_{0n}^{L} < \hat{v}_{0n}^{*} < v_{0n}^{U}, \tag{24}$$

where $v_{0n}^{U} = x_{j^*}$ and $v_{0n}^{L} = x_{k^*}$ satisfy

$$\left\{ j^* \mid \min_{0\le j\le n} \frac{\phi_{jn}^{L}+\xi}{H_{jn}^{L}+\eta} \right\} \text{ and } \left\{ k^* \mid \min_{0\le k\le n} \frac{\phi_{kn}^{U}+\xi}{H_{kn}^{U}+\eta} \right\}, \tag{25}$$

respectively.

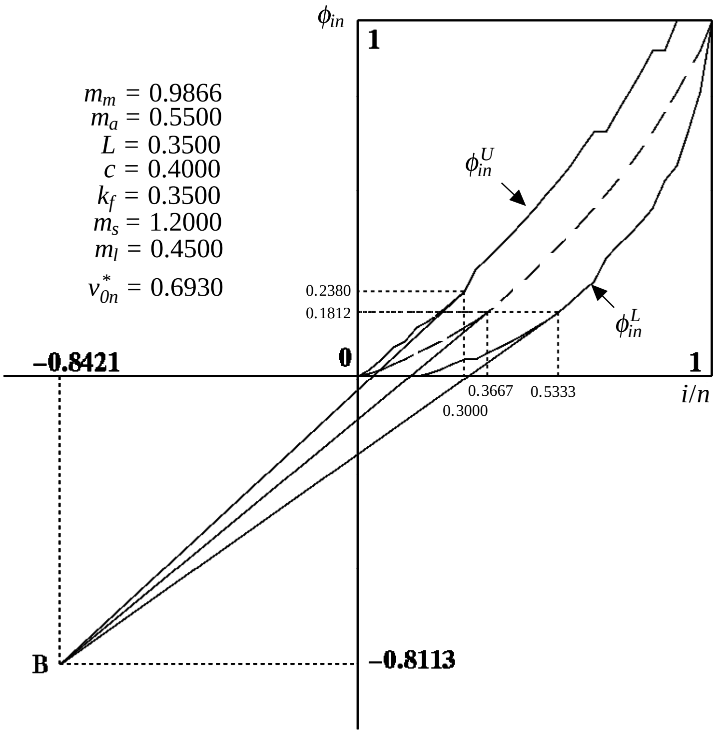

FIGURE 6. Upper and lower bounds of the empirical Lorenz curve.

Example 4.5. Under the same parameters as Example 4.3 except for $m_m = 0.9866$, we determine the asymptotic valid confidence interval for the optimal repair-cost limit. Figure 6 shows the upper and lower bounds of the empirical Lorenz curve. From this figure, we obtain v^L_{0n} and v^U_{0n} which minimize the tangent slopes from the point B to the curves ϕ^U_{in} and ϕ^L_{in}, respectively. In Figures 7 and 8 show the behaviours of the asymptotically valid 95% confidence intervals of the optimal repair-cost limit and its associated minimum expected cost, respectively. These figures tell us that the estimation when the number of data is more than 30 is stable, and that the observation result in Example 4.3 can be also valid taking account of the asymptotically confidence interval.

5. Conclusion

This paper has considered an interesting repair-cost limit replacement problem with imperfect repair and developed a graphical method to determine the optimal repair-cost limit which minimizes the expected cost per unit time in the steady-state, using the Lorenz transform of the underlying repair-cost distribution

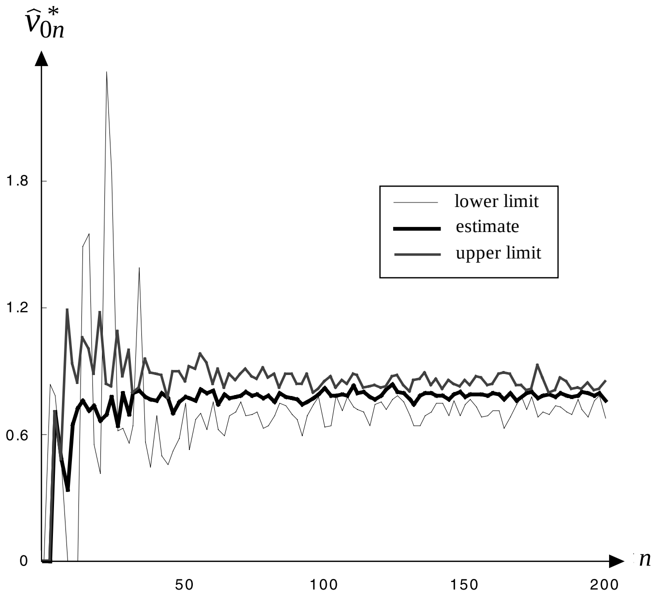

FIGURE 7. The asymptotically valid $100(1-\alpha)\%$ confidence interval of the optimal repair-cost limit.

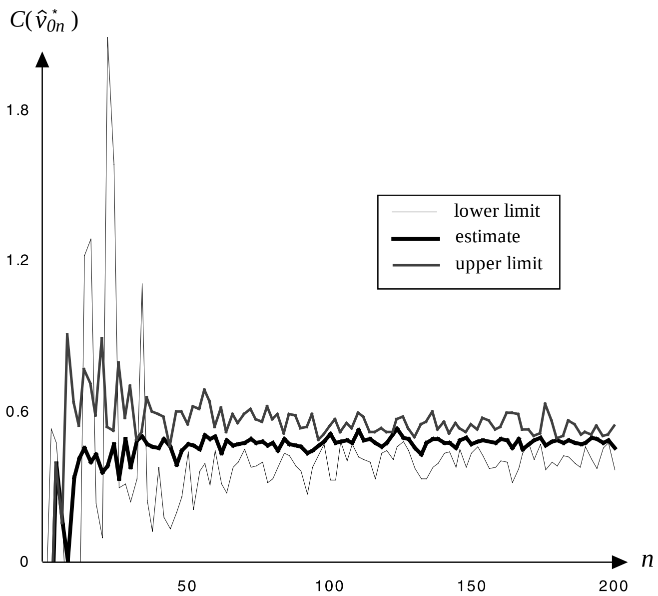

FIGURE 8. The asymptotically valid $100(1-\alpha)\%$ confidence interval of the minimum expected cost.

function. We have examined some properties of the strongly consistent estimator and the asymptotically valid $100(1-\alpha)\%$ confidence interval for the optimal repair-cost limit and the associated minimum expected cost throughout numerical examples.

The main contribution of this paper is to show that the Lorenz statistics as well as TTT statistics is a useful device to estimate the optimal maintenance schedule. This simple but interesting idea should be applied to solve other kinds of stochastic maintenance optimization problems in future.

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