RAIRO Operations Research
RAIRO Oper. Res. **35** (2001) 383-394

A NEW RELAXATION IN CONIC FORM FOR THE EUCLIDEAN STEINER PROBLEM IN $\Re^n$

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Communicated by Pierre Tolla

Abstract. In this paper, we present a new mathematical programming formulation for the Euclidean Steiner Tree Problem (ESTP) in $\Re^n$. We relax the integrality constrains on this formulation and transform the resulting relaxation, which is convex, but not everywhere differentiable, into a standard convex programming problem in conic form. We consider then an efficient computation of an ϵ-optimal solution for this latter problem using interior-point algorithm.

Keywords: Euclidean Steiner tree problem, conic form, interior point algorithms.

1. Introduction

The Euclidean Steiner Tree Problem (ESTP) in $\Re^n$ can be defined as follows: given p points in $\Re^n$, find a minimum tree that spans these points using or not extra points, which are called Steiner points. The distances considered between points are Euclidean. This is a very well known problem in combinatorial optimization, see [6]. It has been considered since the 17$^{\text{th}}$ century, when Fermat proposed the following problem: given three points in the plane, find a fourth point such that the sum of its distance to the three given points is a minimum. Torricelli, in

Received May, 2000.

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1647, proved that the circle circumscribing the equilateral triangles constructed on the sides of and outside the given triangle intersect in the point that is sought. Heinen, in 1837, apparently, is the first to prove that, for a triangle in which one angle is greater than or equal to 120°, the vertex associated with this angle is the minimizing point.

Bellow, we give some well known properties that can be useful when obtaining a solution for the ESTP, see [2,6] and [8].

- Property 1
  Given p points $a^i \in \Re^n$, $i = 1, \ldots, p$, the maximum number of Seiner points is $p - 2$.
- Property 2
  A Steiner point has degree equal to 3.
- Property 3
  The edges emanating from a Steiner point lie in a plane and have mutual angle equal to 120°.

We observe that on the ESTP, we have to determine the number of Steiner points to be used on the minimal Steiner tree, the edges of this tree, and finally, the geometrical position of the Steiner points. We define a topology of a Steiner tree as a tree for which we have fixed the number of Steiner points and the links between points, but we do not know the geometrical position of Steiner points. Based on Property 1, we define a full Steiner topology as a topology with $p - 2$ Steiner points. A tree which corresponds to some topology, but with certain edges shrunk to zero length, is said to be degenerate. All nonfull topologies are thus degenerate full topologies. Some research have been done recently on the computation of the minimal Steiner tree, for a given topology. See Hwang [4], Hwang and Weng [5], Smith [11] and Xue and Ye [12], for example. Xue and Ye [12] present an interior-point algorithm, which efficiently computes an ϵ-optimal solution to the shortest network under a given full Steiner topology interconnecting N points. The problem is given as an application to the problem of minimizing a sum of Euclidean norms, which is initially transformed into a problem in conic form. In this paper we apply a similar transformation to the relaxation proposed for the ESTP. Our goal is to present a relaxation for the problem, for which we can also compute an ϵ-optimal solution efficiently, applying interior-point algorithms.

We consider the Euclidean Steiner tree problem without any given topology. Our final interest would be then, in the computation of a global minimum for the ESTP.

The use of interior-point algorithms in the solution of nonlinear problems, such as semidefinite programming problems, have been considered lately by several authors in the computation of bounds for the solution of difficult combinatorial optimization problems (see [1,3,10]). For the max-cut and max-stable problems, for example, there are well known semidefinite relaxations, as discussed by Lemaréchal and Oustry in [7].

The rest of the paper is organized as follows. In Section 2, we describe the ESTP and present a mixed-integer programming formulation for it. In Section 3, we relax the integrality constrains on the formulation and transform the given relaxation into a standard convex programming problem in conic form. In Section 4, we show that an ϵ-optimal solution for the relaxation can be computed efficiently using interior-point algorithms. We prove the existence of an interior solution for the problem and its dual by constructing one such solution to be used as an initial point for the interior-point algorithms. We conclude the paper in Section 6, and define some matrices used to formulate the problem in the Appendix.

Notation. In the rest of this paper, when we represent a large matrix with several small matrices, we will use semicolons ";" for column concatenation and commas "," for row concatenation. This notation also applies to vectors. We will use the number 0 to represent a matrix with all elements equal to zero and dimension indicated by the context. The letter e will denote the vector of ones, $e = [1 \dots 1]^T$ and e_i denotes the vector with the i-th component equal to one and all the others equal to zero. Their dimensions are also indicated by the context. Given a vector $x \in \Re^n$, x_k denotes the k-th component of x, for $k = 1, \dots, n$.

2. A mathematical programming formulation for ESTP

Following the notation used by Maculan *et al.* [8], we consider a special graph $G = (V, E)$ as follows.

Let $P = \{1, 2, \dots, p-1, p\}$ be the set of indices associated with the given points $a^1, a^2, \dots, a^{p-1}, a^p$ ($a^i \in \Re^n$, for $i = 1, \dots, n$) and $S = \{p+1, p+2, \dots, 2p-3, 2p-2\}$ be the set of indices associated with the Steiner points $x^{p+1}, x^{p+2}, \dots, x^{2p-3}, x^{2p-2}$ ($x^i \in \Re^n$, for $i = 1, \dots, n$). We take $V = P \cup S$. We denote by $[i, j]$, $i, j \in V$, such that, $i < j$, an edge of G. Thus we also consider $E_1 = \{[i, j] | i \in P, j \in S\}$ and $E_2 = \{[i, j] | i \in S, j \in S\}$, and we define $E = E_1 \cup E_2$.

We denote by $\|\cdot\|$ the Euclidean norm and define $y_{ij} \in \{0, 1\}$ for $[i, j] \in E$, where $y_{ij} = 1$ if edge $[i, j]$ is in the Steiner tree solution and $y_{ij} = 0$ otherwise. Finally, we let $M = \text{maximum}\{\|a^i - a^j\| \text{for } 1 \le i < j \le p\}$. Since the Steiner vertices are in the convex hull of the p given points, we have $\|a^i - x^j\| \le M$, $[i, j] \in E_1$ and $\|x^i - x^j\| \le M$, $[i, j] \in E_2$. We consider, without any loss of generality, that $(a^i)_k \ge 0$, for $i = 1, \dots, p$ and $k = 1, \dots, n$. Therefore, we also have $(x^i)_k \ge 0$, for $i = p+1, \dots, 2p-2$ and $k = 1, \dots, n$. We propose then, the following mathematical model for ESTP:

$$
\begin{array}{rlr}
\text{Minimize} & \sum_{(i,j) \in E} d_{ij} & \\
\text{subject to:} & d_{ij} \ge \|a^i - x^j\| - M(1 - y_{ij}), & [i, j] \in E_1 \\
& d_{ij} \ge \|x^i - x^j\| - M(1 - y_{ij}), & [i, j] \in E_2 \\
& d_{ij} \ge 0 & [i, j] \in E \\
& \sum_{j \in S} y_{ij} = 1, & i \in P \\
& \sum_{i < j, i \in S} y_{ij} = 1, & j \in S - \{p+1\} \\
& y_{ij} \in \{0, 1\}, & [i, j] \in E \\
& d_{ij} \in \Re, & [i, j] \in E \\
& x^i \in \Re^n, & i \in S.
\end{array}
\qquad (1)
$$

We observe that Property 1, given above, was considered to determine the size of the set S equal to $p-2$. Maculan *et al.* show in [8] that all full Steiner topologies corresponding to the p given points are feasible solutions of (1). Considering that all nonfull topologies are degenerate full topologies, we see that the feasible set of the given formulation for the ESTP, contains every Steiner tree with at most $p-2$ Steiner points.

The three first constrains on the formulation, determine that the distance between x^i and x^j or between a^i and x^j is only considered on the objective function when the edge $[i,j]$ is in the Steiner tree solution (*i.e.*, when $y_{ij}=1$). The fourth constrain is used to fix each vertex in P with degree equal to 1. And the fifth constrain is used to avoid the formation of subtours among the Steiner vertices, as explained in [8].

3. A new relaxation for ESTP in conic form

In this section, we first relax the integrality constrains of the formulation presented on the previous section. The resulting relaxation for ESTP is then given by the following problem, which is convex, but not everywhere differentiable:

$$\begin{array}{rll} \text{Minimize} & \sum_{(i,j)\in E} d_{ij} & \\ \text{subject to:} & d_{ij} \geq \|a^i - x^j\| - M(1-y_{ij}), & [i,j]\in E_1 \\ & d_{ij} \geq \|x^i - x^j\| - M(1-y_{ij}), & [i,j]\in E_2 \\ & d_{ij} \geq 0 & [i,j]\in E \\ & \sum_{j\in S} y_{ij} = 1, & i\in P \\ & \sum_{i<j, i\in S} y_{ij} = 1, & j \in S - \{p+1\} \\ & 0 \leq y_{ij} \leq 1, & [i,j]\in E \\ & d_{ij}, y_{ij} \in \Re, & [i,j]\in E \\ & x^i \in \Re^n, & i\in S. \end{array} \tag{2}$$

Consider now the slacks variables h_{ij} for $[i,j]\in E$, associated to the constrains $y_{ij}\leq 1$, and the following equivalences:

$$d_{ij} \geq \|x^i - x^j\| - M(1-y_{ij}) \Longleftrightarrow \left\{ \begin{array}{l} z_{ij} = d_{ij} + M(1-y_{ij}) \\ z_{ij} \geq \|x^i - x^j\|, \end{array} \right.$$

and

$$d_{ij} \geq \|a^i - x^j\| - M(1-y_{ij}) \Longleftrightarrow \left\{ \begin{array}{l} z_{ij} = d_{ij} + M(1-y_{ij}) \\ z_{ij} \geq \|a^i - x^j\|, \end{array} \right.$$

where $z_{ij}\in\Re$ for $[i,j]\in E$. The relaxation (2) could then be formulated as the minimization of a linear function subject to affine and convex cone constrains as follows:

$$
\begin{array}{rll}
\text{Minimize} & \sum_{(i,j)\in E} d_{ij} & \\
\text{subject to:} & z_{ij} - d_{ij} + My_{ij} = M, & [i,j] \in E \\
& \sum_{j\in S} y_{ij} = 1, & i \in P \\
& \sum_{i<j, i\in S} y_{ij} = 1, & j \in S - \{p+1\} \\
& y_{ij} + h_{ij} = 1, & [i,j] \in E \\
& z_{ij} \geq \|a^i - x^j\|, & [i,j] \in E_1 \\
& z_{ij} \geq \|x^i - x^j\|, & [i,j] \in E_2 \\
& d_{ij}, y_{ij}, h_{ij} \geq 0 & [i,j] \in E \\
& d_{ij}, y_{ij}, z_{ij}, h_{ij} \in \Re, & [i,j] \in E \\
& x^i \in \Re^n, & i \in S.
\end{array}
\tag{3}
$$

Our goal now is to transform the relaxation (3) into a standard convex programming problem in conic form, where the cone and its associated barrier are self-scaled, as defined by Nesterov and Todd in [9]. We consider two cones mentioned as examples of self-scaled cones in [9]. The first one is the positive orthant defined by

$$K_1 := \{x \in \Re^n : x_k \geq 0, k = 1, \dots, n\}\cdot$$

The interior of K_1 is defined by

$$K_1^0 := \{x \in \Re^n : x_k > 0, k = 1, \dots, n\}\cdot$$

In [9], it is shown that K_1 is a self-scaled cone, which admits an n-self-scaled barrier defined by $F_1(x) := -\sum_{k=1}^{n} \ln x_k$.

The other cone considered in this paper is the second-order cone defined by

$$K_2 := \{(t; s) \in \Re^{n+1} : t \geq \|s\|\},$$

which is shown in [9] to be also a self-scaled cone. Its interior is defined by

$$K_2^0 := \{(t; s) \in \Re^{n+1} : t > \|s\|\}\cdot$$

The function $F_2(t, s) := -\ln(t^2 - \|s\|)$ is a 2-self-scaled barrier for K_2.

Let us consider now the following equivalences for the inequalities constrains of relaxation (3).

$$z_{ij} \geq \|a^i - x^j\| \Longleftrightarrow \begin{cases} s_{ij} = a^i - x^j \\ z_{ij} \geq \|s_{ij}\| \end{cases}, \quad \text{where} \quad [i,j] \in E_1$$

and

$$z_{ij} \geq \|x^i - x^j\| \Longleftrightarrow \begin{cases} s_{ij} = x^i - x^j \\ z_{ij} \geq \|s_{ij}\| \end{cases}, \quad \text{where} \quad [i,j] \in E_2.$$

The relaxation (3) can then be written as

$$
\begin{array}{rlll}
\text{Minimize} & \sum_{(i,j)\in E} d_{ij} & & \\
\text{subject to:} & z_{ij} - d_{ij} + My_{ij} = M, & [i,j]\in E & \\
& s_{ij} + x^j = a^i, & [i,j]\in E_1 & \\
& s_{ij} - x^i + x^j = 0, & [i,j]\in E_2 & \\
& \sum_{j\in S} y_{ij} = 1, & i\in P & \\
& \sum_{i<j, i\in S} y_{ij} = 1, & j\in S-\{p+1\} & (4)\\
& y_{ij} + h_{ij} = 1, & [i,j]\in E & \\
& z_{ij} \geq \|s_{ij}\|, & [i,j]\in E & \\
& d_{ij}, y_{ij}, h_{ij} \geq 0 & [i,j]\in E & \\
& d_{ij}, y_{ij}, z_{ij}, h_{ij} \in \Re, & [i,j]\in E & \\
& x^i \in \Re^n, & i\in S & \\
& s_{ij} \in \Re^n, & [i,j]\in E &
\end{array}
$$

where the second and third equalities denote componentwise equalities. Now let

$$
\mathcal{X} = \begin{pmatrix} x^{p+1} \\ \vdots \\ x^{2p-2} \\ (z_{1,p+1}; s_{1,p+1}) \\ \vdots \\ (z_{2p-2,2p-2}; s_{2p-2,2p-2}) \\ d_{1,p+1} \\ \vdots \\ d_{2p-2,2p-2} \\ y_{1,p+1} \\ \vdots \\ y_{2p-2,2p-2} \\ h_{1,p+1} \\ \vdots \\ h_{2p-2,2p-2} \end{pmatrix} \in \Re^{n|S|+(n+4)|E|},
$$

where $x^i \in \Re^n$ for all $i \in S$, and $z_{ij}, d_{ij}, y_{ij}, h_{ij} \in \Re$ and $s_{ij} \in \Re^n$ for all $[i,j] \in E$.

We also define the vectors $\mathcal{C}$ and $\mathcal{B}$ and the matrix $\mathcal{A}$ as it follows, where the dimensions of each line and column are indicated.

$$
\mathcal{C} = \begin{array}{l} n|S| \\ (n+1)|E| \\ |E| \\ |E| \\ |E| \end{array} \begin{pmatrix} 0 \\ 0 \\ e \\ 0 \\ 0 \end{pmatrix} \in \Re^{n|S|+(n+4)|E|},
$$

$$\mathcal{B} = \begin{array}{l} |E| \\ \\ n|E_1| \\ \\ n|E_2| \\ |P| \\ |S|-1 \\ |E| \end{array} \begin{pmatrix} Me \\ a^1 \\ \vdots \\ a^p \\ 0 \\ e \\ e \\ e \end{pmatrix} \in \Re^{(n+2)|E|+|P|+|S|-1},$$

$$\mathcal{A} = \begin{array}{l} \\ |E| \\ n|E_1| \\ n|E_2| \\ |P| \\ |S|-1 \\ |E| \end{array} \begin{array}{c} \begin{array}{ccccc} n|S| & (n+1)|E| & |E| & |E| & |E| \end{array} \\ \begin{pmatrix} 0 & A_1 & -I & MI & 0 \\ A_2 & A_3 & 0 & 0 & 0 \\ A_4 & A_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_6 & 0 \\ 0 & 0 & 0 & A_7 & 0 \\ 0 & 0 & 0 & I & I \end{pmatrix} \end{array}.$$

The matrix $\mathcal{A} \in \Re^{((n+2)|E|+|P|+|S|-1)\times(n|S|+(n+4)|E|)}$ and the submatrices $A_1, A_2, \ldots, A_7$, which are properly defined by the constrains on (4), are specified on the Appendix.

Finally, we can write the relaxation (4) in the standard primal form

$$\begin{array}{rlr} \text{Minimize} & \mathcal{C}^T\mathcal{X} & \\ \text{subject to:} & \mathcal{A}\mathcal{X} = \mathcal{B} & \\ & x^i \in K_1 & i \in S \\ & (z_{ij}; s_{ij}) \in K_2 & [i,j] \in E \\ & d_{ij} \in K_1 & [i,j] \in E \\ & y_{ij} \in K_1 & [i,j] \in E \\ & h_{ij} \in K_1 & [i,j] \in E. \end{array} \tag{5}$$

Which is equivalent to

$$(P) \quad \begin{array}{rl} \text{Minimize} & \mathcal{C}^T\mathcal{X} \\ \text{subject to:} & \mathcal{A}\mathcal{X} = \mathcal{B} \\ & \mathcal{X} \in \mathcal{K} \end{array} \tag{6}$$

where $\mathcal{K} := K_1^{n|S|+3|E|} \times K_2^{|E|} := K_1 \times \ldots \times K_1 \times K_2 \times \ldots \times K_2$.

The dual problem associated to (P) is given by

$$(D) \quad \begin{array}{rl} \text{Maximize} & \mathcal{B}^T\mathcal{Y} \\ \text{subject to:} & \mathcal{S} = \mathcal{C} - \mathcal{A}^T\mathcal{Y} \\ & \mathcal{S} \in \mathcal{K}. \end{array} \tag{7}$$

The pair of problems (P) and (D) is the one considered by Nesterov and Todd in [9]. In the same paper, the following theorem is given.

Theorem 1. *Let E_1 and E_2 be finite-dimensional linear spaces. If cone $K_i \subseteq E_i$ is self-scaled with ν_i-self-scaled barrier F_i, $i = 1, 2$, then the cone*

$$K := K_1 \times K_2 \subseteq E_1 \times E_2$$

is also self-scaled, with ν-self-scaled barrier given by $F(x_1, x_2) := F_1(x_1) + F_2(x_2)$, where $\nu = \nu_1 + \nu_2$.

We can now apply successively the theorem above and conclude that $\mathcal{K}$ is a self-scaled cone with $\nu = n|S| + 5|E|$. Thus, we can use an interior-point algorithm to compute an ϵ-optimal solution of the relaxation in polynomial time.

4. Solving the relaxation

In recent years rich theories on polynomial-time interior-point algorithms have been developed. These theories and algorithms can be applied to many nonlinear optimization problems to yield better complexity results for various applications. In this paper we have first presented the relaxation (2) for ESTP, which is not everywhere differentiable. It was shown by Xue and Ye, in [12], that by transforming this problem into a standard convex programming problem in conic form, we are able to compute an ϵ-optimal solution efficiently, using interior-point algorithms.

The interior-point algorithms we could apply to approximately solve a problem in conic form are described in [9]. For the algorithms, it is assumed that the interior of the feasible set of the primal and dual problems, (P) and (D), are nonempty. The interior of the feasible sets of (P) and (D) are defined respectively, by

$$\mathcal{F}^0(P) := \{\mathcal{X} \in \mathcal{K}^0 : \mathcal{A}\mathcal{X} = \mathcal{B}\}$$

and

$$\mathcal{F}^0(D) := \{(\mathcal{Y}, \mathcal{S}) \in Y \times \mathcal{K}^0 : \mathcal{S} = \mathcal{C} - \mathcal{A}^T\mathcal{Y}\},$$

where $Y = \Re^{((n+2)|E|+|P|+|S|-1)}$ and $\mathcal{K}^0 := (K_2^0)^{|S|+|E|} \times (K_1^0)^{3|E|} := K_2^0 \times \ldots \times K_2^0 \times K_1^0 \times \ldots \times K_1^0$.

In order to employ interior-point algorithms to approximately solve the relaxation of ESTP, the interior of its feasible set should then be nonempty. To fulfill this assumption, we need to eliminate some variables and constraints from (6), which are defined by the constraints of (4). Considering the formulation of this later problem, we note that the constrain $\sum_{i<j, i \in S} y_{ij} = 1$, for $j \in S - \{p+1\}$ implies that $y_{p+1,p+2} = 1$. Then, from the constrain $y_{ij} + h_{ij} = 1$, for $[i,j] \in E$, we have that $h_{p+1,p+2} = 0$ on every feasible solution. Therefore, no feasible solution satisfies $h_{p+1,p+2} > 0$, which means that the feasible region of the relaxation, has an empty interior. To overcome this difficulty, we reformulate the relaxation, just eliminating the variables $y_{p+1,p+2}$ and $h_{p+1,p+2}$, and taking out the constrains $y_{p+1,p+2} + h_{p+1,p+2} = 1$ $h_{p+1,p+2} \geq 0$ from (4).

Considering the reformulation proposed, we prove on the next section that the interior of the feasible set of (P) and (D) are nonempty by constructing a pair of strictly primal-dual interior feasible solutions. This pair of solutions not only proves the following theorems, but also can be used as an initial point on interior-point algorithms.

5. Initial point

Theorem 2. *Considering the reformulation of problem (P) proposed in the last section, we have that the interior of its feasible set is nonempty.*

Proof. Let x^j be any point in the convex hull of the p given points $a^1, \ldots, a^p$, for $j = p+1, \ldots, 2p-2$, and

$s_{ij} = a^i - x^j$, for $[i,j] \in E_1$,
$s_{ij} = x^i - x^j$, for $[i,j] \in E_2$,
$h_{ij} = 1 - y_{ij}$, for $[i,j] \in E - \{p+1, p+2\}$,
$y_{ij} = \frac{1}{p-2}$, for $[i,j] \in E_1$,
$y_{ij} = \frac{1}{j-(p+1)}$, for $[i,j] \in E_2$,
$d_{ij} = M(y_{ij} + 0.1)$, for $[i,j] \in E$,
$z_{ij} = 1.1M$, for $[i,j] \in E$.

Then, one can verify that $\mathcal{X}$ is an interior feasible solution to (P). □

Theorem 3. *The interior of the feasible set of the dual problem (D) is nonempty.*

Proof. Let:

$$\mathcal{Y} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} (t_1; v_1) \\ \vdots \\ (t_{|E|}; v_{|E|}) \end{pmatrix}$$

where $u_1, u_6 \in \Re^{|E|}$, $u_2 \in \Re^{n|E_1|}$, $u_3 \in \Re^{n|E_2|}$, $u_4 \in \Re^{|P|}$, $u_5 \in \Re^{|S|-1}$, $w_1 \in \Re^{n|S|}$, $w_2 \in \Re^{(n+1)|E|)}$, $w_3, w_4, w_5 \in \Re^{|E|}$, $t_i \in \Re$ and $v_i \in \Re^n$ for $i = 1, \ldots, |E|$.

The dual constraints can then be written as:

$w_1 = -A_2^T u_2 - A_4^T u_3$
$w_2 = -A_1^T u_1 - A_3^T u_2 - A_5^T u_3$
$w_3 = e + u_1$
$w_4 = -Mu_1 - A_6^T u_4 - A_7^T u_5 - u_6$
$w_6 = -u_6$
$w_1, w_3, w_4, w_5, w_6 \in K_1$
$(t_i; v_i) \in K_2$, for $i = 1, \ldots, |E|$.

Now let $u_1 = -0.5e$, $u_3 = 0$, $u_4 = 0$, $u_5 = 0$, $u_6 = -e$ and

$$u_2 = -0.1 \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{|P|} \end{pmatrix} \quad \text{where} \quad e_i \in \Re^{n|S|}.$$

Then, one can verify that $\mathcal{Y}$ and $\mathcal{S}$ form an interior feasible solution to (D). □

6. Conclusion

In this paper, we have first presented a new formulation for the Euclidean Steiner tree problem in $\Re^n$. In order to generate lower bounds for the solution of this problem, we relax the integrality constrains on the formulation. The given relaxation is a convex, but not everywhere differentiable problem. Applying interior-point algorithms, we show that it is possible to get an ϵ-optimal solution for this relaxation by transforming it into a standard convex programming problem in conic form defined by Nesterov and Todd in [9].

The use of interior-point algorithms in the solution of nonlinear relaxations of combinatorial optimization problems have been discussed lately by several authors in the literature, but so far we have not found any application of this approach for the Euclidean Steiner tree problem in $\Re^n$.

Appendix

We now give the submatrices $A_1, A_2, \ldots, A_7$, of the matrix $\mathcal{A}$, used on the formulation of (P). In the following, I represents the $n \times n$ identity matrix, e_1 denotes the vector in $\Re^{n+1}$, with the first component equal to 1 and all the others equal to 0, e^m denotes the vector with all components equal to 1 and dimension equal to m, and OI denotes the $n \times (n+1)$ matrix given by

$$\begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

$$A_1 = \begin{pmatrix} (e_1)^T & & \\ & \ddots & \\ & & (e_1)^T \end{pmatrix}$$

$$A_2 = \begin{pmatrix} I & & \\ & \ddots & \\ & & I \\ & \vdots & \\ I & & \\ & \ddots & \\ & & I \end{pmatrix}$$

$$A_3 = \left(\bar{A}_3|0\right), \quad \text{where} \quad \bar{A}_3 = \begin{pmatrix} OI & & \\ & \ddots & \\ & & OI \end{pmatrix} \in \Re^{n|E_1|\times(n+1)|E_1|}$$

$$A_4 = \begin{pmatrix} -I & I & & & & \\ -I & & I & & & \\ -I & & & I & & \\ \vdots & & & & \ddots & \\ -I & & & & & I \\ & -I & I & & & \\ & -I & & I & & \\ & \vdots & & & \ddots & \\ & -I & & & & I \\ & & \vdots & & & \\ & & & -I & I & \\ & & & -I & & I \\ & & & & -I & I \end{pmatrix}$$

$$A_5 = \left(0|\bar{A}_5\right), \quad \text{where} \quad \bar{A}_5 = \begin{pmatrix} OI & & \\ & \ddots & \\ & & OI \end{pmatrix} \in \Re^{n|E_2|\times(n+1)|E_2|}$$

$$A_6 = \left(\bar{A}_6|0\right), \quad \text{where} \quad \bar{A}_6 = \begin{pmatrix} (e^{|S|})^T & & \\ & \ddots & \\ & & (e^{|S|})^T \end{pmatrix} \in \Re^{|P|\times|E_1|}$$

$$A_7 = \left(0|\bar{A}_7\right), \quad \text{where} \quad \bar{A}_7 = \begin{pmatrix} (e^1)^T & & \\ & \ddots & \\ & & (e^{|S|-1})^T \end{pmatrix} \in \Re^{(|S|-1)\times|E_2|}.$$

Acknowledgements. The authors are grateful to Professor Kurt M. Anstreicher for his very helpful comments. This work was partially supported by CNPq, PRONEX, FAPERJ, FUJB and CTPETRO (ALOPE).

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