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On higher differences. Nota III

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ON HIGHER DIFFERENCES

Nota III (*) di S. C. CHAKRABARTI (*a Calcutta*)

III. - Further definitions and theorems.

1. - Introduction. In earlier two papers [1]), I considered a number of important problems in the Theory of Higher Differences, using 1, a, a^2 etc. as multipliers. I now consider similar problems using the general multipliers w_0, w_1, w_2 etc. The reader will do well if he study the carlier papers before embarking on a study of this paper.

2. - Notations. Besides the notations used in Note I and Note II, a few more notations are used here which are given below: —

(a). ${}^nR_p =$ sum of the products of n arbitrary numbers $w_0, w_1, \dots w_{n-1}$ taken p at a time.

We take ${}^nR_0 = 1$; ${}^nR_p = 0$, if p is < 0 or $> n$.

Evidently ${}^nR_p = {}^nS_p$, if $w_r = a^r$, $r = 0, 1, 2, \dots$.

[For brevity, we often write « $\omega_r = a^r$ » for the expression « $\omega_r = a^r$, $r = 0, 1, 2, \dots$ » and « $x_r = x^r$ » for the expression « $x_r = x^r$, $r = 1, 2, \dots$ »].

(b). $(w_r w_k)_n$ denotes the homogeneous expression of the n^{th} degree with unity as the coefficient of each term, which

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[1]) CHAKRABARTI, S. C., *On Higher Differences*. Note I and Note II. Rend. Sem. Padova, (1954), XXIII, 255-276. *These two papers wiu be referred to as Note I and Note II, respectively.*

can be formed with $w_r, w_{r+1}, w_{r+2}, \dots, w_k$ (k and r are both positive integers, $k > r$).

Thus, for example

$$(w_0 w_2)_3 = w_0^3 + w_0^2 (w_1 w_2)_1 + w_0 (w_1 w_2)_2 + (w_1 w_2)_3,$$

$$(w_1 w_2)_1 = w_1 + w_2.\ (w_1 w_2)_2 = w_1^2 + w_1 w_2 + w_2^2,$$

and

$$(w_1 w_2)_3 = w_1^3 + w_1^2 w_2 + w_1 w_2^2 + w_2^3.$$

We take

$$(w_r w_k)_0 = 1,\ (w_r w_r)_n = w_r^n,\ (w_0 w_k)_n = 0 \text{ if } n \text{ is negative.}$$

(c). $(a^r a^k)_n$ = expression obtained from $(w_r w_k)_n$, if $w_r = a^r$.

3. - The operator V^r.

If u_x be a funtion of x, the operator V^r is defined by the relations

$$V^r u_x = V^{r-1} u_{x+1} - w_{r-1} V^{r-1} u_x,\ V^0 u_x = u_x.$$

In particular

$$V^1 u_x = u_{x+1} - w_0 u_x$$

$$V^2 u_x = \sum_{p=0}^{2} (-)^p u_{x+2-p}\, {}^2R_p$$

Thus, generally, we have

THEOREM.

$$V^n u_x = \sum_{p=0}^{n} (-)^p u_{x+n-p}\, {}^nR_p. \tag{1}$$

This is proved by induction.

NOTE. - The order of the multipliers $w_0, w_1, \dots$ is always to be maintained.

COR.

$$A^n u_x = \sum_{p=0}^{n} (-)^p u_{x+n-p}\, {}^nS_p \tag{2}$$

[Th. (7), Note I]

[Put $w_r = a^r$ in (1)].

4. - The operator V_r.

If (x_0, x_1, x_2) be a function of the variables x_0, x_1 and x_2 which are associated with v_0, v_1 and v_2 respectively, then the operator V_r is defined by the relations

$$V_r(x_0, x_1, x_2) = V_{r-1}(v_0x_0, v_1x_1, v_2x_2)$$
$$- w_{r-1}V_{r-1}(x_0, x_1, x_2), \quad V_0(x_0, x_1, x_2) = (x_0, x_1, x_2).$$

In particular

$$V_1(x_0, x_1, x_2) = (v_0x_0, v_1x_1, v_2x_2) - w_0(x_0, x_1, x_2)$$

$$V_2(x_0, x_1, x_2) = \sum_{p=0}^{2} (-)^p(v_0^{2-p}x_0, v_1^{2-p}x_1, v_2^{2-p}x_2)\,{}^2R_p.$$

Generally, we have

THEOREM.

$$V_n(x_0, x_1, x_2) = \sum_{p=0}^{n} (-)^p(v_0^{n-p}x_0, v_1^{n-p}x_1, v_2^{n-p}x_2)\,{}^nR_p. \quad (3)$$

This is proved by induction.

NOTE. - Here any number of variables x_0, x_1, x_2, x_3 etc. associated with v_0, v_1, v_2, v_3 etc. respectively may be used. But we often deal with three variables only, because if any theorem involving the operator V_r, be true for three variables, it is generally true for any number of variables.

COR.

$$A_nu_x = \sum_{p=0}^{n} (-)^p u_{a^{n-p}x}\,{}^nS_p \quad (4)$$

[Th. 8, Note I]

[The result (4) follows from (3), if $w_r = a^r$ and if (x_0, x_1, x_2) be taken as u_x, a function of x alone, x being associated with a and x_0, x_1, x_2 being functions of x alone or constants].

5. - Operations with V^n.

(i). $$V^r(u_x \pm v_x) = V^ru_x \pm V^rv_x.$$

This is easily proved by (1).

(ii). $V^r c u_x = c V^r u_x$, c being a constant.

(iii). $V^r w_n^x = (w_n - w_{r-1})(w_n - w_{r-2}) \dots (w_n - w_0) w_n^x$ (5).

In particular

$$V^n w_n^x = (w_n - w_{n-1})(w_n - w_{n-2}) \dots (w_n - w_0) w_n^x$$

while

$$V^{n+k} w_n^x = 0, \; k = 1, 2, \dots \tag{6}$$

From (6), by (1), we have

THEOREM. $\sum_{p=0}^{k} (-)^p w_n^{k-p} \, {}^kR_p = 0, \quad k > n$ (7).

COR. 1. $A^r a^{nx} = a^{\frac{1}{2}r(r-1)} (a^n)_r a^{nx}$ [Th. (11), Note I].

[Put $w_r = a^r$ in (5)].

COR. 2.

If $u_x = \lambda_0 w_n^x + \lambda_1 w_{n-1}^x + \dots + \lambda_n w_0^x$, $\lambda s'$, being constants, then

$$V^n u_x = \lambda_0 V^n w_n^x \tag{8}$$

and

$$V^{n+k} u_x = 0, \; k = 1, 2, \dots$$

COR. 3. $A^n (a^x + b)^n = A^n a^{nx}$ (Cor. 1, § 6, Note I)

[Put $\lambda_r = {}^nC_r b^r$ and $w_r = a^r$ in (8)].

COR. 4. $A^n (a^x)_n = A^n a^{nx} / a^{\frac{1}{2}n(n-1)}$ (Cor. 2, § 6, Note I)

[Put $\lambda_r = (-)^r \, {}^nS_r / {}^nS_n$ and $w_r = a^r$ in (8)].

COR. 5. $A^m (a^x)_n = (a^n)_m (a^x)_{n-m} a^{m(x-n+m)}$, $n > m$

[Th. (iv), § 6, Note I].

[In Cor. 2, put $\lambda_r = (-)^r \, {}^nR_r / {}^nR_n$ and operate with V^m applying (6) and (5). Then in the result, if we put $w_r = a^r$, we have

$$A^m(a^x)_n = \frac{1}{{}^nS_n}(a^n)_m {}^mS_m \sum_{p=0}^{n-m}(-)^p a^{(n-p)x}\ {}^{n-m}S_p$$

Hence follows the result by (5), Note I].

6. - Operations with V_r.

(i). If $\varphi(x_0, x_1, x_2)$ and $\psi(x_0, x_1, x_2)$ be two functions of x_0, x_1 and x_2 which are associated with v_0, v_1 and v_2 respectively, then

$$V_r\{\varphi(x_0, x_1, x_2) \pm \psi(x_0, x_1, x_2)\}$$
$$= V_r.\varphi(x_0, x_1, x_2) \pm V_r.\psi(x_0, x_1, x_2).$$

This is easily proved by (3).

(ii). $\qquad V_r C(x_0, x_1, x_2) = CV_r(x_0, x_1, x_2)$

where (x_0, x_1, x_2) is a function of x_0, x_1, x_2 and C is a constant.

(iii). If $u_x = x_n$, x_n being associated wtih w_n, then as an analogue of (5), we have

$$V_r u_x = (w_n - w_0)(w_n - w_1) \dots (w_n - w_{r-1})x_n \tag{9}$$

and

$$V_{n+k}u_x = 0,\ k = 1, 2, \dots \tag{10}$$

Hence by (3), we have

$$\sum_{p=0}^{k}(-)^p w_n^{k-p}\ {}^kR_p = 0,\ k > n \qquad \text{[see (7)].}$$

A few corollaries analogons to those of (5), are also obtained here, viz,

COR. 1. $\qquad A_r x^n = a^{\frac{1}{2}r(r-1)}(a^n)_r x^n \qquad$ [Th. (14), Note 1].

[Put $x_r = x^r$ and $w_r = a^r$ in (9)].

COR. 2. - If $u_x = \lambda_0 x_n + \lambda_1 x_{n-1} + \dots + \lambda_n x_0$, $\lambda s'$ being constants, then $V_n u_x = \lambda_0 V_n x_n$.

COR. 3. $\qquad A_n x^{(n)} = A_n x^n \qquad$ [Cor. 1, § 7, Note I].

[In Cor. 2, put $x_r = x^r$ and $\lambda_p = (-)^p N_p$ where N_p is the sum of the products of $n-1$ natural numbers 1, 2, ..., $n-1$ taken p at a time. $\lambda_n = 0$ evidently].

COR. 4. $A_n(x)_n = (a^n)_n x^n$ [Cor. 2, § 7, Note I].

[In Cor. 2, put $\lambda_p = (-)^p\, {}^nR_p/{}^nR_n$ and operate with V_n. Then in the result put $x_r = x^r$ and $w_r = a^r$].

COR. 5.

$$A_m(x)_n = \frac{(a^n)_m}{a^{m(n-m)}}(x)_{n-m}x^m, \quad n \geq m \quad \text{[Th. (17), Note I]}.$$

[In Cor. 2, put $\lambda_p = (-)^p\, {}^nR_p/{}^nR_n$ and operate with V_m applying (10) and (9). Then in the result if we put $w_r = a^r$ and $x_r = x^r$, we have

$$A_m(x)_n = \frac{{}^mS_m}{{}^nS_n}(a^n)_m \sum_{p=0}^{n-m}(-)^p x^{n-p}\, {}^{n-m}S_p.$$

Hence follows the result by (5), Note I].

7. - THEOREM.

$$\sum_{p=0}^{n}(-)^p(w_0w_k)_{r-p}\, {}^nR_p = 0, \quad k < n \leq k + r. \tag{11}$$

[Consider the series

$$(w_0w_k)_r,\ (w_0w_k)_{r-1} \dots (w_0w_k)_1,\ 1,\ 0,\ 0,\ 0,\ \dots$$

and apply Th. (2), Note I. Notice that in all the orders of differences from $k+1^{\text{th}}$ to $k+r^{\text{th}}$, the first element is zero].

8. - LEMMA (i).

$$\sum_{p=0}^{n} a^p\, {}^{r+p}O_p^- = {}^{n+r+1}O_n^- \tag{12}$$

LEMMA (ii).

$$(w_r w_n)_k = a^{rk}(w_0w_{n-r})_k \quad \text{if} \quad w_r = a^r. \tag{13}$$

For,

$$\begin{aligned}(w_r w_n)_k &= (a^r a^n)_k, \quad \text{if} \quad w_r = a^r \\ &= a^{rk}(a^0a^{n-r})_k = a^{rk}(w_0w_{n-r})_k, \\ &\qquad\qquad \text{for} \quad a^r = w_r.\end{aligned}$$

THEOREM.

$$(w_0 w_r)_n = {}^{n+r}O_n^-, \quad \text{if} \quad w_r = a^r \tag{14}$$

For

$$L \cdot S = \sum_{p=0}^{n} w_0^{n-p} (w_1 w_r)_p$$

$$= \sum_{p=0}^{n} a^p \, {}^{r+p-1}O_p^-, \quad \text{by (13) and if (14) holds for } r-1$$

$$= {}^{n+r}O_n^-, \quad \text{by (12)}.$$

9. - An analogue of Lagrange's formula.

THEOREM. - The polynomial

$$\sum_{p=0}^{n} \alpha_p x^p = \sum_{r=0}^{n} \left\{ \sum_{p=r}^{n} (w_0 w_r)_{p-r} x^p \right\} V^r \alpha_0, \quad r \lesseqgtr p. \tag{15}$$

(A polynomial is expressed here in terms of the differences of its coefficients).

In particular, when $n = 4$, let us assume

$$\sum_{p=0}^{4} \alpha_p x^p = \sum_{p=0}^{4} \lambda_p V^p \alpha_0$$

and substitute for $\alpha_0, \alpha_1, \ldots, \alpha_4$, five sets of values, viz,

(i) $1, \ (w_0 w_0)_1, \ (w_0 w_0)_2, \ (w_0 w_0)_3, \ (w_0 w_0)_4$

(ii) $0, \ 1, \ (w_0 w_1)_1, \ (w_0 w_1)_2, \ (w_0 w_1)_3$

... ...

The first set gives the value of λ_0, the second, that of λ_1, and so on.

The general case may be similarly treated.

COR. 1.

$$\sum_{p=0}^{n} \alpha_p x^p = \sum_{r=0}^{n} \left\{ \sum_{p=r}^{n} {}^pO_r^- x^p \right\} A^r \alpha_0. \tag{16}$$

COR. 2.

$$\sum_{p=0}^{n} \alpha_p x^p = \sum_{r=0}^{n} \left\{ \sum_{p=r}^{n} {}^pC_r x^p \right\} \Delta^r \alpha_0. \tag{17}$$

[(16) is obtained from (15) by (14) and (17) follows from (16) when $a = 1$. Both may also be obtained direct].

10. - THEOREM.

$$u_{x+n} = \{ \sum_{p=0}^{n} (w_0 w_p)_{n-p} V^p \} u_x , \qquad (18)$$

This is proved by induction as follows:

If we apply (1) and proceed exactly as in the case of Th. (26), Note I, we can show that

$$u_{x+n+1} = V^{n+1} u_x + \{ V^0 \sum_{p=0}^{n} (-)^p (w_0 w_0)_{n-p} \, {}^{n+1}R_{1+p} +$$
$$+ V^1 \sum_{p=0}^{n-1} (-)^p (w_0 w_1)_{n-p-1} \, {}^{n+1}R_{1+p} + \dots$$
$$\dots + V^n \sum_{p=0}^{0} (-)^p (w_0 w_n)_{-p} \, {}^{n+1}R_{1+p} \} u_x$$

where the limits go on diminishing, since $(w_0 w_k)_n = 0$ if n is negative.

$$\therefore \quad u_{x+n+1} = \{ \sum_{p=0}^{n+1} (w_0 w_p)_{n+1-p} V^p \} u_x$$

For

$$\sum_{p=0}^{n+1-k} (-)^p (w_0 w_k)_{n+1-k-p} \, {}^{n+1}R_p = 0, \text{ by (11)}. \quad k = 0, 1, 2, \dots$$

COR. 1.

$$u_{x+n} = \{ \sum_{p=0}^{n} {}^nO_p^- A^p \} u_x \qquad (19)$$

[Th. (26), Note I]

[Put $w_r = a^r$ in (18) and apply (14)].

COR. 2.

$$u_x = \{ \sum_{p=0}^{x} (w_0 w_p)_{x-p} V^p \} u_0 \qquad (20)$$

where x is a positive integer.

[Put $n = x$ and $x = 0$ in (18). This may also be obtained from (15) by equating the coefficients of x^p].

11. - As an analogue of (18), we have.

THEOREM

$$(v_0^n x_0 , \; v_1^n x_1 , \; v_2^n x_2) = \{ \sum_{p=0}^{n} (w_0 w_p)_{n-p} V_p \} (x_0 , \; x_1 , \; x_2). \qquad (21)$$

where (x_0, x_1, x_2) is a function of x_0, x_1 and x_2 which are associated with v_0, v_1 and v_2 respectively. The multipliers are w_0, w_1 etc.

The proof is similar to that of (18).

COR. $$u_{a^n x} = (\sum_{p=0}^{n} {}^nO_p^- A_p) u_x \,. \tag{22}$$

[Th. (27), Note I]

[This result is obtained from (21) by (14). We take u_x for (x_0, x_1, x_2) as in (4), § 4, $w_r = a^r$ and x is associated with a].

12. - By equating the coefficients of $(v_0^r x_0,\ v_1^r x_1,\ v_2^r x_2)$ from (21), we have

THEOREM

$$\sum_{p=0}^{n} (-)^p (w_0 w_{r+p})_{n-p}\ {}^{r+p}R_p = 0 \tag{23}$$

PROOF. - Here two self-evident formulae are to be applied, viz,

(i). $$(w_0 w_r)_n = (w_0 w_{r-1})_n + w_r (w_0 w_r)_{r-1} \tag{24}$$

and

(ii). $${}^rR_p = {}^{r-1}R_p + w_{r-1}\ {}^{r-1}R_{p-1} \tag{25}$$

Let C_r denote the $L \cdot S$ of (23), then

$$C_0 = \sum_{p=0}^{n} (-)^p \{ (w_0 w_{p-1})_{n-p}\ {}^pR_p + (w_0 w_p)_{n-p-1}\ {}^{p+1}R_{p+1} \},\ \text{by (24)}$$
$$= 0.$$

Because if we put $p = 0, 1, 2, \ldots, n$ in the expression and add the results together, the sum so obtained, vanishes.

Now

$$C_r = \sum_{p=0}^{n} (-)^p [\{ (w_0 w_{r+p-1})_{n-p} + w_{r+p} (w_0 w_{r+p})_{n-p-1} \} \{ {}^{r+p-1}R_p + $$
$$+ w_{r+p-1}\ {}^{r+p-1}R_{p-1} \},\ \text{by (24) and (25).}$$

$$\therefore \quad C_r - C_{r-1} = \sum_{p=0}^{n} (-)^p \{ (w_0 w_{r+p-1})_{n-p} w_{r+p-1} {}^{r+p-1}R_{p-1}$$
$$+ (w_0 w_{r+p})_{n-p-1} w_{r+p} {}^{r+p}R_p \}$$
$= 0$, similarly substituting as in the case of C_0

So

$$C_r = C_{r-1} = C_{r-2} = \dots = C_0 = 0.$$

This proves (23).

13. - In addition to the operators V^n and V_n, we here introduce two new operators K^n and K_n which stand respectively for

$$\sum_{p=0}^{n} (w_0 w_p)_{n-p} V^p \quad \text{and} \quad \sum_{p=0}^{n} (w_0 w_p)_{n-p} V_p$$

so that

$$K^n u_x = u_{x+n}, \quad \text{by (18)}$$

and

$$K_n(x_0, x_1, x_2) = (v_0^n x_0, v_1^n x_1, v_2^n x_2), \text{ by (21)}.$$

The properties of ks', regarding laws of indices etc., are exactly the same as those of Fs'.

NOTE. - If $w_r = a^r$, then $K^n = F^n$ and $K_n = F_n$.
(§ II, Note I).

IV. - Higher Differences on Calculus.

14. - LEMMA (i). If

$$Z'_{3r} = \begin{vmatrix} (w_0 w_r)_1 & 1 & \\ (w_0 w_r)_2 & (w_0 w_{r+1})_1 & 1 \\ (w_0 w_r)_3 & (w_0 w_{r+1})_2 & (w_0 w_{r+2})_1 \end{vmatrix}_3$$

then

$$Z_{nr} = {}^{n+r}R_n \tag{26}$$
$$[Z'_{0r} = 1]$$

This may be proved by induction, for

$$Z'_{3r} = \sum_{p=0}^{2} (-)^p (w_0 w_{r+2-p})_{1+p} Z'_{(2-p)r}$$

$$= \sum_{p=0}^{2} (-)^p (w_0 w_{r+p})_{3-p}\, {}^{r+p}R_p = {}^{r+3}R_3, \text{ by (23)}$$

COR. $$Z_{nr} = {}^{n+r-1}S_n \qquad \text{[Th. (1), Note II]}$$

LEMMA (ii). If

$$Q'_{4c} = \begin{vmatrix} c & 1 & & \\ c^2 & (w_0w_1)_1 & 1 & \\ c^3 & (w_0w_1)_2 & (w_0w_2)_1 & 1 \\ c^4 & (w_0w_1)_3 & (w_0w_2)_2 & (w_0w_3)_1 \end{vmatrix}_4$$

then

$$Q'_{nc} = (-)^{n-1}(c - w_0)(c - w_1) \dots (c - w_{n-1}) + {}^nR_n. \quad (27)$$

[Show by (26) that

$$Q'_{4c} = -\sum_{p=0}^{4} (-)^p c^{4-p}\, {}^4R_p + {}^4R_4$$

and then apply (2), Note I].

COR.

$$Q_{nr} = (-)^{n-1} \frac{1}{a^r} \{ (a^r)_n\, {}^nS_n + {}^nS_n \}, \quad \text{[Th. (3), Note II]}$$

LEMMA (iii). If

$$B_{4p} = \begin{vmatrix} 1 & 1 & & \\ 2^p & (w_0w_1)_1 & 1 & \\ 3^p & (w_0w_1)_2 & (w_0w_2)_1 & 1 \\ 4^p & (w_0w_1)_3 & (w_0w_2)_2 & (w_0w_3)_1 \end{vmatrix}_4 .$$

Then

$$B_{np} = (-)^{n-1} V^n O^p \quad (28)$$

where V^nO^p = value of V^nx^p when $x = 0$.

[Develop B_{np} in terms of 1, 2^p, 3^p etc. and then apply (26) Notice that the last term of $V^nO^p = 0$ for $x^p = 0$ if $x = 0$].

15. - By differential Calculus and by § 13,

$$u_{x+n} = e^{n\frac{d}{dx}} u_x = K^n u_x \tag{29}$$

and

$$(v_0^n x_0,\ v_1^n x_1,\ v_2^n x_2) = \left(e^{v_0^n x_0 \frac{\partial}{\partial x_0} + v_1^n x_1 \frac{\partial}{\partial x_1} + v_2^n x_2 \frac{\partial}{\partial x_2}}\right) u_0 = K_n(x_0, x_1, x_2) \tag{30}$$

where

$$\left(v_0^n x_0 \frac{\partial}{\partial x_0} + v_1^n x_1 \frac{\partial}{\partial x_1} + v_2^n x_2 \frac{\partial}{\partial x_2}\right)^m u_0$$

$$= \text{value of } \left(v_0^n x_0 \frac{\partial}{\partial x_0} + v_1^n x_1 \frac{\partial}{\partial x_1} + v_2^n x_2 \frac{\partial}{\partial x_2}\right)^m (x_0,\ x_1,\ x_2)$$

when x_0, x_1, x_2 are replaced by 0 in (x_0, x_1, x_2) and in any differential coefficient of (x_0, x_1, x_2).

Thus the operators Ks', like Fs' are related to the operators in Differential Calculus.

16. - *$V^n u_x$ may be expressed in terms of the differential coefficients of u_x.*

THEOREM. - If u_x be a rational and integral function of x of degree l in x, then

$$V^n u_x = \left\{ \sum_{p=0}^{l} \frac{1}{p\,!} V^n 0^p \right\} \frac{d^p u_x}{dx^p} \tag{31}$$

where $V^n 0^p$ = value of $V^n x^p$ when $x = 0$.

$$x^m = 1, \text{ if } x = m = 0\,;\ x^m = 0 \text{ if } x = 0$$

and m is a positive integer;

$$\frac{d^p u_x}{dx^p} = u_x \text{ if } p = 0.$$

By (18) and (29), we have

$$\left\{ \sum_{p=0}^{n} (w_0 w_p)_{n-p} V^p \right\} u_x = e^{n\frac{d}{dx}} u_x .$$

In this equation if we put $n = 1$, 2, 3 and 4, we have four equations from which eliminating V^1u_x, V^2u_x and V^3u_x we *have*

$$(-)^4V^4u_x = \begin{vmatrix} w_0 - e^{\frac{d}{dx}} & 1 & & \\ w_0^2 - e^{2\frac{d}{dx}} & (w_0w_1)_1 & 1 & \\ w_0^3 - e^{3\frac{d}{dx}} & (w_0w_1)_2 & (w_0w_2)_1 & 1 \\ w_0^4 - e^{4\frac{d}{dx}} & (w_0w_1)_3 & (w_0w_2)_2 & (w_0w_3)_1 \end{vmatrix}_4 u_x$$

$$= Z'_{40}u_x - \sum_{p=0}^{l} \frac{1}{p\,!} B_{4p} \frac{d^pu_x}{dx^p} \text{ [Lemmas (i) and (iii), §14]}$$

$$= {}^4R_4u_x + \sum_{p=0}^{l} \frac{1}{p\,!} V^4O^p \frac{d^pu_x}{dx^p}$$

$$= \sum_{p=0}^{l} \frac{1}{p\,!} V^4O^p \frac{d^pu_x}{dx^p}$$

if $x^p = 1$ when $x = p = 0$ and if $x^p = 0$ when $x = 0$ and p is a positive integer.

The general case may be similarly treated.

COR. - If u_x is a rational and integral function of x of degree l in x, then

$$A^nu_x = \sum_{p=0}^{l-1} \frac{1}{(1+p)\,!} A^nO^{1+p} \frac{d^{1+p}u_x}{dx^{1+p}} \qquad \text{[Th. (5), Note II]}$$

where A^nO^m = value of A^nx^m when $x = 0$

[Put $w_r = a^r$ in (31) and note that

$$A^nO^0 = \sum_{p=0}^{n} (-)^{p\,n}S_p = 0] \qquad \text{[Th. (6), Note I]}$$

17. - $V_n(x_0, x_1, x_2)$ *may also be expressed in terms of the differential coefficients of* (x_0, x_1, x_2).

Here we are to employ some new

NOTATIONS.

$\left\{ \begin{matrix} v \\ n \end{matrix} \right\} = (v - w_0)(v - w_1)(v - w_2)$ ... n factors, ws' being the usual moltipliers. With respect to this notation, the following conventions will be used:

(i). $$\Big\}\genfrac{}{}{0pt}{}{v}{n}\Big\{_k = \Big\}\genfrac{}{}{0pt}{}{v^k}{n}\Big\{ = (v^k - w_0)(v^k - w_1) \dots n \text{ factors},$$

Thus

$$\Big\}\genfrac{}{}{0pt}{}{v}{n}\Big\{_0 = \Big\}\genfrac{}{}{0pt}{}{v^0}{n}\Big\{ = \Big\}\genfrac{}{}{0pt}{}{1}{n}\Big\{ = (1 - w_0)(1 - w_1) \dots n \text{ factors}.$$

(ii). $$\left[\Big\}\genfrac{}{}{0pt}{}{v_0}{n}\Big\{ x_0 \frac{\partial}{\partial x_0} + \Big\}\genfrac{}{}{0pt}{}{v_1}{n}\Big\{ x_1 \frac{\partial}{\partial x_1} + \Big\}\genfrac{}{}{0pt}{}{v_2}{n}\Big\{ x_2 \frac{\partial}{\partial x_2}\right]_2$$

$$= \Big\}\genfrac{}{}{0pt}{}{v_0^2}{n}\Big\{ x_0^2 \frac{\partial^2}{\partial x_0^2} + \Big\}\genfrac{}{}{0pt}{}{v_1^2}{n}\Big\{ x_1^2 \frac{\partial^2}{\partial x_1^2} + \Big\}\genfrac{}{}{0pt}{}{v_2^2}{n}\Big\{ x_2^2 \frac{\partial^2}{\partial x_2^2} + 2 \Big\}\genfrac{}{}{0pt}{}{v_0 v_1}{n}\Big\{ x_0 x_1 \frac{\partial^2}{\partial x_0 \partial x_1}$$
$$+ 2 \Big\}\genfrac{}{}{0pt}{}{v_0 v_2}{n}\Big\{ x_0 x_2 \frac{\partial^2}{\partial x_0 \partial x_2} + 2 \Big\}\genfrac{}{}{0pt}{}{v_1 v_2}{n}\Big\{ x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2}.$$

Similarly

$$\left[\Big\}\genfrac{}{}{0pt}{}{v_0}{n}\Big\{ x_0 \frac{\partial}{\partial x_0} + \Big\}\genfrac{}{}{0pt}{}{v_1}{n}\Big\{ x_1 \frac{\partial}{\partial x_1} + \Big\}\genfrac{}{}{0pt}{}{v_2}{n}\Big\{ x_2 \frac{\partial}{\partial x_2}\right]_p$$

may be developed as if by the multinomial theorem.

We take

$$\left[\Big\}\genfrac{}{}{0pt}{}{v_0}{n}\Big\{ x_0 \frac{\partial}{\partial x_0} + \Big\}\genfrac{}{}{0pt}{}{v_1}{n}\Big\{ x_1 \frac{\partial}{\partial x_1} + \Big\}\genfrac{}{}{0pt}{}{v_2}{n}\Big\{ x_2 \frac{\partial}{\partial x_2}\right]_0 = \Big\}\genfrac{}{}{0pt}{}{1}{n}\Big\{.$$

In the place of x_0, x_1, x_2, any number of variables may be used.

Theorem. - If $u = (x_0, x_1, x_2)$, then

$$V_n u = \sum_{p=0}^{l} \frac{1}{p!}\left[\Big\}\genfrac{}{}{0pt}{}{v_0}{n}\Big\{ x_0 \frac{\partial}{\partial x_0} + \Big\}\genfrac{}{}{0pt}{}{v_1}{n}\Big\{ x_1 \frac{\partial}{\partial x_1} + \Big\}\genfrac{}{}{0pt}{}{v_2}{n}\Big\{ x_2 \frac{\partial}{\partial x_2}\right]_p u_0 \quad (32)$$

where l is the highest degree of the variable whose degree is the highest in u among the variables and

$$\frac{\partial^{k+n+r}}{\partial x_0^k \partial x_1^n \partial x_2^r} u_0 = \text{value of } \frac{\partial^{k+n+r}}{\partial x_0^k \partial x_1^n \partial x_2^r} u_r \text{ if } x_0 = x_1 = x_2 = 0.$$

Let us consider the particular ease when $n=3$, $l=2$ and only two variables x_0 and x_1 are used. From (21) and (30) we have

$$\{ \sum_{p=0}^{n} (w_0 w_p)_{n-p} V_p \} u = (e^{v_0^n x_0 \frac{\partial}{\partial x_0} + v_1^n x_1 \frac{\partial}{\partial x_1}}) u_0$$

where $u=(x_0, x_1)$, a function of x_0 and x_1 only.

In this equation if we substitute $n=1, 2, 3$, we have three equations from which eliminating $V_1 u$ and $V_2 u$ we have

$$(-)^3 V_3 u = \begin{vmatrix} w_0 u - (e^{v_0 x_0 \frac{\partial}{\partial x_0} + v_1 x_1 \frac{\partial}{\partial x_1}}) u_0 & 1 & \\ w_0^2 u - (e^{v_0^2 x_0 \frac{\partial}{\partial x_0} + v_1^2 x_1 \frac{\partial}{\partial x_1}}) u_0 & (w_0 w_1)_1 & 1 \\ w_0^3 u - (e^{v_0^3 x_0 \frac{\partial}{\partial x_0} + v_1^3 x_1 \frac{\partial}{\partial x_1}}) u_0 & (w_0 w_1)_2 & (w_0 w_2)_1 \end{vmatrix}_3$$

$$= Q'_{3w_0} u - \left[Q'_{31} + \frac{1}{1!} Q'_{3v_0} x_0 \frac{\partial}{\partial x_0} + \cdots + \frac{1}{2!} Q'_{3v_1^2} x_1^2 \frac{\partial^2}{\partial x_1^2} \right] u_0$$

[Lemma (ii), § 14]

$$\therefore \quad V_3 u = \sum_{p=0}^{2} \frac{1}{p!} \left[\left\{ \begin{matrix} v_0 \\ 3 \end{matrix} \right\} x_0 \frac{\partial}{\partial x_0} + \left\{ \begin{matrix} v_1 \\ 3 \end{matrix} \right\} x_1 \frac{\partial}{\partial x_1} \right]_p u_0$$

for, the coefficient of

$${}^3R_3 = u - u_0 - \frac{1}{1!} x_0 \frac{\partial u_0}{\partial x_0} \cdots \frac{1}{2!} x_1^2 \frac{\partial^2 u_0}{\partial x_1^2} = 0 .$$

The general case when $n=n$, $l=l$ and any number of variables $x_0, x_1, \ldots, x_r$ are used, may be similarly treated.

COR.

$$A_n u_x = {}^n S_n \left\{ \sum_{p=0}^{l-n} \frac{x^{n+p}}{(n+p)!} (a^{n+p})_n \frac{d^{n+p}}{dx^{n+p}} \right\} u_0, \qquad (33)$$

[Th. (6), Note II]

where u_x is rational integral function of $\bar{x}$ of degree l in x and $\frac{d^m u_0}{dx^m} =$ value of $\frac{d^m u_x}{dx^m}$ if $x=0$.

[If u_x, a function of x alone, stand for (x_0, x_1, x_2) when the variables x_0, x_1, x_2 *are* functions of x or constants, then from (32), if x is associated with a, we have

$$V_n u_x = \sum_{p=0}^{l} \frac{1}{p\,!} \left[\left\{ \begin{matrix} a \\ n \end{matrix} \right\} x \frac{d}{dx} \right]_p u_0$$

which, if $w_r = a^r$, reduces to

$$A_n u_x = \sum_{p=0}^{l} \frac{1}{p\,!} \, {}^nS_n(a^p)_n x^p \frac{d^p u_0}{dx^p}.$$

This result is the same as (33)].

18. $\dfrac{du_x}{dx}$ *may be expressed in terms of* V^1, V^2, V^3 etc.

LEMMA (i). If

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix}_3$$

then

$$D_n = n\,!\,(n-1)\,!\,(n-2)\,!\,\ldots\,2\,!\,1\,!. \tag{34}$$

LEMMA (ii).

$$D_{n-1, r} = \frac{1}{r} \, {}^nC_r D_n \tag{35}$$

Where $D_{n-1,r}$ = determinant of the $n-1^{th}$ order obtained from D_n by deleting the first column and the r^{th} row.

THEOREM. - If u_x be a rational and integral function of x of degree n in x, then

$$\frac{du_x}{dx} = \sum_{r=0}^{n} \left[\sum_{p=1}^{n} (-)^{p-1} \{ (w_0 w_r)_{p-r} - (w_0 w_r)_{-r} \} \frac{1}{p} \, {}^nC_p V^r u_x \right]. \tag{36}$$

By (18) and (29), we have

$$\sum_{p=0}^{n} (w_0 w_p)_{n-p} V^p u_x = e^{n\frac{d}{dx}} u_x. \tag{37}$$

Let us consider the particular case when u_x is of degree 3 in x. If we put $n = 1$ in (37) we have

$$\{ (w_0 w_0)_1 - 1 \} u_x + V^1 u_x = \left\{ \frac{d}{dx} + \frac{1}{2!} \frac{d^2}{dx^2} + \frac{1}{3!} \frac{d^3}{dx^3} \right\} u_x .$$

If $n = 2$ and 3, two similar equations may be obtained. From these three equations, by eliminaling $\frac{d^2 u_x}{dx^2}$ and $\frac{d^3 u_x}{dx^3}$ and simplifying by (34) and (35), we have

$$\begin{aligned} \frac{du_x}{dx} &= \sum_{p=0}^{2} (-)^p \{ (w_0 w_0)_{1+p} - 1 \} \frac{1}{1+p} \, {}^3C_{1+p} u_x \\ &+ \sum_{p=0}^{2} (-)^p \left\{ (w_0 w_1)_p \frac{1}{1+p} \, {}^3C_{1+p} \right\} V^1 u_x \\ &+ \sum_{p=1}^{2} (-)^p \left\{ (w_0 w_2)_{p-1} \frac{1}{1+p} \, {}^3C_{1+p} \right\} V^2 u_x \\ &+ \sum_{p=2}^{2} (-)^p \left\{ (w_0 w_3)_{p-2} \frac{1}{1+p} \, {}^3C_{1+p} \right\} V^3 u_x \end{aligned}$$

ie

$$\frac{du_x}{dx} = \sum_{r=0}^{3} \left[\sum_{p=1}^{3} (-)^{p-1} \{ (w_0 w_r)_{p-r} - (w_0 w_r)_{-r} \} \frac{1}{p} \, {}^3C_p V^r u_x \right].$$

The general case may be similarly treated.

COR.

$$\frac{du_x}{dx} = \sum_{r=1}^{n} \left\{ \sum_{p=r}^{n} (-)^{p-1} \, {}^nC_p \frac{1}{p} \, {}^pO_r^- A^r u_x \right\} \tag{38}$$

[Th. (8), Note II].

[(36) may be written

$$\begin{aligned} \frac{du_x}{dx} &= \sum_{p=1}^{n} (-)^{p-1} \{ w_0 w_0)_p - 1 \} \frac{1}{p} \, {}^nC_p u_x \\ &+ \sum_{r=1}^{n} \left[\sum_{p=r}^{n} (-)^{p-1} (w_0 w_r)_{p-r} \frac{1}{p} \, {}^nC_p V^r u_x \right] \end{aligned}$$

in which, when $w_x = a^r$, the first summation vanishes and the second summation reduces to (38), by (14)].

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