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ON NEW PARTITION OF NUMBERS

Nota (*) *di* MAHADEB DUTTA (*a Calcutta*)

INTRODUCTION. - In recent investigations of statistical physics, the theory of partition of numbers plays a very important role. Temperly [1]) and others have used methods of partitions of numbers with convenience and advantage in cases when the usual method of statistical mechanics, viz., method of steepest descent, does not yield desired accuracy, or is not valid. In an essential statistical approach to thermodynamic problems, from general statistical considerations, Dutta [2]) has obtained some general formulae from which different statistics, viz., those of Bose, Fermi, Gentile, Maxwell-Boltzmann can be deduced by using different partitions of numbers. This has necessitated the introduction and a thorough investigation of a new type of partitions of numbers in which the repetition of any part of the number in partition is restricted suitably.

In the existing theory, unrestricted partitions have been extensively and thoroughly investigated by Hardy and Ramanujan [3]), Rademacher [4]) and many others. Some investigations [5])

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Indirizzo dell'A.: College of Science, University, Calcutta (India).

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2) DUTTA, M., *An Essential Statistical Approach to Thermodynamic Problem.* Proc. Nat. Inst. Sc. India, *19*, (1935), 109-26.

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3) HARDY, G. H. & RAMANUJAN, S., *Asymptotic Formula in Combinatory Analysis.* Procl, Lond. Math. Soc. (2), *17*, (1918), 75-115, also Collected Papers of S. Raumanujan, 276-309.

4) RADEMACHER, H., *A Convergent Series for the Partition Function* $p(n)$, Proc. Nat. Acad. Sc., U.S.A. *23*, (1937), 78-84.

5) HUA, L. K., *On the Number of Partitions of a Number into*

have also been made of partitions of numbers in unequal parts, these into parts not exceeding a given number, and the like. Kothari and his associates [6]) have discussed partitions of numbers occurring in statistical physics, viz., partition in powers of integers, that in odd multiples of half, and the like, by the methods of statistical mechanics, which are only repetitions of usual calculations of statistical mechanics by dropping the physical interpretation of quantities involved in those calculations. But no attempt has yet been made to define and to investigate partitions of numbers in which the repetition of a part is restricted, though these are of great importance for applications in statistical physics. Mathematical problems of statistics of Bose, Gentile and Fermi are those of partitions of numbers (energy) into parts in which repetiton of parts are restricted differently .In partitions corresponding to Bose statistics any part can be repeated any number of times, that to Gentile statistics, any part can be repeated upto « d » times where d is any fixed number, and that to Fermi statistics no part can repeat, i. e. $d = 1$.

In this note, a partition of number in which any part can be repeated upto « d » times has been introduced and its simple algebraic properties have been investigated. A rough approximate value for this partition has been calculated by a Tauberian theorem. Evidently the formulae for partition of unequal parts and unrestricted partition are obtained when $d = 1$ and $d \rightarrow \infty$ respectively.

Unequal Parts. Bull. Amer. Math. Soc. *49*, (1940), 419, abs. No. 279.

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GUPTA, H., *A Formula in Partition.* J. Ind. Math. Soc. *6*, (1942), 115-17.

6) AULUCK, F. C. and KOTHARI, D. S., *Statistical Mechanics an the Partitions of Numbers.* Proc. Camb. Phil. Soc. *42*, (1946), 272-277.

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Definition and investigation of simple algebraic properties:

DEFINITION. - ${}_dp(n)$ is the partition of a number N into any number parts, in which no part is repeated more than d times.

The generating funtion this partition is

$$f(x) = \Sigma\, {}_dp(n)x^n = \prod_n^{\infty} (1 + x^n + x^{2n} + \dots + x^{dn}) =$$

$$= \prod_n^{\infty} \frac{1 - x^{n(d+1)}}{1 - x^n} = \frac{\prod_n^{\infty} (1 - x^{n(d+1)})}{\prod_n^{\infty} (1 - x^n)} = \frac{\Sigma\, p(n)x^n}{\Sigma\, p(n)x^{n(d+1)}},$$

where $p(n)$ is unrestricted partition.

$$\therefore \{\Sigma\, p(n)x^{n(d+1)}\}\{\Sigma\, {}_dp(n)x^n\} = \Sigma\, p(n)x^n .$$

Equating co-efficients of like powers of x^n, we get

$$n < d + 1, \; {}_dp(n) = p(n);$$

$$(d + 1) \leq n < (d + 1), \; {}_dp(n) = p(n) - p(1)\,{}_dp(n - d - 1) =$$
$$= p(n) - p(1)p(n - d - 1);$$

$$2(d + 1) \leq n < 3(d + 1), \; {}_dp(n) = p(n) - p(1)\,{}_dp(n - d - 1) -$$
$$- p(2)\,{}_dp(n - 2d - 2) = p(n) - p(1)p(n - d - 1) -$$
$$- \{p(2) - p^2(1)\}\, p(n - 2d - 2);$$

$$3(d + 1) \leq n < 4(d + 1), \; {}_dp(n) = p(n) - p(1)\,{}_dp(n - d - 1) -$$
$$- p(2)\,{}_dp(n - 2d - 2) - p(3)\,{}_dp(n - 3d - 3) =$$
$$= p(n) - p(1)p(n - d - 1) - \{p(2) - p^2(1)\}\, p(n - 2d - 2) -$$
$$- \{p(3) - 2p(2)p(1) + p^3(1)\}\, p(n - 3d - 3);$$

$$4(d + 1) \leq n < 5(d + 1),$$

$${}_dp(n) = p(n) - p(1)\,{}_dp(n - d - 1) - p(2)\,{}_dp(n - 2d - 2) -$$
$$- p(3)\,{}_dp(n - 3d - 3) - p(4)\,{}_dp(n - 4d - 4)$$
$$= p(n) - p(1)p(n - d - 1) - \{p(2) - p^2(2)\}\, p(n - 2d - 2) -$$
$$- \{p(3) - 2p(2)p(1) + p^3(1)\}\, p(n - 3d - 3) -$$
$$- \{p(4) - 2p(3)p(1) - p^2(2) + 3p(2)p^2(1) - p^4(1)\}\, p(n - 4d - 4);$$

$$5(d+1) \leq n < 6(d+1),$$

$$
\begin{aligned}
{}_dp(n) &= p(n) - p(1)\,{}_dp(n-d-1) - p(2)\,{}_dp(n-2d-2) - \\
&- p(3)\,{}_dp(n-3d-3) - p(4)\,{}_dp(n-4d-4) - p(5)\,{}_dp(n-5d-5) = \\
&= p(n) - p(1)p(n-d-1) - \{p(2) - p^2(1)\}\,p(n-2d-2) - \\
&- \{p(3) - 2p(2)p(1) + p^3(1)\}\,p(n-3d-3) - \\
&- \{p(4) - 2p(3)p(1) - p^2(2) + 3p(2)p^2(1) - p^4(1)\}\,p(n-4d-4) - \\
&- \{p(5) - 2p(4)p(1) - 2p(3)p(2) + 3p(3)p^2(1) + \\
&+ 3p^2(2)p(1) - 4p(2)p^3(1) + p^5(1)\}\,p(n-5d-5).
\end{aligned}
$$

Thus, numerical values of ${}_dp(n)$ can be calculated successively from values of $p(n)$ and so ultimately from Euler's table.

Now, from general values of n, when $m(d+1) \leq n < (m+1)(d+1)$, the expression can also be written as

$$
\begin{aligned}
{}_dp(n) &= p(n) - p(1)\,{}_dp(n-d-1) - p(2)\,{}_dp(n-2d-2) - \dots \\
&\dots - p(m)\,{}_dp(n-md-m) = \\
&= p(n) - p(1)p(n-d-1) - \{p(2) - p^2(1)\}\,p(n-2d-2) - \\
&- \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot - \\
&- [p(m) + (-1)^{2-1}\cdot 2!\,p(m-1)p(1) + \\
&+ p(m-2)\{(-1)^{2-1}\cdot 2!\,p(2) - (-1)^{3-1}\cdot \frac{3!}{2!}p^2(1)\} + \dots + \\
&+ p(m-r)\{(-1)^{2-1}\cdot 2!\,p(r) + (-1)^{3-1}\frac{3!}{2!}p(r-1)p(1) + \dots\} + \\
&+ \dots]\,p(n-md-m).
\end{aligned}
$$

If this formula is true for this n and all its preceeding values then the expression for n, where $(m+1)(d+1) \leq n < (m+2)(d+1)$, follows as

$$
\begin{aligned}
{}_dp(n) &= p(n) - p(1)p(n-d-1) - \{p(2) - p^2(1)\}\,p(n-2d-2) - \\
&- \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot - \\
&- [p(m+1) + (-1)^{2-1}\cdot 2!\,p(m)p(1) + \\
&+ p(m-1)\{(-1)^{2-1}\cdot 2!\,p(2) + (-1)^{3-1}\frac{3!}{2!}p^2(1)\} + \dots + \\
&+ p(m-r)\{(-1)^{2-1}\cdot 2!\,p(r) + (-1)^{3-1}\cdot\frac{3!}{2!}p(r-1)p(1) + \dots\} + \\
&+ \dots]\,p(n - \overline{m+1}\cdot\overline{d+1}).
\end{aligned}
$$

Thus, the formula is established by the second principle of finite induction [7]).

An approximate valuation of ${}_dp(n)$ by a Tauberian Theorem:

An approximate value correct upto the exponential order alone for ${}_dp(n)$ for large n can be bery easily obtained by the help of a Tauberian theorem already used by Hardy and Ramanujan [8]). The theorem is:

« *If* $g(x) = \Sigma\, a_n x^n$ *is a power series with positive co-efficients and*

$$\log g(x) = \frac{A}{1-x} \quad \textit{when } x \to 1,$$

then, $\log s_n = \log (a_0 + a_1 + \dots + a_n) = 2\sqrt{(An)}$ *when* $n \to \infty$ ».

Here let

$$g(x) = (1-x)f(x) = \Sigma \{ {}_dp(n) - {}_dp(n-1) \} x^n =$$
$$= \prod_n^{\infty} \frac{(1-x^{n(d+1)})}{(1-x^n)} \cdot (1-x),$$

i. e.,

$$a_n = {}_dp(n) - {}_dp(n-1), \quad n \neq 0, \quad {}_dp(n)$$

and $a_0 = {}_dp(0) = 1$ by definition.

Then,

$$\log g(x) = \sum_1^{\infty} \log \frac{1}{1-x^n} - \sum_1^{\infty} \frac{1}{1-x^{n(d+1)}} =$$
$$= \sum_1^{\infty} \frac{1}{\nu} \frac{x^{2\nu}}{1-x^{\nu}} - \sum_1^{\infty} \frac{1}{\nu} \frac{x^{d+1}}{1-x^{\nu(d+1)}} =$$
$$= \frac{1}{1-x} \sum_1^{\infty} \frac{1}{\nu^2} - \frac{1}{1-x} \sum_1^{\infty} \frac{1}{\nu^2} \cdot \frac{1}{d+1} =$$
$$= \frac{d}{d+1} \cdot \frac{1}{1-x} \cdot \left(\Sigma \frac{1}{\nu^2} \right) = \frac{d}{d+1} \cdot \frac{\pi^2}{6} \cdot \frac{1}{1-x};$$

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[7]) Cf. e.g.: BIRKHOFF, G. and MACLANE, S., *A Survey of Modern algebra.* (1953), p. 13.

i. e.

$$A = \frac{\pi^2}{6} \cdot \frac{d}{d+1},$$

$$\therefore \; {}_d p(n) = \exp\left(\pi \sqrt{\frac{2}{3} n \frac{d}{d+1}}\right).$$

When $d = 1$,

$$p(n) = \exp\left(\pi \sqrt{\frac{1}{3} n}\right).$$

When

$$d \to \infty, \; \frac{d}{d+1} \to 1,$$

$$\therefore \; p(n) = \exp\left(\pi \sqrt{\frac{2}{3} n}\right).$$

The last two results for partition into unequal parts and unrestricted partition are in agreement with the approximate expression obtained by Hardy and Ramanujan [3]) upto the exponential order.

The result for unrestricted partition may be used to find out the dominating term in the expression for entropy of the corresponding thermodynamic system. This is important for applications.

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