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SOME THEOREMS ON RECURRENT AND RICCI-RECURRENT SPACES

Nota () di MANINDRA CHANDRA CHAKI (a Calcutta)*

Introduction. - Ruse [4] and Walker [5] introduced the recurrent spaces for which the curvature tensor satisfies a relation of the form

$$(1) \quad R_{hijk} \lambda_i = R_{hijk} \lambda_i$$

where λ_i is a non-zero vector and comma denotes covariant derivative. An n -space of this kind was denoted by K_n . In a recent paper [3] Patterson considered a Riemannian space of more than two dimensions whose Ricci tensor satisfies

$$(2) \quad R_{ij} \lambda_k = 0, \quad R_{ij,k} = R_{ij} \lambda_k$$

for some non-zero vector λ_k . He called such a space Ricci-Recurrent and denoted an n -space of this kind by R_n . Obviously every K_n is an R_n but the converse is not true. The question as to when an R_n can be a K_n has been considered in section 1 of this paper. In the remaining sections some necessary conditions for a space V_n ($n > 2$) to be conformal to a recurrent and to a Ricci-recurrent space have been given.

1. - It is known that the conformal tensor C_{ijkl} of a Riemannian space V_n with metric tensor g_{ij} is given by

$$(1.1) \quad C_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{jk}g_{il} - R_{jl}g_{ik} + R_{il}g_{jk} - R_{ik}g_{jl}) \\ + \frac{R}{(n-1)(n-2)} (g_{jk}g_{il} - g_{jl}g_{ik}).$$

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Now let the V_n be a Ricci-recurrent space R_n specified by a non-zero vector λ_i . Then since

$$R_{ij,p} = R_{ij}\lambda_p \quad \text{and} \quad R_{,p} = R\lambda_p$$

it follows from (1.1) that

$$(1.2) \quad C_{ijkl,p} = R_{ijkl,p} + (C_{ijkl} - R_{ijkl})\lambda_p$$

or $C_{ijkl,p} - C_{ijkl}\lambda_p = R_{ijkl,p} - R_{ijkl}\lambda_p$.

Conversely, if (1.2) holds, multiplying both sides of (1.2) by g^{jk} and summing for j and k we get

$$C_{u,p} - C_{u\lambda_p} = R_{u,p} - R_{u\lambda_p}$$

which reduces, in virtue of $C_{ij} = 0$, to

$$(1.3) \quad R_{u,p} = R_{u\lambda_p}$$

From (1.2) and (1.3) we have therefore the following theorem:

THEOREM 1. A necessary and sufficient condition that a V_n be an R_n is that (1.2) holds.

In particular, if the V_n is conformal to a flat space or if $n = 3$ then $C_{ijkl} = 0$. In the first case it follows from (1.2) that the R_n is a K_n . In the second case it follows that the R_3 is a K_3 . Thus we have the following theorem:

THEOREM 2. Every R_n ($n > 3$) is a K_n if it is conformal to a flat space and every R_3 is a K_3 .

Again for an R_n specified by a non-zero vector λ_i we have

$$R_{ij,kl} - R_{ij,ik} = (\lambda_{k,l} - \lambda_{l,k})R_{ij}$$

or by Ricci's identity

$$(1.4) \quad R_{pj}R_{ikl}^p + R_{ip}R_{jkl}^p = (\lambda_{k,l} - \lambda_{l,k})R_{ij}$$

If λ_i be a gradient then $\lambda_{k,l} = \lambda_{l,k}$ and so the lefthand side of (1.4) is zero. Conversely if the lefthand side of (1.4) is zero then λ_i is a gradient. From this we have the following theorem:

THEOREM 3. A necessary and sufficient condition that the vector of recurrence of an R_n be a gradient is that at every point of the R_n

$$(1.5) \quad R_{pj}R_{ikl}^p + R_{ip}R_{jkl}^p = 0.$$

Since the vector of recurrence of every K_n is a gradient it follows in virtue of Theorem 2 that the relation (1.5) holds at every point of an R_n ($n > 3$) which is conformal to a flat space and at every point of an R_3 .

2. - In this section we deduce a necessary condition that a space V_n ($n > 3$) with metric tensor g_{ij} be conformal to a recurrent space.

Let \bar{V}_n be a space with metric tensor

$$(2.1) \quad \bar{g}_{ij} = e^{2\sigma} g_{ij}$$

Then since $\bar{C}_{ijk}^t = C_{ijk}^t$ we have

$$(2.2) \quad \bar{C}_{hijk} = e^{2\sigma} C_{hijk}$$

Now suppose \bar{V}_n is a recurrent space specified by a non-zero vector λ_i and let a comma and a semi-colon denote covariant derivatives with respect to g_{ij} and \bar{g}_{ij} respectively. Then from (1.1) we get

$$\bar{C}_{hijk;l} = \bar{C}_{hijk}\lambda_l$$

Writing $\sigma_i = \frac{\partial \sigma}{\partial x_i}$, $\sigma^j = g^{ij}\sigma_i$, we obtain from (2.2)

$$(2.3) \quad e^{2\sigma} C_{hijk}\lambda_l = e^{2\sigma} C_{hijk;l} + 2e^{2\sigma} C_{hijk}\sigma_l$$

$$\text{or } C_{hijk}\lambda_l = 2C_{hijk}\sigma_l + C_{hijk;l}$$

But

$$C_{hijk;l} = C_{hijk,l} - 4C_{hijk}\sigma_l - (C_{iijk}\sigma_h + C_{hlij}\sigma_i + C_{hukl}\sigma_j + C_{hijl}\sigma_k) \\ + \sigma^a(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{ki}C_{hija}).$$

Therefore, since C_{hijk} possesses the properties of the indi-

ces of R_{hijk} , the equation (2.3) reduces to

$$(2.4) \quad \begin{aligned} C_{hijk}\lambda_l &= C_{hijk,l} - 2C_{hijk}\sigma_l \\ &\quad - (C_{lijk}\sigma_h + C_{lhkj}\sigma_i + C_{lkhj}\sigma_j + C_{ljih}\sigma_k) \\ &\quad + \sigma^s(g_{hl}C_{sijk} + g_{il}C_{shkj} + g_{jl}C_{skhi} + g_{kl}C_{sjih}). \end{aligned}$$

We may therefore state the following theorem :

THEOREM 4. A necessary condition, given in terms of the conformal tensor, that a V_n be conformal to a K_n is that the equation (2.4) holds, provided that the V_n is not conformal to a flat space.

Further, substituting from (2.1) in the last two lines of (2.4) and simplifying we obtain the condition (2.4) as

$$(2.5) \quad \begin{aligned} C_{hijk}\lambda_l &= C_{hijk,l} - 2C_{hijk}\sigma_l \\ &\quad - [(R_{lijk}\sigma_h + R_{lhkj}\sigma_i + R_{lkhj}\sigma_j + R_{ljih}\sigma_k) \\ &\quad + \frac{1}{n-2} \{ (g_{ik}R_{lj} - g_{ij}R_{lk})\sigma_h + (g_{hj}R_{li} - g_{hk}R_{lj})\sigma_i \\ &\quad + (g_{ik}R_{lh} - g_{hk}R_{li})\sigma_j + (g_{hj}R_{li} - g_{ij}R_{lh})\sigma_k \}] \\ &\quad + \sigma^s[(g_{hl}R_{sijk} + g_{il}R_{shkj} + g_{jl}R_{skhi} + g_{kl}R_{sjih}) \\ &\quad + \frac{1}{n-2} \{ g_{hl}(g_{ik}R_{sj} - g_{ij}R_{sk}) + g_{il}(g_{hj}R_{sk} - g_{hk}R_{sj}) \\ &\quad + g_{jl}(g_{ik}R_{sh} - g_{hk}R_{si}) + g_{kl}(g_{hj}R_{si} - g_{ij}R_{sh}) \}]. \end{aligned}$$

If now the vector σ_i be supposed to be parallel in the space V_n and therefore $\sigma^s R_{stjk} = 0 = \sigma^s R_{sk}$, then the last three lines in the above equation vanish. On the otherhand if the space V_n be supposed to be a recurrent space specified by σ_i then the second line of the above equation reduces to $-2R_{hijk}\sigma_l$.

3. - In this section we deduce a necessary condition that a V_n be conformal to a Ricci-recurrent space.

With reference to the conformal relation (2.1) we have the following equations

$$(3.1) \quad \begin{aligned} e^{-2\sigma}R_{hijk} &= R_{hijk} + g_{hk}(\sigma_{i,j} - \sigma_j\sigma_i) + g_{ij}(\sigma_{h,k} - \sigma_h\sigma_k) \\ &\quad - h_j(\sigma_{i,k} - \sigma_i\sigma_k) - g_{ik}(\sigma_{h,j} - \sigma_h\sigma_j) + (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma \end{aligned}$$

whence

$$(3.2) \quad \sigma_{i,j} - \sigma_i \sigma_j = \frac{1}{n-2} (\bar{R}_{ij} - R_{ij}) \\ - \frac{1}{2(n-1)(n-2)} (\bar{g}_{ij} \bar{R} - g_{ij} R) - \frac{1}{2} g_{ij} \Delta_1 \sigma.$$

Now let \bar{V}_n with metric tensor \bar{g}_{ij} be supposed to be a Ricci-recurrent space. Differentiating (3.2) covariantly with respect to g_{ij} and substituting

$$(3.3) \quad \left\{ \begin{array}{l} \bar{R}_{ij,k} = \bar{R}_{ij} \lambda_k + 2\bar{R}_{ij} \sigma_k + \bar{R}_{jk} \sigma_i + \bar{R}_{ik} \sigma_j - g^{sm} \sigma_m (\bar{R}_{sj} g_{ik} + \bar{R}_{si} g_{jk}) \\ \bar{g}_{ij,k} = 2\bar{g}_{ij} \sigma_k \quad , \quad \bar{R}_{,k} = R \lambda_k \end{array} \right.$$

we get

$$(3.4) \quad \sigma_{i,jk} - (\sigma_i \sigma_{j,k} + \sigma_j \sigma_{i,k}) \\ = \frac{1}{n-2} \{ \bar{R}_{ij} \lambda_k + 2\bar{R}_{ij} \sigma_k + \bar{R}_{jk} \sigma_i + \bar{R}_{ik} \sigma_j \\ - g^{sm} \sigma_m (\bar{R}_{sj} g_{ik} + \bar{R}_{si} g_{jk}) - R_{ij,k} \} \\ - \frac{g_{ij}}{2(n-1)(n-2)} \{ e^{2\sigma} \bar{R} (\lambda_k + 2\sigma_k) - R_{,k} \} - \frac{1}{2} g_{ij} (\Delta_1 \sigma)_{,k}.$$

But from (3.2)

$$\sigma_i \sigma_{j,k} + \sigma_j \sigma_{i,k} - 2\sigma_i \sigma_j \sigma_k = \frac{1}{n-2} \{ \sigma_i (\bar{R}_{jk} - R_{jk}) + \sigma_j (\bar{R}_{ik} - R_{ik}) \} \\ - \frac{1}{2(n-1)(n-2)} \{ \sigma_i (\bar{g}_{jk} \bar{R} - g_{jk} R) + \sigma_j (\bar{g}_{ik} \bar{R} - g_{ik} R) \} \\ - \frac{1}{2} \Delta_1 \sigma (\sigma_i g_{jk} + g_j g_{ik}).$$

Substituting in (3.4), we obtain

$$(3.5) \quad \sigma_{i,jk} - 2\sigma_i \sigma_j \sigma_k = \frac{1}{n-2} \{ \bar{R}_{ij} \lambda_k + 2(\bar{R}_{ij} \sigma_k + \bar{R}_{jk} \sigma_i + \bar{R}_{ik} \sigma_j) \\ - \sigma_m (\bar{R}_j^m g_{ik} + \bar{R}_i^m g_{jk}) - (R_{ij,k} + R_{jk} \sigma_i + R_{ik} \sigma_j) \}$$

$$\begin{aligned}
 & - \frac{1}{2(n-1)(n-2)} [e^{2\sigma} \bar{R} \{ g_{ij}(\lambda_k + 2\sigma_k) + g_{jk}\sigma_i + \sigma_{ik}\sigma_j \} \\
 & \quad - \{ g_{ij}R_{,k} + R(g_{jk}\sigma_i + g_{ik}\sigma_j) \}] \\
 & - \frac{1}{2} \{ g_{ij}(\Delta_1\sigma)_{,k} + \Delta_1\sigma(g_{jk}\sigma_i + g_{ik}\sigma_j) \}.
 \end{aligned}$$

This is therefore a necessary condition to be satisfied by the function σ in order that a V_n with metric tensor g_{ij} be conformal to the Ricci-recurrent \bar{V}_n with metric tensor $g_{ij} = e^{2\sigma}g_{ij}$.

If V_n is supposed to be an Einstein space we have simply to put $R_{ij,k} = R_{,k} = 0$ and $R_{ij} = \frac{R}{n} g_{ij}$ in (3.5).

4. - The condition (3.5) which has been obtained by straightforward method is not simple enough. We look for a simpler condition. For this purpose we first obtain the condition of integrability of the equation (3.2).

Differentiating (3.2) covariantly with respect to x^k , interchanging the indices j and k and subtracting, the lefthand side becomes, after the operation _{i.e.},

$$\sigma_{i,jk} - \sigma_{i,kj} + \sigma_{i,j}\sigma_k - \sigma_{i,k}\sigma_j = \sigma_s \bar{R}_{ijk}^s + \sigma_{i,j}\sigma_k - \sigma_{i,k}\sigma_j.$$

From (3.1) we obtain

$$\begin{aligned}
 \sigma_s \bar{R}_{ijk}^s &= \sigma_s [\bar{R}_{ijk}^s - \delta_k^s(\sigma_{i,j} - \sigma_i\sigma_j) - g_{ij}g^{hs}(\sigma_{h,k} - \sigma_h\sigma_k) \\
 & + \delta_j^s(\sigma_{i,k} - \sigma_i\sigma_k) + g_{ik}g^{hs}(\sigma_{h,j} - \sigma_h\sigma_j) - (\delta_k^s g_{ij} - \delta_j^s g_{ik})\Delta_1\sigma] \\
 &= \sigma_s \bar{R}_{ijk}^s - \sigma_k(\sigma_{i,j} - \sigma_i\sigma_j) - \frac{1}{2} g_{ij}(\Delta_1\sigma)_{,k} + g_{ij}\sigma_k\Delta_1\sigma \\
 & + \sigma_j(\sigma_{i,k} - \sigma_i\sigma_k) + \frac{1}{2} g_{ik}(\Delta_1\sigma)_{,j} - g_{ik}\sigma_j\Delta_1\sigma - (\sigma_k g_{ij} - \sigma_j g_{ik})\Delta_1\sigma \\
 &= \sigma \bar{R}_{ijk}^s + \sigma_j\sigma_{i,k} - \sigma_k\sigma_{i,j} + \frac{1}{2} \{ g_{ik}(\Delta_1\sigma)_{,j} - g_{ij}(\Delta_1\sigma)_{,k} \}.
 \end{aligned}$$

Therefore after the operation we obtain from (3.2)

$$\begin{aligned}
 (4.1) \quad & \sigma_s \bar{R}_{ijk}^s + \frac{1}{2} \{ g_{ik}(\Delta_1 \sigma)_{,j} - g_{ij}(\Delta_1 \sigma)_{,k} \} \\
 & = \frac{1}{n-2} \{ (R_{ij,k} - \bar{R}_{ik,j}) - (R_{ij,k} - R_{ik,j}) \} \\
 & - \frac{1}{2(n-1)(n-2)} \{ \bar{R}(\bar{g}_{ij,k} - \bar{g}_{ik,j}) + (\bar{g}_{ij}\bar{R}_{,k} - \bar{g}_{ik}\bar{R}_{,j}) \\
 & - (g_{ij}R_{,k} - g_{ik}R_{,j}) \} - \frac{1}{2} \{ g_{ij}(\Delta_1 \sigma)_{,k} - g_{ik}(\Delta_1 \sigma)_{,j} \}.
 \end{aligned}$$

Now apply (3.3). Then (4.1) reduces to

$$\begin{aligned}
 \sigma_s \bar{R}_{ijk}^s & = \frac{1}{n-2} [\bar{R}_{ij,k} - \bar{R}_{ik,j} + \bar{R}_{ij}\sigma_k - \bar{R}_{ik}\sigma_j + \sigma_i \bar{g}^{st} (\bar{g}_{ij}\bar{R}_{sk} - \bar{g}_{ik}\bar{R}_{sj}) \\
 & - \frac{1}{2(n-1)} \{ 2\bar{R}(\bar{g}_{ij}\sigma_k - \bar{g}_{ik}\sigma_j) + g_{ij}\bar{R}_{,k} - \bar{g}_{ik}\bar{R}_{,j} \}] \\
 & - \frac{1}{n-2} [R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)} (g_{ij}R_{,k} - g_{ik}R_{,j})].
 \end{aligned}$$

Writing as usual (Eisenhart, 1926, P. 91)

$$(4.2) \quad \left\{ \begin{aligned} R_{ijk} & = R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)} (g_{ij}R_{,k} - g_{ik}R_{,j}) \\ \bar{R}_{ijk} & = \bar{R}_{ij,k} - \bar{R}_{ik,j} - \frac{1}{2(n-1)} (\bar{g}_{ij}\bar{R}_{,k} - \bar{g}_{ik}\bar{R}_{,j}) \end{aligned} \right.$$

the above equation reduces to

$$\begin{aligned}
 \sigma_s \bar{R}_{ijk}^s & - \frac{1}{n-2} [\bar{R}_{ij}\sigma_k - \bar{R}_{ik}\sigma_j + \sigma_i \bar{g}^{st} (\bar{g}_{ij}\bar{R}_{sk} - \bar{g}_{ik}\bar{R}_{sj}) \\
 & - \frac{R}{n-1} (\bar{g}_{ij}\sigma_k - \bar{g}_{ik}\sigma_j)] = \frac{1}{n-2} [\bar{R}_{ij,k} - R_{ij,k}].
 \end{aligned}$$

Now from the conformal tensor (1.1) we get

$$\begin{aligned} \sigma_h C_{ijk}^h &= \sigma_h \bar{C}_{ijk}^h = \sigma_h \bar{R}_{ijk}^h \\ &+ \frac{1}{n-2} \{ \bar{R}_{ik} \sigma_j - \bar{R}_{ij} \sigma_k + \sigma_h \bar{g}^{ht} (\bar{g}_{ik} \bar{R}_{tj} - \bar{g}_{ij} \bar{R}_{tk}) \} \\ &+ \frac{\bar{R}}{(n-1)(n-2)} (\bar{g}_{ij} \sigma_k - \bar{g}_{ik} \sigma_j). \end{aligned}$$

Substituting this in the above equation we obtain finally

$$(4.3) \quad \sigma_s C_{ijk}^s = \frac{1}{n-2} (\bar{R}_{ijk} - R_{ijk})$$

This is the condition of integrability of equation (3.2). This reduces to a known result given by Brinkmann [1] in the case when \bar{V}_n is an Einstein space.

Now suppose that \bar{V}_n is a Ricci-recurrent space specified by a vector λ_i . Then from (4.2)

$$\bar{R}_{ijk} = \bar{R}_{ij} \lambda_k - \bar{R}_{ik} \lambda_j + \frac{\bar{R}}{2(n-1)} (g_{ik} \lambda_j - g_{ij} \lambda_k).$$

Substituting in (4.3) we get

$$(4.4) \quad \begin{aligned} \sigma_s C_{ijk}^s &+ \frac{1}{n-2} R_{ijk} \\ &= \frac{1}{n-2} \left\{ \bar{R}_{ij} \lambda_k - \bar{R}_{ik} \lambda_j + \frac{\bar{R} e^{2\sigma}}{2(n-1)} (g_{ik} \lambda_j - g_{ij} \lambda_k) \right\}. \end{aligned}$$

We may therefore state the following theorem:

THEOREM 5. A necessary condition that a space V_n be conformal to a Ricci-recurrent \bar{V}_n is that there exists a function σ satisfying (4.4).

If V_n be an Einstein space, we have simply to put $R_{ijk} = 0$ in (4.4).

And if V_n be a Ricci-recurrent space specified by a vector μ_i we have simply to put in (4.4)

$$R_{ijk} = R_{ij\mu_k} - R_{ik\mu_j} + \frac{R}{2(n-1)}(g_{ik\mu_j} - g_{ij\mu_k}).$$

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