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SATISH CHANDRA CHAKRABARTI

On higher differences. Nota IV

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ON HIGHER DIFFERENCES

Nota IV (*) di SATISH CHANDRA CHAKRABARTI (a Calcutta)

1. - Introduction.

In an earlier communication⁽¹⁾, it was shown how the differential coefficient $\frac{du_x}{dx}$ can be expressed in terms of the operators A^n of Higher Differences. I now consider the general case, viz. $\frac{d^k u_x}{dx^k}$ and express it in terms of the same operators A^n . In the study of this problem, I come across a few algebraic identities which are also considered here.

Note.

We always deal with integers, if not otherwise stated.

2. - Notations used here.

a). $\binom{l, n}{r}_p$ = sum of the products of the elements, taken p at a time, of the series

$$l, l+1, l+2, \dots, n \quad (l < r < n)$$

in which the element r is absent.

Thus

$$\begin{aligned} \binom{1, 5}{3}_2 &= 1 \cdot 2 + 1 \cdot 4 + 1 \cdot 5 + 2 \cdot 4 + 2 \cdot 5 + 4 \cdot 5 \\ &= 49 \end{aligned}$$

the series here being 1, 2, 4, 5.

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Indirizzo dell'A.: Jadabpur University, Calcutta (India).

¹⁾ CHAKRABARTI, S. C. «On Higher Differences, Note II». Rend. Sem. Padova, XXIII (1954), 270-76.

We take $\binom{l, n}{r}_0 = 1$ and $\binom{l, n}{r}_p = 0$ if $p > n - l$ or negative or if $r > n$.

b). $\binom{l, n}{r, s}_p =$ sum of the products of the elements, taken p at a time, of the series

$$l, l + 1, l + 2, \dots, n \quad (l \leq r < s \leq n)$$

in which the elements from r to s are absent.

Thus,

$$\binom{2, 7}{3, 5}_2 = 2 \cdot 6 + 2 \cdot 7 + 6 \cdot 7 = 68,$$

the series here, being 2, 6, 7.

We take $\binom{l, n}{r, s}_0 = 1$ and $\binom{l, n}{r, s}_p = 0$ if $p > n + r - l - s$

or negative:

c).

$$\Phi(r, n) = r^n - {}^r C_1 (r-1)^n + \dots + (-)^{n-1} {}^r C_{r-1} 1^n.$$

[Evidently

$$\Phi(r, 0) = (-)^{r-1}, \Phi(1, n) = 1, \Phi(r, n) = 0 \text{ if } n < r,$$

$$\Phi(0, n) = 0 \text{ and } \Phi(n, n) = n!$$

The last one is a well-known finite difference formula. A table for the values of $\Phi(r, n)$ may be obtained by the formula

$$\Phi(r, n) = r\Phi(r, n-1) + r\Phi(r-1, n-1)]$$

d). ${}^n N_p =$ sum of the products of the first n natural numbers taken p at a time.

${}^n N_0 = 1$ and ${}^n N_p = 0$, if $p > n$ or negative.

All other notations used here, are the same as employed in previous Notes.

3. - *Lemma.* If

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix}_3$$

then similarly formed

$$(1) \quad D_n = n! (n-1)! \dots 2! 1!$$

4. - *Theorem.*

If $\{n\}_{r,m}$ denotes a determinant of the n^{th} order obtained from D_{n+1} by deleting the r^{th} row and the m^{th} column, then

$$(2) \quad \{n\}_{r,n} = D_n {}^{n+1}C_r \binom{1, n+1}{r}_{n+1-m}.$$

5. - *Theorem.*

$$(3) \quad \sum_{p=0}^{n-r} (-)^p \binom{1, n}{\pi+p}_t {}^{n-r}C_p = (n-r)! \binom{1, r}{r}_{t-\dots+r} \\ t \leq n-1.$$

[Use self-evident formulae like

$$\binom{1, n}{r}_t = \binom{1, n}{r, r+1}_t + (r+1) \binom{1, n}{r, r+1}_{t-1}$$

and get the result by Finite Differences].

6. - *Theorem.*

$$(4) \quad \sum_{p=0}^{n-r} (-)^p \Phi(r, n-p) {}^rN_p = 0 \\ n > r.$$

[Here the left-side is the first element of the r^{th} order of differences, $u_p = \Phi(r, n-p)$ and multipliers: 1, 2, 3 etc. In particular when $r=3$ and $n=6$, the first element of the first order of differences is $\Phi(3, 6) - \Phi(3, 5) = 2 \cdot 3^5 - {}^3C_1 1 \cdot 2^5$. Proceed thus and obtain the first element of the third order of differences].

7. - *Theorem.*

$$(5) \quad \sum_{p=0}^n \Phi(r, n-p) {}^n C_p = \sum_{p=0}^{r-1} (-)^p (r+1-p)^n {}^r C_p.$$

[Here L. S. is the first element of the n^{th} order of sums, $u_p = \Phi(r, n-p)$, multipliers: 1, 1, 1, ... In particular, if $r=3$, $n=6$, the first element of the first order of sums is $\Phi(3, 6) + \Phi(3, 5) = 4 \cdot 3^5 - {}^3 C_1 3 \cdot 2^5 + {}^3 C_2 2 \cdot 1^5$. Proceed in this way].

Cor.

$$(6) \quad \sum_{p=1}^{n-r} \Phi(r, n-p) {}^n C_p = \Phi(r+1, n)$$

$$n > r.$$

For, L. S.

$$= \sum_{p=0}^n \Phi(r, n-p) {}^n C_p - \Phi(r, n) - \Phi(r, 0)$$

$$= \Phi(r+1, n), \text{ by (5).}$$

8. - *Theorem.*

The coefficient of x^m in $A^r x^n$ is

$$(7) \quad {}^n C_m \sum_{p=0}^{r-1} \Phi(1+p, n-m) {}^{r-1} L_{r-1-p}$$

where ${}^n L_p =$ sum of the products of n numbers $1-a, 1-a^2, \dots (1-a^n)$ taken p at a time; we take ${}^n L_0 = 1$ for all values of n , positive, zero or negative; ${}^n L_p = 0$, if $p > n$ or negative.

[Obtain the coefficients of x^{n-1}, x^{n-2} etc. in Ax^n, A^2x^n etc. and reduce them to the proper forms (each as a function of L 's) with the help of (6) and the identity

$${}^n C_p {}^{n-p} C_k = {}^n C_{p+k} {}^{p+k} C_p.$$

Thus the coefficient of x^{n-k} is found to be

$$(8) \quad {}^n C_k \sum_{p=0}^{r-1} \Phi(1+p, k) {}^{r-1} L_{r-1-p}.$$

Hence the result].

9. - *Theorem.*

$$(9) \quad \sum_{p=0}^{r-1} (-)^p (r-p)^k r S_p = \sum_{p=0}^{r-1} \Phi(1+p, k) r^{-1} L_{r-1-p}.$$

[See § 4, Higher Differences, Note I].

10. - *Lemma.*

$$(10) \quad a^{m+1} {}^m L_k + {}^{m+1} L_{k+1} = {}^m L_k + {}^m L_{k+1}.$$

[It is self-evident. It holds even when k and m are zero and negative integers].

Theorem.

$$(11) \quad \sum_{p=0}^{k-r} {}^k O_{r+p}^{-} r^{-1+p} L_p = {}^k C_r.$$

Proof. Let us consider a particular case viz,

$$\sum_{p=0}^2 \frac{{}^{1+p} L_p}{{}^{2+p} S_{2+p}} {}^4 S_{2+p} = {}^4 C_2$$

where the left-side is the first element of the fourth order of differences (multipliers: 1, a , a^2 etc.), obtained from the series

$$(12) \quad \frac{{}^{-1} L_{-2}}{{}^0 S_0}, \quad - \frac{{}^0 L_{-1}}{{}^1 S_1}, \quad \frac{{}^1 L_0}{{}^2 S_2}, \quad \dots \quad \frac{{}^3 S_2}{{}^4 S_4}.$$

(Here the first two terms are zero. In the general case, the first r terms are zero). From (12), obtain the successive orders of differences, simplifying each step by (10). Then the first element of the 4th order of differences will be

$$= \sum_{p=0}^4 {}^4 C_p \quad {}^{-1} L_{2+p}$$

which is $= {}^4 C_2$ for ${}^{-1} L_0 = 1$ and all other L 's are zero. The general case may be similarly treated.

11. - Lemma.

$$(13) \quad \sum_{p=0}^k \Phi(p, m) {}^k C_p = k^m.$$

[Find the successive orders of sums, $u_p = \Phi(p, m)$, multipliers: 1, 1, 1, ...].

Theorem.

$$(14) \quad \sum_{r=1}^k \left\{ \sum_{p=0}^{r-1} \Phi(1+p, m) r^{-1} L_{r-1-p} {}^k O_r^- \right\} = k^m.$$

[From the left side, pick up the coefficients of $\Phi(1, m)$, $\Phi(2, m)$ etc. and then apply (11) and (13)].

12. - Theorem.

$$(15) \quad \sum_{p=0}^{l-k} (-)^p \frac{\Phi(k+p, l)}{(k+p)^{(p)}} \binom{1, k+p}{(k+p)}_p = 0.$$

where $n^{(r)} = n(n-1) \dots (n-r+1)$.

13. - Theorem.

$$(16) \quad \frac{d^k u_x}{dx^k} = (-)^{k-1} \frac{k!}{n!} \sum_{r=1}^n \left\{ \sum_{p=r}^n (-)^{p-1} {}^n C_p {}^p O_r^- \binom{1, n}{p} \right\}_{n-k} A^r u_x$$

n being the highest power of x in u_x ; $k < n$.

[Just as in Art. 6, Note II, we can have n equations of the form

$$A u_x = \left(\frac{d}{dx} + \frac{1}{2!} \frac{d^2}{dx^2} + \dots + \frac{1}{n!} \frac{d^n}{dx^n} \right) u_x.$$

Then eliminating we have $(-)^{k-1} \frac{1}{k!} \frac{d^k u_x}{dx^k} \cdot D_n = a$ determinant which develops into

$$[A \{n-1\}_{1, k} - ({}^k O^-, A + A^2) \{n-1\}_{2, k} + \dots] u_x.$$

Hence follows (16) by (2)].

Cor. 1.

$$(17) \quad \frac{du_x}{dx} = \sum_{r=1}^n \left\{ \sum_{p=r}^n (-)^{p-1} {}^n C_p \frac{{}^p O_r}{p} \Delta^r u_x \right\}.$$

[Th (8), Note II].

Cor. 2.

$$(18) \quad \frac{d^k u_x}{dx^k} = \sum_{p=0}^{n-k} (-)^p \frac{1}{(k+p)^{(p)}} \binom{1, k+p}{k+p}_p \Delta^{k+p} u_x.$$

[If $a \rightarrow 1$, the right-side of (16)

$$\begin{aligned} &= (-)^{k-1} \frac{k!}{n!} \sum_{r=1}^n \left\{ \sum_{p=r}^n (-)^{p-1} {}^n C_p {}^p C_r \binom{1, n}{p}_{n-k} \Delta^r u_x \right\} \\ &= (-)^{k-1} \frac{k!}{n!} \sum_{r=1}^n \left\{ (-)^{r-1} {}^n C_r \sum_{p=0}^{n-r} (-)^p \binom{1, n}{r+p}_{n-k} {}^{n-r} C_p \cdot \Delta^r u_x \right\} \\ &= (-)^{k-1} k! \sum_{r=1}^n (-)^{r-1} \frac{1}{r!} \binom{1, r}{r}_{r-k} \Delta^r u_x, \text{ by (3).} \end{aligned}$$

Hence follows (18)].

Cor. 3.

$$(19) \quad \frac{du_x}{dx} = \sum_{p=1}^n (-)^{p-1} \frac{1}{p} \Delta^p u_x.$$

This is a theorem of Finite Differences. It follows readily from (18).