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## **Some multiplication formulas**

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## SOME MULTIPLICATION FORMULAS

*Nota (\*) di LEONARD CARLITZ (a Durham)\*\**

Some years ago Rainville [5] proved the formula

$$(1) \quad \left(\frac{\sin \beta}{\sin \alpha}\right)^n P_n(\cos \alpha) = \sum_{r=0}^n \binom{n}{r} \left(\frac{\sin(\beta - \alpha)}{\sin \alpha}\right)^r P_{n-r}(\cos \beta),$$

where  $P_n(x)$  is the Legendre polynomial. The writer [1] showed that

$$(2) \quad \left(\frac{\sin \beta}{\sin \alpha}\right)^n C_n^\lambda(\cos \alpha) = \sum_{r=0}^n \binom{2\lambda + n - 1}{r} \left(\frac{\sin(\beta - \alpha)}{\sin \alpha}\right)^r C_{n-r}^{\lambda, r}(\cos \beta),$$

where  $C_n^\lambda(x)$  is the ultraspherical polynomial defined by

$$(1 - 2xz + z^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) z^n.$$

For  $\lambda = 1/2$ , (2) reduces to (1).

Chatterjea [2] has recently proved the following formulas:

$$(3) \quad (\cot \alpha \sin \beta)^n \Phi_n(x \tan \alpha) = \sum_{r=0}^n \binom{n}{r} \left(\frac{\sin(\beta - \alpha)}{\sin \alpha}\right)^r (\cos \beta)^{n-r} \Phi_{n-r}(x \cot \alpha \tan \beta),$$

where  $\Phi_n(x)$  is defined by

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$$e^t I_0(xt) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!},$$

and

$$(4) \quad (\cot \alpha \sin \beta)^n L_n(\tan \alpha) \\ = \sum_{r=0}^n \binom{n}{r} \left( \frac{\sin(\beta - \alpha)}{\sin \alpha} \right)^r (\cos \beta)^{n-r} L_{n-r}(\tan \beta),$$

where  $L_n(x)$  is the Laguerre polynomial.

These results and in particular (3) suggest the consideration of the functional equation

$$(5) \quad (\cot \alpha \sin \beta)^n f_n(x \tan \alpha) \\ = \sum_{r=0}^n \binom{n}{r} \left( \frac{\sin(\beta - \alpha)}{\sin \alpha} \right)^r (\cos \beta)^{n-r} f_n(x \tan \beta).$$

If we take  $\beta = \pi/4$  it is easily verified that (5) reduces to

$$(\cot \alpha)^n f_n(x \tan \alpha) = \sum_{r=0}^n \binom{n}{r} \left( \frac{\cos \alpha - \sin \alpha}{\sin \alpha} \right)^r f_n(x).$$

For  $\lambda = \tan \alpha$  this becomes

$$(6) \quad f_n(\lambda x) = \sum_{r=0}^n \binom{n}{r} (1 - \lambda)^r \lambda^{n-r} f_{n-r}(x).$$

For  $x = 1$ , (6) becomes

$$(7) \quad f_n(\lambda) = \sum_{r=0}^n \binom{n}{r} (1 - \lambda)^r \lambda^{n-r} c_{n-r},$$

where

$$(8) \quad c_r = f_r(1) \quad (r = 0, 1, 2, \dots).$$

Conversely if  $f_n(\lambda)$  is defined by (7), where the  $c_r$  are arbitrary constants, it is easily verified that (6) holds. Indeed we have

$$f_n(\lambda x) = \sum_{r=0}^n \binom{n}{r} [1 - \lambda + \lambda(1 - x)]^r (\lambda x)^{n-r} c_{n-r}$$

$$\begin{aligned}
 &= \sum_{r=0}^n \binom{n}{r} (\lambda x)^{n-r} \sum_{s=0}^r \binom{r}{s} (1-\lambda)^s \lambda^{r-s} (1-x)^{r-s} c_{n-r} \\
 &= \sum_{s=0}^n \binom{n}{s} (1-\lambda)^s \lambda^{n-s} \sum_{r=s}^n \binom{n-s}{r-s} (1-x)^{r-s} x^{n-r} c_{n-r} \\
 &= \sum_{s=0}^n \binom{n}{s} (1-\lambda)^s \lambda^{n-s} f_{n-s}(x) .
 \end{aligned}$$

We see therefore that (6) is equivalent to (7) and that the general solution of (6) is given by (7) with arbitrary  $c_r$ . Moreover it follows from (6) with  $x$  replaced by  $\mu x$  and  $\lambda$  replaced by  $\lambda/\mu$  that

$$\begin{aligned}
 f_n(\lambda x) &= \sum_{r=0}^n \binom{n}{r} \left(1 - \frac{\lambda}{\mu}\right)^r \left(\frac{\lambda}{\mu}\right)^{n-r} f_{n-r}(\mu x) , \\
 \mu^n f_n(\lambda x) &= \sum_{r=0}^n \binom{n}{r} (\mu - \lambda)^r \lambda^{n-r} f_{n-r}(\mu x) .
 \end{aligned}$$

If now take  $\lambda = \tan \alpha$ ,  $\mu = \tan \beta$  we get (5). Thus (5) is equivalent to (7).

We recall that Feldheim [3] has proved that the Laguerre polynomials  $L_n^{(\alpha)}(x)$  are the only orthogonal polynomials with a multiplication theorem of the form

$$\lambda^n f_n(x/\lambda) = \sum_{r=0}^n A_{nr} (\lambda - 1)^{n-r} f_r(x) .$$

It may be of interest to note that (7) implies

$$\sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!} = e^{(1-x)z} \sum_{n=0}^{\infty} c_n \frac{(xz)^n}{n!}$$

or, if we prefer,

$$(9) \quad \sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!} = e^z \sum_{n=0}^{\infty} d_n \frac{(xz)^n}{n!}$$

with

$$(10) \quad d_n = \sum_{s=0}^n (-1)^{n-s} \binom{n}{s} c_s = (c-1)^n ,$$

where the last is symbolic. Rainville [4, p. 244] has noted that the polynomials  $f_n(x)$  defined by (9) satisfy (6).

It follows from (9) that the polynomials

$$(11) \quad g_n(x) = x^n f_n\left(\frac{1}{x}\right)$$

constitute an Appell set, that is

$$(12) \quad g'_n(x) = n g_{n-1}(x) .$$

Also (6) becomes

$$(13) \quad g_n(\lambda x) = \sum_{r=0}^n \binom{n}{r} (\lambda - 1)^r x^r g_{n-r}(x) .$$

Conversely if the Appell set  $\{g_n(x)\}$  is assigned then the polynomials  $\{f_n(x)\}$  defined by (11) satisfy (5).

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