Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 32 (1962), p. 295-303

<http://www.numdam.org/item?id=RSMUP_1962__32__295_0>

ON THE BESSEL POLYNOMIALS

Nota (*) *di* S. K. CHATTERJEA (*a Calcutta*)

1. Introduction.

In connection with certain solutions of the wave equation Krall and Frink [1] considered a system of polynomials $y_n(x)$, $(n = 0, 1, 2, ...)$, known as the Bessel Polynomials, where $y_n(x)$ is defined as the polynomial solution

$$(1.1) \qquad y_n(x) = \sum_{\nu=0}^{n} \frac{(n+\nu)!}{(n-\nu)!\,\nu!} (x/2)^\nu ,$$

of the differential equation

$$(1.2) \qquad x^2y'' + 2(x+1)y' - n(n+1)y = 0 .$$

Recently Rajagopal [2] has given an interesting representation for $y_n(x)$, viz.,

$$(1.3) \qquad y_n(x) = (x^{2n}/2^n)[d/dx + \{2(1+nx)/x^2\}]^n \cdot 1 ,$$

in the iterative sense.

(*) Pervenuta in redazione il 19 febbraio 1962.
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In this paper we present two determinant representations for $y_n(x)$, one is derived from the Rodrigues formula and the other from the recursion relation. These determinants have been studied in some details. A continued fraction-expression for $y_n(x)/y_{n-1}(x)$ is derived. Two determinantal representations for the Laguerre polynomials $x^n L_n^{(-2n-1)}(2/x)$ are also given. Moreover some recursion relations of $y_n(x)$, which are supposed to be new, are obtained.

2. Determinant representation from Rodrigues formula.

It is well-known that for the sequence of the Bessel polynomials $\{y_n(x)\}$, there is a generalised Rodrigues formula through which the nth. element $y_n(x)$, of the sequence is determined by the relation

$$(2.1) \qquad 2^n y_n(x) = e^{2/x} \frac{d^n}{dx^n} (e^{-2/x} x^{2n}) .$$

In a recent paper, Pandres [3] has shown that if the nth. element of a sequence of polynomials is given by the relation

$$(2.2) \qquad P_n = \frac{1}{\omega} \frac{d^n}{dx^n} (\omega F^n) ,$$

then

$$(2.3) \qquad P_n = \Delta_n ,$$

where

$$(2.4) \qquad \Delta_n = \begin{vmatrix} D_{1,n} & -1 & 0 & 0 & 0 & \cdots & 0 \\ D_{2,n} & D_{1,n} & -2 & 0 & 0 & \cdots & 0 \\ D_{3,n} & D_{2,n} & D_{1,n} & -3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & (1-n) \\ D_{n,n} & D_{n-1,n} & \cdots & \cdots & \cdots & D_{2,n} & D_{1,n} \end{vmatrix}$$

and

$$D_{k,n} = \frac{F^k}{(k-1)!}\frac{d^k}{dx^k}(\log \omega F^n) \,. \tag{2.5}$$

Now for the Bessel polynomials, we have

$$\omega = e^{-2/x}\,; \qquad F = x^2\,.$$

It follows, therefore, that

$$\begin{aligned} D_{k,n} &= \frac{x^{2k}}{(k-1)!}\cdot\frac{d^k}{dx^k}\left(-\frac{2}{x} + 2n\log x\right) \\ &= 2\cdot(-x)^{k-1}\cdot(k+nx) \end{aligned} \tag{2.6}$$

Thus we have the following determinant representation for the Bessel polynomials:

$$2^{n-1}y_n(x) = \tag{2.7}$$

$$\begin{vmatrix} 2(1+nx) & -1 & 0 & 0 & 0\ \ldots & 0 \\ -2x(2+nx) & 2(1+nx) & -2 & 0 & 0\ \ldots & 0 \\ 2x^2(3+nx) & -2x(2+nx) & 2(1+nx) & -3 & 0\ \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & (1-n) \\ (-x)^{n-1}(n+nx) & (-x)^{n-2}(n-1+nx) & \cdots & \cdots & \cdots & (1+nx) \end{vmatrix}$$

Next using $\Delta_n = n!\,H_n$, we easily derive the following system of equations:

$$\begin{cases} H_1 = D_{1.n} \\ 2H_2 = D_{2.n} + H_1 D_{1.n} \\ 3H_3 = D_{3.n} + H_1 D_{2.n} + H_2 D_{1.n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ nH_n = D_{n.n} + H_1 D_{n-1.n} + \ldots + H_{n-1} D_{1.n}\,. \end{cases} \tag{2.8}$$

Now since $D_{k,n}$ depends on n, H_k also depends on n and thus it is better to write $H_{k,n}$ for H_k. Consequently from (2.8) we immediately derive the following solutions:

(2.9)
$$D_{1,n} =$$
$$\begin{vmatrix} H_{1,n} & 0 & 0 & \dots & 0 \\ 2H_{2,n} & 1 & 0 & \dots & 0 \\ 3H_{3,n} & H_{1,n} & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \dots & \dots & \dots & \dots & 0 \\ nH_{n,n} & H_{n-2,n} & H_{n-3,n} & \dots & 1 \end{vmatrix}$$

(2.10)
$$-D_{2,n} =$$
$$\begin{vmatrix} H_{1,n} & 1 & 0 & \dots & 0 \\ 2H_{2,n} & H_{1,n} & 0 & \dots & 0 \\ 3H_{3,n} & H_{2,n} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ nH_{n,n} & H_{n-1,n} & H_{n-3,n} & \dots & 1 \end{vmatrix}$$

and so on.

Generally we have

(2.11)
$$D_{k,n} =$$
$$(-)^{k-1} \cdot \begin{vmatrix} H_{1,n} & 1 & 0 \; \dots & 0 & 0 \; \dots & 0 \\ 2H_{2,n} & H_{1,n} & 1 & 0 & 0 & 0 \\ 3H_{3,n} & H_{2,n} & H_{1,n} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ nH_{n,n} & H_{n-1,n} & H_{n-2,n} & H_{n-k+1,n} & H_{n-k-1,n} & 1 \end{vmatrix}$$

Lastly we have

$$(2.12)\qquad D_{n,n} = (-)^{n-1}\cdot\begin{vmatrix} H_{1,n} & 1 & 0 & \cdots & 0 \\ 2H_{2,n} & H_{1,n} & 1 & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ nH_{n,n} & H_{n-1,n} & H_{n-2,n} & \cdots & H_{1,n} \end{vmatrix}$$

where

$$(2.13)\qquad H_{k,n} = \frac{1}{k!}\cdot\begin{vmatrix} 2(1+nx) & -1 & 0 & \cdots & 0 \\ -2x(2+nx) & 2(1+nx) & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & (1-k) \\ 2(-x)^{k-1}\cdot(k+nx) & 2(-x)^{k-2}\cdot(k-1+nx) & \cdots & \cdots & 2(1+nx) \end{vmatrix}$$

and

$$(2.14)\qquad H_{n,n} = \frac{1}{n!}\Delta_n\,.$$

3. Determinant representation from recursion relation.

The recursion relation

$$(3.1)\qquad \begin{cases} y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x)\,; & (n \geqslant 1) \\ y_0(x) = 1\,, \quad y_1(x) = 1 + x\,. \end{cases}$$

together with the given values of $y_0(x)$ and $y_1(x)$ determine uniquely the value of $y_n(x)$. Indeed from (3.1) we have

$$(3.2)\qquad y_n(x) = \begin{vmatrix} (2n-1)x & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & (2n-3)x & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & (2n-5)x & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 3x & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & x & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}$$

which is a continuant determinantal representation (of order $n+1$) for $y_n(x)$.

Now using E. Pascal's result [4]

$$\begin{vmatrix} a_1 & -b_2 & 0 & \dots & 0 & 0 \\ 1 & a_2 & -b_3 & \dots & 0 & 0 \\ 0 & 1 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & a_{n-1} & -b_n \\ 0 & 0 & 0 & & 1 & a_n \end{vmatrix} : \begin{vmatrix} a_2 & -b_3 & 0 & \dots & 0 & 0 \\ 1 & a_3 & -b_4 & \dots & 0 & 0 \\ 0 & 1 & a_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & a_{n-1} & -b_n \\ 0 & 0 & 0 & & 1 & a_n \end{vmatrix}$$

$$= a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{a_n}, \tag{3.3}$$

we obtain from (3.2)

$$\begin{aligned} \frac{y_n(x)}{y_{n-1}(x)} &= (2n-1)x + \frac{1}{(2n-3)x +} \frac{1}{(2n-5)x +} \cdots \frac{1}{x +} \frac{1}{1} \\ &= (2n-1)x + \frac{1}{(2n-3)x +} \frac{1}{(2n-5)x +} \cdots \frac{1}{x+1}\,. \end{aligned} \tag{3.4}$$

which also follows from (3.1).

Again noticing the following relation between the Bessel polynomials and the Laguerre polynomials

$$y_n(x) = n\,!\,(-x/2)^n L_n^{(-2n-1)}(2/x) \tag{3.5}$$

we at once derive the following determinant representation for some particular cases of Laguerre polynomials:

$$n\,!\,(-x/2)^n L_n^{(-2n-1)}(2/x) = \tag{3.6}$$

$$\begin{vmatrix} (2n-1)x & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & (2n-3)x & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & (2n-5)x & -1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & & 1 & 3x & -1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & x & -1 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 1 & 1 \end{vmatrix}$$

Again from (2.7) we have

$$n!\,(-x)^n L_n^{(-2n-1)}(2/x) = \tag{3.7}$$

$$\begin{vmatrix} 2(1+nx) & -1 & 0 & 0 & 0 \dots & 0 \\ -2x(2+nx) & 2(1+nx) & -2 & 0 & 0 \dots & 0 \\ 2x^2(3+nx) & -2x(2+nx) & 2(1+nx) & -3 & 0 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & (1-n) \\ 2(-x)^{n-1}\cdot(n+nx) & 2(-x)^{n-2}\cdot(n-1+nx) & & & \dots & 2(1+nx) \end{vmatrix}$$

4. Some recurrence relations.

It is well-known that the Bessel polynomials give rise to the following properties:

$$x^2y_n'' + 2(x+1)y_n' = n(n+1)y_n \tag{4.1}$$

$$y_{n+1} = (2n+1)xy_n + y_{n-1} \tag{4.2}$$

$$x^2y_n' = (nx-1)y_n + y_{n-1} \tag{4.3}$$

$$x^2y_{n-1}' = y_n - (nx+1)y_{n-1} \tag{4.4}$$

$$x(y_n' + y_{n-1}') = n(y_n - y_{n-1}) \tag{4.5}$$

$$(nx+1)y_n' + y_{n-1}' = n^2y_n\,. \tag{4.6}$$

Now differentiating both members of (4.2) with respect to x, we get

$$y_{n-1}' = y_{n+1}' - (2n+1)xy_n' - (2n+1)y_n\,. \tag{4.7}$$

Again differentiating (4.3), we derive

$$x^2y_n'' = \{(n-2)x-1\}\,y_n' + y_{n-1}' + ny_n\,. \tag{4.8}$$

Next eliminating y_{n-1}' between (4.7) and (4.8) and using (4.1) we finally obtain

$$y'_{n+1} - \{(n+1)x - 1\}y'_n = (n+1)^2 y_n \tag{4.9}$$

Changing n into $n-1$, we derive

$$y'_n - (nx-1)y'_{n-1} = n^2 y_{n-1} \tag{4.10}$$

which may be compared with (4.6). It may be noted that (4.5) is readily obtained by subtracting (4.10) from (4.6). Consequently (4.10) will be readily derived by multiplying both members of (4.5) by n and then subtracting the result from (4.6).

Now considering the recurrence relation (4.10) and differentiating k times with respect to x we obtain

$$y_n^{k+1} - (nx-1)y_{n-1}^{k+1} = n(n+k)y_{n-1}^k \, , \tag{4.11}$$

where

$$y_n^k(x) \equiv y_n^k = \frac{d^k}{dx^k}\{y_n(x)\} \, .$$

One can easily compare (4.11) with the following result obtained by C. K. Chatterjea [5]:

$$(nx+1)y_n^{k+1} + y_{n-1}^{k+1} = n(n-k)y_n^k \, . \tag{4.12}$$

Here we also notice the result [5, p. 69[:

$$x(y_n^{k+1} + y_{n-1}^{k+1}) = (n-k)y_n^k - (n+k)y_{n-1}^k \, . \tag{4.13}$$

From (4.11) and (4.13) it follows that

$$nx^2 y_{n-1}^{k+1} + (n+k)(nx+1)y_{n-1}^k = (n-k)y_n^k \, , \tag{4.14}$$

which may well be compared with [5, p. 69]:

$$nx^2 y_n^{k+1} + (n-k)(1-nx)y_n^k = (n+k)y_{n-1}^k \, . \tag{4.15}$$

Next using $x = -1/n$, we derive from (4.14)

$$\frac{1}{n}\left\{ y_{n-1}^{k+1}\left(-\frac{1}{n}\right) = (n-k)y_n^k\left(-\frac{1}{n}\right)\right. \tag{4.16}$$

When $k = 0$, (4.16) shapes into

$$y'_{n-1}\left(-\frac{1}{n}\right) = n^2 y_n\left(-\frac{1}{n}\right) \tag{4.17}$$

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