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Operational formulae for certain classical polynomials - II

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OPERATIONAL FORMULAE FOR CERTAIN CLASSICAL POLYNOMIALS - II

Nota *) *di* SANTI KUMAR CHATTERJEA (*a Calcutta*)

1. INTRODUCTION

In a recent paper [1]), we have found the operational formula:

$$\prod_{j=1}^{n} \{x^2D + (2j + a)x + b\} \tag{1.1}$$
$$= \sum_{r=0}^{n} \binom{n}{r} b^{n-r}x^{2r}y_{n-r}(x, a + 2r + 2, b)D^r .$$

where $y_n(x, a, b)$ is the generalised Bessel polynomials defined by

$$y_n(x, a, b) = b^{-n}x^{2-a}e^{b/x}D^n(x^{2n+a-2}e^{-b/x}) . \tag{1.2}$$

Using (1.1) we have derived the following results:

$$b^ny_n(x, a + 2, b) = \prod_{j=1}^{n} \{x^2D + (2j + a)x + b\} \cdot 1 \tag{1.3}$$

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*) Pervenuta in redazione il 9 agosto 1962.
Indirizzo dell'A.: Department of mathematics, Bangabasi College, Calcutta (India).

1) CHATTERJEA S. K.: *Operational formulae for certain classical polynomials*, I, Communicated to the Quart. J. Math. Oxford.

$$
\begin{aligned}
(1.4) \qquad & b^2\{y_{n+1}(x, a, b) - y_n(x, a, b)\} \\
& = (2n + a)x\,\{by_n(x, a, b) + nxy_{n-1}(x, a + 2, b)\}
\end{aligned}
$$

$$
\begin{aligned}
(1.5) \qquad & y_{n+m}(x, a, b) \\
= \sum_{r=0}^{min(m,n)} & \binom{m}{r}\binom{n}{r} r\,!\,(m + 2n + a - 1)_r (x/b)^{2r} y_{n-r}(x, a + 2r, b) \cdot \\
& \cdot y_{m-r}(x, a + 2n + 2r, b)\ .
\end{aligned}
$$

In the same paper we have derived the formula:

$$
(1.6) \qquad \sum_{m=0}^{n} \binom{\alpha + n}{n - m} \frac{x^m}{m\,!} (D - 1)^m = \sum_{r=0}^{n} \frac{x^r}{r\,!} L_{n-r}^{(\alpha+r)}(x) D^r\ .
$$

where $L_n^{(\alpha)}(x)$ is the generalised Laguerre polynomials of degree n.

The object of the present paper is to derive some more operational formulae for the special case of Bessel polynomials $y_n(x, a, b)$, obtained by taking $a = b = 2$, sometimes written as $y_n(x)$, and for the generalised Laguerre polynomials. Lastly we study the formula (1.5) for the special case of the Bessel polynomials just now mentioned.

2. The Bessel polynomials of Krall and Frink:

The polynomials $y_n(x)$ are defined by

$$
(2.1) \qquad y_n(x) = 2^{-n} e^{2/x} D^n (x^{2n} e^{-2/x})\ .
$$

In [1]), we have derived

$$
\begin{aligned}
(2.2) \qquad & e^{2/x} D^n (x^{2n} e^{-2/x} Y) \\
& = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2 + 2r, 2) D^r Y
\end{aligned}
$$

where Y is an arbitrarily differentiable function of x.

Now using the operational formula [2]),

$$
D^n \cdot e^{\Phi(x)} \equiv e^{\Phi(x)}[D + \Phi'(x)]^n
$$

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[2]) Stephens E.: *The elementary theory of operational mathematics* (1937), 25.

we derive

$$e^{2/x}D^n(x^{2n}e^{-2/x}Y) = x^{2n}\left[D + \frac{2(nx+1)}{x^2}\right]^n \cdot Y \tag{2.3}$$

Thus it follows from (2.2) and (2.3)

$$\begin{aligned} & x^{2n}\left[D + \frac{2(nx+1)}{x^2}\right]^n \cdot Y \\ &= \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2+2r, 2) D^r Y \end{aligned} \tag{2.4}$$

As a special case of (2.4) we notice that

$$x^{2n}\left[D + \frac{2(nx+1)}{x^2}\right]^n \cdot 1 = 2^n y_n(x, 2, 2) \equiv 2^n y_n(x)\,, \tag{2.5}$$

which was derived by Rajagopal [3]).

Again we observe

$$\begin{aligned} & D^n(x^{2n}e^{-2/x}Y) \\ &= \sum_{k=0}^{n} \binom{n}{k} D^k(x^{2n}) D^{n-k}(e^{-2/x}Y) \\ &= e^{-2/x} \sum_{k=0}^{n} \binom{n}{k}\binom{2n}{k} k!\, x^{2n-k} \left(D + \frac{2}{x^2}\right)^{n-k} Y \\ &= e^{-2/x} \sum_{m=0}^{n} A_{n,m} x^{n+m} \left(D + \frac{2}{x^2}\right)^m Y \end{aligned} \tag{2.6}$$

where

$$A_{n,m} = \frac{n!\,(2n)!}{(n-m)!\,(n+m)!\,m!}$$

Thus from (2.2) and (2.6) we obtain

$$\Omega_n Y = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2+2r, 2) D^r Y \tag{2.7}$$

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[3]) RAJAGOPAL A. K.: *On some of the classical orthogonal polynomials.* Amer. Math. Monthly, 67 (1960), 166-169.

where

$$\Omega_n \equiv \sum_{m=0}^{n} A_{n,m} x^{n+m} \left(D + \frac{2}{x^2}\right)^m .$$

As a special case of (2.7) we note that

$$(2.8) \qquad 2^n y_n(x, 2, 2) = \sum_{m=0}^{n} A_{n,m} x^{n+m} \left(D + \frac{2}{x^2}\right)^m \cdot 1 .$$

3. The Bessel polynomials of Burchnall: Burchnall [4]), found additional properties of the Bessel polynomials $\theta_n(x)$ defined by

$$\theta_n(x) = x^n y_n(1/x) .$$

More generally he defined

$$(3.1) \qquad \theta_n(x, a, b) = (-b)^{-n} e^{bx} x^{a+2n-1} D^n(x^{-a-n+1} e^{-bx})$$

Now we observe from (3.1)

$$\theta_n(x, 2, 2) \equiv \theta_n(x) = (-2)^{-n} e^{2x} x^{2n+1} D^n(x^{-n-1} e^{-2x}) .$$

Thus we obtain

$$\begin{aligned} & D^n(x^{-a-n+1} e^{-bx} Y) \\ &= \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(x^{-a-n+1} e^{-bx}) D^r Y \\ &= e^{-bx} x^{-a-2n+1} \sum_{r=0}^{n} \binom{n}{r} (-b)^{n-r} x^r \theta_{n-r}(x, a+r, b) D^r Y \end{aligned}$$

In other words,

$$\begin{aligned} & e^{bx} x^{a+2n-1} D^n(x^{-a-n+1} e^{-bx} Y) \\ (3.2) \qquad &= \sum_{r=0}^{n} \binom{n}{r} (-b)^{n-r} x^r \theta_{n-r}(x, a+r, b) D^r Y . \end{aligned}$$

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[4]) Burchnall J. L.: *The Bessel polynomials*. Canad. J. Math., 3 (1951), 62-68.

In particular, when $a = b = 2$, we have

$$e^{2x}x^{2n+1}D^n(x^{-n-1}e^{-2x}Y)$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-2)^{n-r}x^r\theta_{n-r}(x, 2+r, 2)D^rY \,. \tag{3.3}$$

Next we notice that

$$D^n(x^{-n-1}e^{-2x}Y)$$

$$= x^{-n-1}e^{-2x}\left[D - \frac{2x+n+1}{x}\right]^n \cdot Y \,. \tag{3.4}$$

It follows therefore from (3.3) and (3.4) that

$$\begin{aligned} &x^n\left[D - \frac{2x+n+1}{x}\right]^n \cdot Y \\ &= \sum_{r=0}^{n} \binom{n}{r} (-2)^{n-r}x^r\theta_{n-r}(x, 2+r, 2)D^rY \,. \end{aligned} \tag{3.5}$$

As a special case of (3.5) we observe

$$x^n\left[D - \frac{2x+n+1}{x}\right]^n \cdot 1 = (-2)^n\theta_n(x, 2, 2) \,. \tag{3.6}$$

Again we notice that

$$\begin{aligned} &D^n(x^{-n-1}e^{-2x}Y) \\ &= \sum_{k=0}^{n} \binom{n}{k} D^k(x^{-n-1})D^{n-k}(e^{-2x}Y) \\ &= e^{-2x}\sum_{k=0}^{n} A_{n,k}x^{-n-1-k}(D-2)^{n-k}Y \\ &= x^{-2n-1}e^{-2x}\sum_{k=0}^{n} A_{n,k}x^{n-k}(D-2)^{n-k}Y \end{aligned} \tag{3.7}$$

where

$$A_{n,k} = (-)^k\frac{(n+k)\,!}{(n-k)\,!\,k\,!} \,.$$

Thus we derive from (3.3) and (3.7) that

$$\sigma_n Y = \sum_{r=0}^{n} \binom{n}{r} (-2)^{n-r} x^r \theta_{n-r}(x, 2+r, 2) D^r Y \tag{3.8}$$

where

$$\sigma_n \equiv \sum_{k=0}^{n} A_{n,k} x^{n-k} (D-2)^{n-k} .$$

In particular, we obtain

$$(-2)^n \theta_n(x, 2, 2) = \sum_{k=0}^{n} (-)^k \frac{(n+k)!}{(n-k)!\,k!} x^{n-k} (D-2)^{n-k} \cdot 1 \tag{3.9}$$

which implies the explicit representation of $\theta_n(x, 2, 2)$ viz.,

$$\theta_n(x, 2, 2) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k (n-k)!\,k!} x^{n-k} .$$

4. The Laguerre polynomials:

Carlitz [5]), has recently proved that

$$\frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x} Y) = \sum_{r=0}^{n} \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r Y \tag{4.1}$$

But we notice that

$$D^n (x^{\alpha+n} e^{-x} Y)$$

$$= x^{\alpha+n} e^{-x} \left[D + \frac{\alpha+n-x}{x} \right]^n \cdot Y$$

$$\therefore x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x} Y) = x^n \left[D + \frac{\alpha+n-x}{x} \right]^n \cdot Y . \tag{4.2}$$

Thus we obtain from (4.1) and (4.2)

$$\frac{x^n}{n!} \left[D + \frac{\alpha+n-x}{x} \right]^n Y = \sum_{r=0}^{n} \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r Y \tag{4.3}$$

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[5]) Carlitz L.: *A note on the Laguerre polynomials*. Michigan Math. J., 7 (1960), 219-223.

As a special case of (4.3) we notice that

$$L_n^{(\alpha)}(x) = \frac{x^n}{n\,!}\left[D + \frac{\alpha + n - x}{x}\right]^n \cdot 1 \, . \tag{4.4}$$

5. Special case of the formula (1.5):

Using $a = b = 2$, $n = 1$ and $m > 1$, we derive from (1.5) the following

$$y_{m+1}(x)$$

$$= \sum_{r=0}^{1}\binom{m}{r}\binom{1}{r} r\,!\,(m+3)_r (x/2)^{2r} y_{1-r}(x, 2+2r, 2) y_{m-r}(x, 4+2r, 2) \tag{5.1}$$

Now making use of the relation

$$\frac{d}{dx}\{y_n(x, a, b)\} = \frac{n(n + a - 1)}{2} y_{n-1}(x, a + 2, b) \tag{5.2}$$

we obtain from (5.1)

$$x^2 y''_{m+1}(x) + 2(1 + x) y'_{m+1}(x) = (m + 1)(m + 2) y_{m+1}(x) \; ;$$

which is the differential equation satisfied by the polynomial $y_{m+1}(x)$.

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