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OPERATIONAL FORMULAE
FOR CERTAIN CLASSICAL POLYNOMIALS - III

*Nota ** di SANTI KUMAR CHATTERJEA (a Calcutta)

1. INTRODUCTION

In an earlier paper [1] we found the operational formula

$$(1.1) \quad \prod_{j=1}^n \{x^2 D + (2j + a)x + b\} \\ = \sum_{r=0}^n \binom{n}{r} b^{n-r} x^{2r} y_{n-r}(x, a + 2r + 2, b) D^r .$$

where $y_n(x, a, b)$ is the generalised Bessel polynomials as defined by Krall and Frink [2]. In [1] we also noticed the following consequences of (1.1):

$$(1.2) \quad b^n y_n(x, a + 2, b) = \prod_{j=1}^n \{x^2 D + (2j + a)x + b\} \cdot 1$$

$$(1.3) \quad 2^n y_n(x) = \prod_{j=1}^n (x^2 D + 2jx + 2) \cdot 1$$

where $y_n(x)$ is the special case of the polynomials $y_n(x, a, b)$.

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obtained by taking $a = b = 2$.

$$(1.4) \quad \begin{aligned} & b^2 \{y_{n+1}(x, a, b) - y_n(x, a, b)\} \\ &= (2n + a)x \{by_n(x, a, b) + nxy_{n-1}(x, a + 2, b)\} \end{aligned}$$

which implies two well-known formulae:

$$(i) \quad b \{y_n(x, a + 1, b) - y_n(x, a, b)\} = nx y_{n-1}(x, a + 2, b)$$

$$(ii) \quad b y'_n(x, a, b) = n(n + a - 1)y_{n-1}(x, a + 2, b);$$

$$(1.5) \quad \begin{aligned} & y_{n+m}(x, a, b) \\ &= \sum_{r=0}^{\min(m,n)} \binom{m}{r} \binom{n}{r} r! (m + 2n + a - 1)_r (x/b)^{2r} y_{n-r}(x, \\ & \quad \quad \quad a + 2r, b) y_{m-r}(x, a + 2n + 2r, b). \end{aligned}$$

Later in a recent paper [3] we have obtained the operational formula

$$(1.6) \quad x^{2n} \left[D + \frac{2(nx + 1)}{x^2} \right]^n = \sum_{r=0}^n \binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2 + 2r, 2) D^r.$$

which generalises the operational formula derived by Rajagopal [4]:

$$(1.7) \quad x^{2n} \left[D + \frac{2(nx + 1)}{x^2} \right]^n \cdot 1 = 2^n y_n(x).$$

In [3] we have also derived the following formulae:

$$(1.8) \quad x^n \left[D - \frac{2x + n + 1}{x} \right]^n = \sum_{r=0}^n \binom{n}{r} (-2)^{n-r} x^r \theta_{n-r}(x, 2 + r, 2) D^r.$$

where $\theta_n(x, a, b)$ are those polynomials defined by Burchnell [5]:

$$(1.9) \quad \begin{aligned} & \theta_n(x, a, b) = (-b)^{-n} e^{bx} x^{a+2n-1} D^n (x^{-a-n+1} e^{-bx}) \\ & \frac{x^n}{n!} \left[D + \frac{\alpha + n - x}{x} \right]^n = \sum_{r=0}^n \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r. \end{aligned}$$

where $L_n^{(\alpha)}(x)$ is the generalised Laguerre polynomials. In this connection we like to mention that we have been inspired by Carlitz's work [6]. The interesting result of Carlitz is

$$(1.10) \quad \prod_{j=1}^n (xD - x + \alpha + j) = n! \sum_{r=0}^n \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r .$$

wherefrom he obtains $n! L_n^{(\alpha)}(x) = \prod_{j=1}^n (xD - x + \alpha + j) \cdot 1$.

Thus far we have tried to give a systematic development of the operational formulae derived in [1] and [3], for certain classical polynomials. The object of this paper is to discuss in the same line the polynomials $\theta_n(x, a, b)$ as defined by Burchnell.

2. Burchnell defined the polynomials $\Phi_n(x, a, b)$ by

$$\Phi_n(x, a, b) = x^n y_n(x^{-1}, a, b) .$$

He showed that $\Phi_n(x, a, b)$ was a solution of

$$(2.1) \quad \delta(\delta + 1 - a - 2n)z = bx(\delta - n)z ; \quad (\delta \equiv xD)$$

and that $e^{-bx}\Phi_n(x, a, b)$ was a solution of

$$(2.2) \quad \delta(\delta + 1 - a - 2n)\omega = -bx(\delta - n - a + 2)\omega .$$

Further he showed that the equation (2.2) had the solution

$$(2.3) \quad \omega = (\delta - n - a + 1)(\delta - n - a) \dots (\delta - 2n - a + 2)e^{-bx},$$

wherefrom he deduced that

$$(2.4) \quad \Phi_n(x, a, b) = (-b)^{-n} e^{bx} x^{a+2n-1} D^n (x^{-a-n+1} e^{-bx})$$

In particular, when $a = b = 2$, he pointed out that

$$(2.5) \quad \theta_n(x) \equiv \Phi_n(x, 2, 2) = \left(-\frac{1}{2}\right)^n e^{2x} x^{2n+1} D^n (x^{-n-1} e^{-2x})$$

We first mention that we shall write $\theta_n(x, a, b)$ for $\Phi_n(x, a, b)$ throughout this paper. Now we have for any arbitrarily differentiable function y of x :

$$\begin{aligned}
 & x^n D^n(x^{-a-n+1}y) \\
 &= \delta(\delta - 1) \dots (\delta - n + 1)(x^{-a-n+1}y) \\
 &= x^{-a-n+1}(\delta - a - n + 1)(\delta - a - n) \dots (\delta - a - 2n + 2)y \\
 &\therefore x^{a+2n-1}D^n(x^{-a-n+1}y) \\
 (2.6) \quad &= (\delta - a - n + 1)(\delta - a - n) \dots (\delta - a - 2n + 2)y
 \end{aligned}$$

Now since the linear operators on the right of (2.6) are commutative, we can write (2.6) as

$$(2.7) \quad x^{a+2n-1}D^n(x^{-a-n+1}y) = \prod_{j=1}^n (\delta - a - 2n + j + 1)y$$

Thus we easily have

$$\begin{aligned}
 (2.8) \quad & x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y) \\
 &= \prod_{j=1}^n (\delta - a - 2n + j + 1)e^{-bx}y
 \end{aligned}$$

But in [3] we have proved that

$$\begin{aligned}
 (2.9) \quad & e^{bx}x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}y) \\
 &= \sum_{r=0}^n \binom{n}{r} (-b)^{n-r} x^r \theta_{n-r}(x, a+r, b) D^r y
 \end{aligned}$$

It therefore follows from (2.8) and (2.9)

$$\begin{aligned}
 (2.10) \quad & e^{bx} \prod_{j=1}^n (\delta - a - 2n + j + 1) e^{-bx} y \\
 &= \sum_{r=0}^n \binom{n}{r} (-b)^{n-r} x^r \theta_{n-r}(x, a+r, b) D^r y
 \end{aligned}$$

As a special case of (2.10) we notice that

$$(2.11) \quad e^{bx} \prod_{j=1}^n (\delta - a - 2n + j + 1) e^{-bx} = (-b)^n \theta_n(x, a, b)$$

which may be compared with the remark made by Burchall in (2.2) and (2.3).

Now we shall find a more interesting operational formula for $\theta_n(x, a, b)$.

From (2.8) we again derive

$$(2.12) \quad \begin{aligned} & e^{bx} x^{a+2n-1} D^n (x^{-a-n+1} e^{-bx} y) \\ &= \prod_{j=1}^n (xD - bx - a - 2n + j + 1) y . \end{aligned}$$

Now a comparison of (2.9) and (2.12) yields our desired result:

$$(2.13) \quad \begin{aligned} & \prod_{j=1}^n (xD - bx - a - 2n + j + 1) y \\ &= \sum_{r=0}^n \binom{n}{r} (-b)^{n-r} x^r \theta_{n-r}(x, a+r, b) D^r y . \end{aligned}$$

When $a = b = 2$, we get from (2.13)

$$(2.14) \quad \begin{aligned} & \prod_{j=1}^n (xD - 2x - 2n + j - 1) y \\ &= \sum_{r=0}^n \binom{n}{r} (-2)^{n-r} x^r \theta_{n-r}(x, 2+r, 2) D^r y . \end{aligned}$$

As particular cases of (2.13) and (2.14) we note that

$$(2.15) \quad (-b)^n \theta_n(x, a, b) = \prod_{j=1}^n (xD - bx - a - 2n + j + 1) \cdot 1$$

$$(2.16) \quad (-2)^n \theta_n(x) = \prod_{j=1}^n (xD - 2x - 2n + j - 1) \cdot 1$$

In this connection it is interesting to note that a comparison

of (1.10) with the result [7]:

$$(-2)^n \theta_n \left(\frac{x}{2} \right) = n! L_n^{(-2n-1)}(x);$$

implies that

$$(2.17) \quad (-2)^n \theta_n \left(\frac{x}{2} \right) = \prod_{j=1}^n (xD - x - 2n + j - 1) \cdot 1.$$

Lastly we like to mention a consequence of the formula (2.16). To this end, we observe from (2.16)

$$(2.18) \quad -2(xD - 2x - n)\theta_n(x) = (xD - 2x - 2n)(xD - 2x - 2n + 1)\theta_{n-1}(x)$$

which implies

$$(2.19) \quad 2 \{ x\theta'_n - (2x + n)\theta_n \} + x^2\theta''_{n-1} - 2(2x^2 + 2nx - x)\theta'_{n-1} + 2 \{ 2x^2 + 2(2n - 1)x + 2n^2 - n \} \theta_{n-1} = 0.$$

To verify the truth of (2.19) we observe [5, formulae (15), (16)]

$$(2.20) \quad \theta'_n - \theta_n = -x\theta_{n-1}$$

$$(2.21) \quad \theta_{n+1} - x^2\theta_{n-1} = (2n + 1)\theta_n$$

At first we shall prove the formula

$$(2.22) \quad x\theta'_{n-1} + \theta_n = (x + 2n - 1)\theta_{n-1}$$

which is not mentioned in Burchinal's paper. For this, we easily notice from (2.20) and (2.21)

$$\theta_{n+1} + x(\theta'_n - \theta_n) = (2n + 1)\theta_n$$

or,
$$x\theta'_n + \theta_{n+1} = (x + 2n + 1)\theta_n.$$

Now we have

$$\begin{aligned}
 (2.23) \quad & 2 \{x\theta'_n - (2x + n)\theta_n\} \\
 & = 2x(\theta_n - x\theta_{n-1}) - 2(2x + n)\theta_n \\
 & = -2x^2\theta_{n-1} - 2(n + x)\theta_n
 \end{aligned}$$

Thus eliminating θ_n between (2.19) and (2.22) with the help of (2.23), we obtain

$$(2.24) \quad x\theta''_{n-1} - 2(x + n - 1)\theta'_{n-1} + 2(n - 1)\theta_{n-1} = 0$$

which is the differential equation for $\theta_{n-1}(x)$ and which may be compared with Burchnell's form:

$$(2.25) \quad \delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n .$$

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