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W. W. A. AL-SALAM

Remarks on some operational formulas

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REMARKS ON SOME OPERATIONAL FORMULAS

Nota *) *di* W. W. A. AL-SALAM (*a Lubboch*)**

Let $y_n(x, a, b)$ be the Bessel polynomial of Krall and Frink defined by means of

$$y_n(x, a, b) = b^{-n}x^{2-a}e^{b/x}D^n(x^{2n+a-2}e^{-b/x}) .$$

where $D = d/dx$. Chatterjea proved in [1] the operational formula

$$(1) \quad \prod_{j=1}^{n} (x^2D + (2j + a)x + b) = \sum_{r=0}^{n} \binom{n}{r} b^{n-r}x^{2r}y_{n-r}(x, a + 2r + \\ + 2, b) \cdot D^r$$

where the terms of the operator on the left hand side of (1) do not commute among themselves and the operator must be interpreted to mean

$$\left\{ \prod_{j=1}^{n-1} (x^2D + (2j + a)x + b) \right\} \cdot (x^2D + (2n + a)x + b) .$$

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Indirizzo dell'A.: Dept. of Maths. and Astr. Texas Technological College - Lubboch, Texas, USA.

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He also gave in [2] the formula

$$(2) \quad x^{2n}\left[D+\frac{2(nx+1)}{x^2}\right]^n=\sum_{r-0}^{n}\binom{n}{r}2^{n-r}x^{2r}y_{n-r}(x,2+2r,2)D^r\,.$$

We remark that (2) can be generalized to involve the generalized Bessel polynomial $y_n(x, a, b)$. In fact we prove first the following theorem.

THEOREM 1.

$$(3) \quad x^{2n}\left\{D+\frac{(a+2n)x+b)}{x^2}\right\}^n=\prod_{j=1}^{n}(x^2D+(2j+a)x+b)\,.$$

To prove this theorem we first note that

$$(4) \qquad \left\{x^\lambda D+g(x)\right\}^n x^v Y=x^v\left\{x^\lambda D+g(x)+\frac{v}{x}\right\}^n Y$$

which can be proved by induction.

It is now easy to prove (3) by induction making use of (4).

Combining (1) and (3) we get

$$(5) \quad x^{2n}\left\{D+\frac{(a+2n)x+b}{x^2}\right\}^n=\sum_{r=0}^{n}\binom{n}{r}b^{n-r}x^{2r}y_{n-r}(x,\,a+2r+$$
$$+\,2,\,b)D^r\,.$$

We next show that (1) and (5) (and hence (2)) are special cases of some results of Gould and Hopper [3].

Let

$$(6) \qquad \mathfrak{D}=D-prx^{r-1}+\frac{a}{x}$$

and

$$(7) \qquad H_n^r(x,\,a,\,p)=(-1)^n x^{-a}e^{px^r}D^n(x^a e^{-px^r})\,.$$

Gould and Hopper proved, among other results,

(8) $$\mathfrak{D}^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H^r_{n-k}(x, a, p) D^k$$

and

(9) $$x^n \mathfrak{D}^n = \prod_{j=0}^{n-1} (xD - p + x^q - a - j) .$$

It is easy to see that (7) and

(10) $$y_n(x, a, b) = b^{-n} x^{2-a} e^{b/x} D^n (x^{2n+a-2} e^{-b/x})$$

imply the relation

$$y_n(x, a, b) = (-1)^n b^{-n} x^{2n} H_n^{-1}(x, an + a - 2, b)$$

so that

(11) $$y_{n-r}(x, a + 2r + 2, b) = (-1)^{n-r} b^{r-n} x^{2n-2r} H^{-1}_{n-r}(x, 2n+a, b) .$$

Consequently putting $r = -1$ and $a = 2n +$ a in formula (8) we get

$$\left(D + \frac{(2n + \alpha)x + b}{x^2}\right)^n = \sum_{r=0}^{n} \binom{n}{r} b^{r-n} x^{2r-2n} y_{n-r}(x, \alpha + 2r + 2, b) D^r$$

which is formula (5). Putting $\alpha = 0$, $b = 2$ we get (2).

Another operational formula similar to (3) is contained in the following theorem.

THEOREM 2.

(12) $$(x^{v+1} D + x^v g(x))^n = x^{nv} \prod_{j=1}^{n} (xD + g(x) + (n - j)v) .$$

Proof. (By induction). Formula (12) is obviously true for $n = 1$. Assume it is true for $n = m$ and consider

$$(x^{v+1}D + x^v g(x))^{m+1} Y =$$
$$= x^v (xD + g(x)) x^{mv} \prod_{j=1}^{m} (xD + g(x) + (m - j)v) Y =$$
$$= x^{(m+1)v} (xD + g(x) + mv) \prod (xD + g(x) + (m-j)v) Y =$$
$$= x^{(m+1)v} \prod_{j=0}^{n} (xD + g(x) + (m - j)v) \; Y =$$
$$= x^{(m+1)v} \prod_{j=0}^{m+1} (xD + g(x) + (m + 1 - j)v) Y \, .$$

Thus (12) is also true for $n = m + 1$. This completes the proof of (12).

One interesting special case of (12), in terms of the Gould-Hopper operator $\mathfrak{D}$, is obtained by taking $g(x) = a - px^q$ we get

$$(13) \qquad (x^{v+1}\mathfrak{D})^n = x^{nv} \prod_{j=1}^{n} (xD - px^r + a + (n - j)v) \, .$$

which may be compared with (9).

REFERENCE

[1] Chattercea S. K.: *Operational formulae for certain classical polynomials*. Quarterly Journal of Mathematics, Oxford, (2), vol. 14 (1963), pp. 241-246.

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[3] Gould H. W. and Hopper A. T.: *Operational Formulas connected with two generalizations of Hermite polynomials*. Duke Mathematical Journal, vol. 29 (1962), pp. 51-64.