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ON THE SUBSPACE OF L^p INVARIANT UNDER MULTIPLICATION OF TRANSFORM BY BOUNDED CONTINUOUS FUNCTIONS

Memoria *) *di* ALESSANDRO FIGÀ-TALAMANCA (*a Cambridge, Mass.*)

§ 1. Let $f \in L^p(0,2\pi)$, $1 \leqslant p \leqslant 2$ and let $\widehat{f}(n)$ $(n = 0, \pm 1, \pm 2, ...)$ be the (complex) Fourier coefficients of f. It is known that if for every bounded sequence $\{a(n) : n = 0, \pm 1, \pm 2, ...\}$, $a(n)\widehat{f}(n)$ are still the Fourier coefficients of a p-integrable function, then $\Sigma|\widehat{f}(n)|^2 < \infty$ [**8**, vol. I, p. 214]. That is the subspace of $L^p(0,2\pi)$ which is invariant under multiplication of the Fourier sequences by bounded sequences is exactly $L^2(0,2\pi)$. Helgason has proved in [**3**](1) that if $f \in L^1(\mathbf{R})$ and for any continuous bounded function φ on $\mathbf{R}$ $\widehat{f\varphi}$ is still the Fourier transform of an integrable function, then $\widehat{f} \equiv 0$ (here $\widehat{f}$ denotes the Fourier transform of f). If $1 < p \leqslant 2$, elements of $L^p(\mathbf{R})$ still have transformers which belong to $L^q(\mathbf{R})$ $(1/p + 1/q = 1)$ and which coincide with the Fourier transform for elements of $L^1(\mathbf{R}) \cap$

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1) Helgason's results apply to a much wider class of groups and they are based on the study of L^1 as a Banach algebra. See also [4] where his results are extended to a large class of non commutative groups.

$\cap\, L^p(\mathbf{R})$ (by Hausdorff-Young theorem [**7**, p. 96]). It is natural, therefore, to investigate the analogous situation for L^p at least in the case $1 < p \leqslant 2$. As the Fourier transform can be extended to a unitary transformation in L^2 (after appropriate normalization of the Fourier integral), it is clear that every element of L^2 is mapped into L^2 when its transform is multiplied by a bounded continuous function. I shall prove in this paper that if $f \in L^p(\mathbf{R})$ $(1 < p < 2)$ and $\widehat{f}\varphi$ is the transform of an element of $L^p(\mathbf{R})$ forevery bounded continuous function φ, then $f \equiv 0$. That is: *the subspace of* L^p $(1 < p < 2)$ *which is invariant under multiplication of transforms by bounded continuous functions is the zero subspace.* The proof of Theorem 3 below is a modification of Helgason's proof for L^1 [**3**]; Lemma 1 and the related Lemma 2 are needed to make Helgason's proof applicable to our case and for a discussion of the discrete case (see Remark 4 below).

In § 3 Theorem 3 is applied to multipliers to yield a result of Hörmander [**5**]. In § 4 the case $p > 2$ is discussed. In § 5 a question raised by Helgason concerning $L^p(0,2\pi)$ $p > 2$ is answered. All the results of this paper which are stated for the real line $\mathbf{R}$ are valid (with the same proof) for $\mathbf{R}_n$, indeed most of the results are more general as their proofs apply to various classes of locally compact Abelian groups, this will be indicated case by case.

Throughout the paper, unless otherwise stated, p will be a real number $1 < \mathrm{p} < \infty$ and q will be defined by $1/p + 1/q = 1$, $L^p = L^p(\mathbf{R})$ will be the space of (almost everywhere defined) complex valued functions whose p-th power is absolutely integrable (with respect to the Lebesgue measure). L^p will be endowed with the usual norm which will be denoted by $\| \cdot \|_p \cdot C_0 = C_0(\mathbf{R})$ and $C_{00} = C_{00}(\mathbf{R})$, will be respectively the space of (complex valued) functions on $\mathbf{R}$ vanishing at infinity and the space of functions with compact support. The conjugate space M of C_0 can be identified with the space of complex valued completely additive set functions of bounded variation defined on the σ-ring generated by the compact sets. I shall refer to the elements of M as *regular bounded measures* or simply, when the context allows it, as measures. The variation $\| \mu \|$ of an element $\mu \in M$ coincides with the norm of μ as an element of the conjugate space of C_0. The weak*topology of M, with respect to C_0, will also be used extensively; it is the smallest topology with respect to which elements of C_0 define continuous linear functionals on M.

The simbol « $\frown$ » will denote the Fourier transform, the Hausdorff-Young extension of the Fourier transform or the Fourier-Stieltjes transform when applied respectively to an element of L^1, an element of L^p $(1 < p \leqslant 2)$ or a measure.

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§ 2. It is convenient at first to treat the situation symmetrically for $p \leqslant 2$ and $p \geqslant 2$; the case of $p > 2$ will be discussed in more detail in § 3.

DEFINITION: 1. *For* $f \in L^p$ *define*

$$\| f \|_0 = \sup \{ \| h * f \|_p : h \in L^1 \| \widehat{h} \|_\infty \leqslant 1 \} .$$

The space $(L^p)_0$ *is defined as the subspace of* L^p *consisting of elements satisfying* $\| f \|_0 < \infty$.

Let $f \in L^p$, $p \leqslant 2$, and suppose that for each bounded continuous function φ, $\widehat{f}\varphi$ is the transform of an element of L^p, then $f \in (L^p)_0$. Indeed under the above metioned hypothesis f defines a linear transformation T mapping the space of bounded continuous functions C into L^p, by $T\varphi = g$, where $\widehat{g} = \widehat{f}\varphi$. T is a continuous transformation, in fact it suffices to show that the graph of T is closed [**1**, II, 2.4] and if $\lim \varphi_n = \varphi$ (in the supremum norm), $\lim\limits_n Tf_n = g$ (in the L^p norm), then by Hausdorff-Young theorem, $\| \widehat{f}\varphi_n - \widehat{g} \|_q \leqslant \| T\varphi_n - g \|_p$ while $\| \widehat{f}\varphi_n - \widehat{f}\varphi \|_q \leqslant \| \varphi_n - \varphi \|_\infty \| \widehat{f} \|_q$. Thus $\lim\limits_n \widehat{f}\varphi_n = \widehat{g}$ and $\lim \widehat{f}\varphi_n = \widehat{f}\varphi$ in the norm of L^q implies that $\widehat{f}\varphi = \widehat{g}$. This concludes the proof of the continuity of T. We have then

$$\| h * f \|_p = \| \widehat{Th} \|_p \leqslant \| T \| \, \| \widehat{h} \|_\infty$$

and therefore $\| f \|_0 < \infty$. The fact that if $f \in (L^p)_0$, $p < 2$, $\widehat{f\varphi} = \widehat{g}$ with $g \in L^p$, for any $\varphi \in C$ is of course an obvious consequence of Theorem 3. In § 4 we shall see that something analogous can be said about $(L^p)_0$ for $p > 2$.

If f is a measurable function on $\mathbf{R}$,we shall define, for $x \in \mathbf{R}$, $\tau_x f(y) = f(x + y)$. The operators τ_x (traslation by x) are isometries of L^p onto L^p, $1 \leqslant p \leqslant \infty$. If $\widehat{\mu}$ is the Fourier-Stieltjes transform of a measure μ, $\tau_x \widehat{\mu}$ is also the Fourier-Stieltjes transform of a measure ν with the same norm; indeed for any $f \in C_0$

$$\int_{-\infty}^{+\infty} f(t) d\nu(t) = \int_{-\infty}^{+\infty} f(t) e^{-ixt} d\mu(t) .$$

It is convenient to write $e^{ixt} = e_x(t)$ when e^{ixt} is considered as a function of t. Accordingly we shall also write $\nu = e_{-x}\mu$ when $\widehat{\nu} = \tau_x \widehat{\mu}$.

Lemma 1. *Let $f \in L^p$, then $f \in (L^p)_0$ if and only if $f * g$ is the Fourier-Stieltjes transform of a measure μ, for every $g \in L^q$. Conversely if T is a bounded linear operator mapping L^q into M with the property that $T(\tau_x g) = e_{-x} T(g)$, then there exists $f \in (L^p)_0$ such that $f * g = T(g)$ for every $g \in L^q$.*

Proof. Let $f \in (L^p)_0$ and $g \in L^q$. For each $h \in L^1$ define $F(h) = (h * f * g)(0)$; F is then a linear functional defined on the algebra A of Fourier transforms of elements of L^1. Hölder's inequality and the fact that $f \in (L^p)_0$ imply that

$$|F(h)| \leqslant \| h * f \|_p \|g \|_q \leqslant \| \widehat{h} \|_\infty \| f \|_0 \|g \|_q .$$

Thus F is continuous with respect to the supremum norm on A, therefore it can be extended to a continuous linear functional with the same norm defined on the space C_0. That is, there exists a measure μ such that

$$h * f * g(0) = \int_{-\infty}^{+\infty} \widehat{h} d\mu .$$

If h_n is an approximate identity in L^1 (*i.e.* $\|h_n \| \leqslant 1$ and

$\lim_n h_n * h = h$ for each $h \in L^1$) then

$$\widehat{\mu}(t) = \int_{-\infty}^{+\infty} e^{-ixt} d\mu(x) = \lim_n \int \widehat{h}_n e^{-ixt} d\mu(x) =$$

$$= \lim_n (h_n * \tau_t f * g)(0) = (\tau_t f * g)(0) = f * g(t) \,,$$

because $\lim h_n * \tau_t f = \tau_t f$ in the norm of L^p. Thus $\widehat{\mu}(t) = f*g(t)$ Let now $f * g$ be the Fourier-Stieltjes transform of a measure μ, for every $g \in L^q$. Then the transformation $Tg = \mu$ defined by $f * g = \widehat{\mu}$ is a linear transformation of L^q into M. An application of the closed graph theorem [**1**, II. 2.4] shows that T is a bounded transformation, thus

$$\| \mu \| \leqslant \| T \| \, \| g \|_q \,.$$

Let $h \in L^1$, then

$$\| h * f \|_q = \sup \{ |(h * f * g)(0)| : \| g \|_q \leqslant 1 \}$$

but

$$|(h * f * g)(0)| = |h * \widehat{\mu}(0)| =$$

$$= \Big| \int_{-\infty}^{+\infty} \widehat{h}(x) d\mu(x) \Big| \leqslant \| h \|_\infty \| \mu \| \leqslant \| \widehat{h} \|_\infty \| T \| \,.$$

Thus $\| f \|_0 \leqslant \| T \| < \infty$. Now let T be a continuous linear transformation mapping L^q into M and satisfying $T(\tau_x g) = e_{-x} T(g)$ If $\mu = Tg$ define $F(g) = \int_{-\infty}^{+\infty} 1 d\mu$. F is a linear functional defined on L^q and $|F(g)| \leqslant \| \mu \| \leqslant \| T \| \, \| g \|_q$; therefore there exists $f \in L^p$ such that for all $g \in L^q$

$$F(g) = \int_{-\infty}^{+\infty} 1 d\mu = \int_{-\infty}^{+\infty} f(-t) g(t) dt$$

The property $T(\tau_x g) = e^{-ixt}T(g)$ implies

$$\int_{-\infty}^{+\infty} f(-t)g(t+x)dt = F(\tau_x g) = \int_{-\infty}^{+\infty} e^{-ixt} d\mu(y) = \mu(x)\,.$$

Thus $f * g = \widehat{\mu}$ and $f \in (L^p)_0$.

It is clear from the Lemma just proved that $(L^p)_0$ can be identified with the Banach space of continuous linear maps T from L^q to M which satisfy $T(\tau_x g) = e_{-x}T(g)$. We have seen that if T corresponds to $f \in (L^p)_0$, $\| f \|_0 \leqslant \| T \|$; it is not difficult to prove that $\| T \| \leqslant \| f \|_0 + \epsilon$ for any $\epsilon > 0$ and therefore $\| T \| = \| f \|_0$.

It should also be noted that the preceding lemma is valid for any locally compact Abelian group G provited that M is interpreted as the space measures on the character group Γ of G.

LEMMA 2. *Let $f \in (L^p)_0$ with $p \leqslant 2$ and let $h \in L^1$, $\| h \|_1 = 1$; then $\widehat{h}f \in (L^p)_0$.*

PROOF. Let $f \in (L^p)_0$; for any $x \in \mathbf{R}$ let $e_x(t) = e^{ixt}$; then for every x, $e_x f \in (L^p)_0$ indeed if $g \in L^q$, $(e_x f * g)(y) = e_x(-y)(f * e_x g)(y)$ so that by Lemma 1 $e_x f \in (L^p)_0$, it is also clear that $\| e_x f \|_0 = \| f \|_0$. To prove that $\widehat{h}f \in (L^p)_0$ we shall find a sequence of trigonometric polynomials $p_n(t) = \Sigma_j a_{jn} e_{x_{jn}}(t)$ satisfying $\Sigma |a_{jn}| = 1$ and such that the operators T_n from L^q to M associated with

$$\Sigma a_{jn} e_{x_{jn}}(t) f(t) = f_n(t) \in (L^p)_0\,,$$

converge in the strong operator topology [**1**, VI.1.2] to an operator T which will correspond in the sense of Lemma 1 to $\widehat{h}f$. To do this we first consider h as an (absolutely continuous) member of M of norm 1. Then by Krein-Milman theorem [**1**, V.8.4 and V.4.2] there exists a sequence of convex combinations of measures each supported at a point converging to h in the weak * topology of M; that is a sequence of measures $\mu_n = \Sigma c_{jn} \sigma_{y_{jn}}$, where $\Sigma |c_{jn}| = 1$ and $\sigma_{y_{jn}}$ is the positive measure of mass one concentrated at the point y_{jn}. (The fact that we can consider a sequence rather than a generalized sequence follows from [**1**, V.5.1], actually the whole reasoning would go through using generalized sequences instead). Now the measures μ_n dermine linear operators L_{μ_n} mapping L^q into L^q defined by

$$(L_{\mu_n} g)(x) = \mu_n * g(x) = \int_{-\infty}^{+\infty} g(x-t) d\mu_n(t) .$$

As μ_n converges weak* to h, L_{μ_n} converge in the weak operator topology [**1**, VI.1.3] to the operator L_h from L^q to L^q, where $L_h = h * g$. (This fact can be seen directly noticing that

$$\int_{-\infty}^{+\infty} (L_n f_1)(x) f_2(-x) dx = (L_\mu f_1 * f_2)(0) = \int_{-\infty}^{+\infty} (f_1 * f_2)(-x) d\mu(x)$$

for $f_1 \in L^q$, $f_2 \in L^p$ and that $f_1 * f_2 \in C_0$; it is also an obvious consequence of [**2**, Th.1]). If $f \in (L^p)_0 \subseteq L^p$, then $\widehat{f} \in L^q$, so that $L_{\mu_n} \widehat{f} = \mu_n * \widehat{f}$ converges weakly to $h * \widehat{f} = (\widehat{h} f)^{\frown}$. As weak and strong topologies in L^q have the same closed convex sets [**1**, V.3.13] there exists a sequence of convex combinations of $\mu_n * \widehat{f}$ converging strongly to $h * \widehat{f}$; we obtain in this way a sequence of discrete measures $\nu_n = \Sigma a_{jn} \delta_{\mu_{jn}}$ such that $\nu_n * \widehat{f}$ converges in the norm of L^q to $h * \widehat{f}$ and satisfying $\Sigma |a_{jn}| = 1$. $\widehat{\nu}_n(t) = \Sigma a_{jn} e_{x_{jn}}(t)$, where $x_{jn} = -y_{jn}$; let $f_n = \Sigma a_{jn} e_{x_{jn}} f$ and let T_n be the operator from L^q to M associated with f_n. I shall prove that T_n converges in the strong operator topology to an operator T: we have that $\| T_n \| = \| f_n \|_0 \leqslant \Sigma |a_{nj}| \, \| f \|_0 = \| f \|_0$, let S be the subspace of L^q consisting of q-integrable functions which are transforms of elements of L^p; then S is dense in L^q (it contains for instance all rapidly decreasing infinitely differentiable functions); if $g \in S$, $g = \widehat{\varphi}$, $\varphi \in L^p$ therefore $(T_n g)^{\frown} = f_n \nu g = (\widehat{f_n \varphi})^{\frown}$ so that $T_n g$ is an absolutely continuous measure, that is $T_n g = \widehat{f_n \varphi} \in L^1$;

$$\begin{aligned} \| T_n g - (h * \widehat{f}) \varphi \|_1 &= \| \widehat{f}_n \varphi - (h * \widehat{f}) \varphi \|_1 \leqslant \\ &\leqslant \| \widehat{f}_n - h * \widehat{f} \|_q \, \| \varphi \|_p \to 0 \end{aligned}$$

as $\widehat{f_n}$ converges to $h * \widehat{f}$ in the L^q norm. Thus $\lim\limits_n T_n g = (h * \widehat{f})\varphi$ in the norm of L^1 (wich coincides with the norm of M) for every $\varphi \in S$, therefore the principle of uniform boundedness [**1**, II.1.18] can be applied and $\lim T_n g = Tg$ exists, in the L^1 norm, for every $g \in L^q$. As $(Tg)\widehat{\ } = \widehat{hf} * g$ for every $g \in S$ and T is continuous $(Tg)\widehat{\ } = \widehat{hf} * g$, for every $g \in L^q$. Therefore by Lemma 1 $\widehat{hf} \in (L^p)_0$.

Lemma 2 extends also to the case of a locally compact Abelian group G (the L^1 function appearing in the statement will of course be defined on the charactei group Γ of G).

Lemma 1 and 2 allow us now to modify Helgason's proof to cover the case of L^p, $p < 2$.

THEOREM 3. *Let $p < 2$ and $f \in (L^p)_0$, then $f \equiv 0$.*

PROOF. Let $f \in (L^p)_0$ and suppose that $f \neq 0$; in view of Lemma 2 we may assume that f vanishes outside a compact set C. As C is compact, for some $r > 0$, $C \subseteq [-r/4, r/4]$; let $V = (-\delta, \delta)$ with $0 < \delta < r/4$, then for $n \neq m$ and $n, m = 0, \pm 1, \pm 2, \ldots$ $(C + V + rn) \cap (C + V + rm) = \varnothing$. Let u be a nonnegative function vanishing outside V satisfying

$$\| u \|_1 = 1 \text{ and } \big| \| f * u \|_p - \| f \|_p \big| < \epsilon < \frac{1}{2} \| f \|_p .$$

(Such a u certainly exists as for any approximate identity $\{u_n\}$ of L^1, $\lim u_n * f = f$ in the norm of L^p). Let $g(t) = \Sigma b_n u(t - nr)$ where the sum is taken over a finite number of integers. By simple calculation it is not difficult to see that

$$\| f * g \|_1 = \| f * u \|_p (\Sigma |b_n|^p)^{1/p},$$

therefore

$$\begin{aligned} &\big| \| f * g \|_p - \| f \|_p (\Sigma |b_n|^p)^{1/p} \big| = \\ &= \big| \| f * u \|_p (\Sigma |b_n|^p)^{1/p} - \| f \|_p (\Sigma |b_n|^p)^{1/p} \big| \leqslant \\ &\leqslant \epsilon (\Sigma |b_n|^p)^{1/p} < \frac{1}{2} \| f \|_p (\Sigma |b_n|^p)^{1/p} , \end{aligned}$$

so that

$$\| f \|_p (\Sigma |b_n|^p)^{1/p} \leqslant 2 \| f * g \|_p \leqslant 2 \| f \|_0 \| \widehat{g} \|_\infty$$

which implies

$$(\Sigma|b_n|^p)^{1/p} \leqslant \frac{2\,\|f\|_0}{\|f\|}\,\|\hat{g}\|_\infty .$$

Let now $B = \dfrac{2\,\|f\|_0}{\|f\|_p}$ so that $(\Sigma|b_n|^p)^{1/p} \leqslant B\,\|\hat{g}\|_\infty$. We have that

$$\hat{g}(t) = \Sigma b_n \int_{-\infty}^{+\infty} u(x - nr)e^{-itx}dx = \Sigma b_n \int_{-\infty}^{+\infty} u(x)e^{-it(nr+x)dx} =$$

$$= \Sigma b_n e^{itnr} \int_{-\infty}^{+\infty} u(x)e^{-itx}dx = \hat{u}(t)\Sigma b_n e^{inrt}.$$

Therefore

$$(\Sigma|b_n|^p)^{1/p} \leqslant B\,\|\hat{u}\|\,\sup_t |\Sigma b_n e^{inrt}|,$$

that is $(\Sigma|b_n|^p)^{1/p} \leqslant B \sup_t |\Sigma b_n e^{inrt}|$ for any choice of (finitely many) b_n. The functions $\Sigma b_n e^{inrt}$ can be considered as continuous functions of period $2\pi/r$ and ideed any continuous function of period $2\pi/r$ can be approximated uniformly by trigonometric polynomials of the form $\Sigma b_n e^{inrt}$. The last inequality implies that any continuous function φ of period $2\pi/r$ has Fourier coefficients $\hat{\varphi}(n)$ satisfying $\Sigma|\hat{\varphi}(n)|^p < \infty$ with $p < 2$. This is known to be false [**8**, v.I, p. 200] and we are led to a contradiction.

The proof of the preceding theorem applies as well to the case of a a group $G = \mathbf{K} \times \mathbf{R}^n (n \leqslant 1)$ where $\mathbf{K}$ is a compact group, that is to the case of a connected, locally compact, non-compact Abelian group [**3**, Th. 5].

Remark. Let $\mathbf{Z}$ be the group of integers and let $l^p = L^p(\mathbf{Z})$ be the space of sequences $a(n)$, $n = 0, \pm 1, \pm 2, \ldots$ satisfying $\Sigma|a(n)|^p < \infty$. If $b(n) \in l^1$ one defines

$$(a * b)(n) = \Sigma_k a(n - k)b(k),$$

$(l^p)_0$ can then be defined in analogy to Definition 1 and Lemma 2 is readily seen to apply to this case in the following formulation: *if* $a \in l^p$, $p > 2$, *and* $h \in L^1(0,2\pi)$, $\| h \|_1 = 1$, then $\widehat{h}a \in (l^p)_0$. Taking $\widehat{h} = 1$ on $(0,2\pi)$ we obtain that if $(l^p)_0 \neq \{0\}$ the sequence satisfying $a(0) = 1$, $a(n) = 0$ for $n \neq 0$ belongs to $(l^p)_0$. In view of Lemma 1 this implies that every element of l^q is the Fourier-Stieltjes transform of a measure on $(0,2\pi)$, this is again known to be false for $q > 2$ [**6**, 7.8.5 and 7.8.6] and we conclude that $(l^p)_0 = \{0\}$ for $p < 2$. This remark is also valid when $\mathbf{Z}$ is replaced by any discrete Abelian group.

§ 3. Let $\varphi \in L^\infty$ (the space of essentially bounded measurable functions) then φ defines a bounded operator T on L^2 by the relation $(Tf)^\frown = \varphi \widehat{f}$ for $f \in L^2$. T has the property:

$$T\tau_x = \tau_x T \quad \text{for every } x \in \mathbf{R}\,. \tag{1}$$

Conversely envery bounded T satisfying (1) corresponds biuniquely to an element $\varphi \in L^\infty$ [**5**]. The space of bounded operators on L^2 satisfying (1) is called the space of *multipliers* in L^2 and denoted by M_2. The operator norm in M_2 is equivalent to the L^∞ norm. If we let M_p be the space of bounded operators on L^p satisfying (1) it is not difficult to see, considering the ajoint of each member of M_p, that $M_p = M_q$ $(1/p + 1/q = 1)$. This, together with the Riesz convexity theorem [**1**, VI.10.11], implies that $M_p \subseteq M_2$ and therefore each element T of M_p corresponds biuniquely to an element $\varphi \in L^\infty$ with the property $(Tf)^\frown = \varphi \widehat{f}$ for $f \in L^p \cap L^2$ (cf. [**2**] and [**5**]). The space M_p (or equivalently the subspace of L^∞ consisting of those elements of L^∞ which correspond to members of M_p) is called the space of multipliers in L^p.

The following is an important special case of a theorem proved by Hörmander [**5**, Th. 1.12, p. 106].

THEOREM 5. *Let $\varphi \in L^\infty$ be a multiplier in L^p with the property that if $|\psi| \leqslant |\varphi|$, ψ is also a multiplier in L^p, then $\varphi \equiv 0$.*

PROOF. As $M_p = M_q$ we may assume $p < 2$. Let φ satisfy the hypothesis of the theorem and let T be the operator corresponding to φ. Then $Tf \in (L^p)_0$ for each $f \in L^p$. Indeed $(Tf)^\frown = \varphi \widehat{f}$

and if ψ is a bounded continuous function, $|\psi\varphi| \leqslant \| \psi \|_\infty |\varphi|$ so that $\widehat{\varphi\psi f} = \widehat{g}$, where $g \in L^p$; as this holds for any bounded continuous function φ in view of the remarks following Definition 1, $Tf \in (L^p)_0$; but $(L^p)_0 = \{0\}$ implies $Tf \equiv 0$ and hence $\varphi \equiv 0$.

It should be noted that with the same method one can prove the full strenght of Hörmander's result. Obviously the proof above applies to all cases in which Theorem 3 is valid.

§ 4. We have seen that $p < 2$ implies $(L^p)_0 = \{0\}$. The situation is completely different for $p > 2$: in this case $(L^p)_0$ contains at least all functions which are transforms of elements of L^q (these are L^p functions because of the Hausdorff-Young theorem [7, p. 96, Th. 74]). Indeed, if $f \in L^p$ and $f = \widehat{g}$, $g \in L^q$, then for $h \in L^1$, $(h * f)\widehat{\ } = \widehat{hg}$ so that

$$\| h * f \|_p \leqslant \| \widehat{hg} \|_q \leqslant \| \widehat{h} \|_\infty \| g \|_q ;$$

thus $f \in (L^p)_0$ and $\| f \|_0 \leqslant \| g \|_r$. Let S be the subspace of L^p $(p > 2)$ consisting of transforms of elements of L^q. the following questions arise naturally: (a) is $S = (L^p)_0$? (b) is S dense in $(L^p)_0$? In § 5 a negative answer is provided to the first question for $L^p(0,2\pi)$, but both questions remain open for $L^p(\mathbf{R})$. It should be noted that an affirmative answer to question $b)$ would imply that $f * g$ is the Fourier transform of an absolutely continuous measure for $f \in (L^p)_0$, $g \in L^q$.

While it is not clear that elements of $(L^p)_0$, $p > 2$ have locally integrable transforms (considering for example their transform in the distribution sense) the following result can still be established:

PROPOSITION 5. *Let φ be a bounded continuous function on $\mathbf{R}$ then there exists a bounded operator T on $(L^p)_0$ such that $T\tau_x = \tau_x T$, for all $x \in \mathbf{R}$, and $(\varphi h)\widehat{\ } = Tf$ if $h \in L^q$, $\widehat{h} = f$.*

PROOF. By Lemma 1, $f * g = \widehat{\mu}$ for some $\mu \in M$ if $f \in (L^p)_0$ and $g \in L^q$. Define, fixing $f \in (L^p)_0$, $F(g) = \int_{-\infty}^{+\infty} \varphi d\mu$. Then F is a linear functional defined on L^q and

$$|F(g)| \leqslant \| \varphi \|_\infty \| \mu \| \leqslant \| \varphi \|_\infty \| f \|_0 \| g \|_q.$$

Thus F is bounded and there exist $Tf \in L^p$ such that $F(g) = \int_{-\infty}^{+\infty} Tf(-x)g(x)dx$. But $F(\tau_y g) = \int_{-\infty}^{+\infty} \varphi(t)e^{i}{}_{tt}d\mu(t)$ therefore

$$(Tf * g)(y) = \int_{-\infty}^{+\infty} Tf(-x)g(x+y)dy = F(\tau_y g) =$$

$$= \int_{-\infty}^{+\infty} \varphi(t)e^{iyt}d\mu(t) = \widehat{\nu}(y)$$

were r is the measure whose element of length is $d\nu = \varphi(t)d\mu$ Thus $Tf \in (L^p)_0$ and $\| Tf \|_0 \leqslant \| \varphi \|_\infty \| \mu \| \leqslant \| \varphi \|_\infty \| f \|_0$. It is easy to verify that T commutes with translations.

REMARK 6. Let $(L^\infty)_0$ be the set of elements $f \in L^\infty$ such that $\| f \|_0 = \sup \{ \| h * f \|_\infty : \| \widehat{h} \|_\infty \leqslant 1 \} < \infty$. It is not difficult to see that $(L^\infty)_0$ *consists of all Fourier-Stieltjes transforms of measures*; thus, in this case the questions (a) and (b) raised earlier have a negative answer and moreover elements of $(L^\infty)_0$ will not in general have locally integrable transforms.

§ 5. Definition 1 applies as well to $L^p(0,2\pi)$ ($\| \widehat{h} \|_\infty = \sup |\widehat{h}(n)|$ for $h \in L^1(0,2\pi)$). I have remarked earlier that $L^2(0,2\pi) = (L^p(0,2\pi))_0$ for $p < 2$. For $p > 2$ the situation is more complex: Lemma 1 still applies but no simpler characterization of $(L^p)_0$ seems to be known. We have that all elements of L^p satisfying $\Sigma |\widehat{f}(n)|^q < \infty$ belong to $(L^p)_0$; the fact that $(L^p)_0$ contains functions satisfying $\Sigma |\widehat{f}(n)|^q = \infty$ is easily established: let $E = \{ \pm 2^n : n = 0, 1, ...\}$ and let $\{a(n) : n = 0, \pm 1, ...\}$ be a sequence satisfying $a(n) = 0$ for $n \notin E$, $\Sigma |a(n)|^2 < \infty$ and $\Sigma |a(n)|^q = \infty$. Then by the properties of lacunary Fourier series [**8**, v.I, p. 215, Th. 8./20]

$f \sim \Sigma a(n)e^{inx}$ belongs to L^p for every $p < \infty$. If $\varphi(n)$ is a bounded sequence, the sequence $\varphi(n)a(n) = b(n)$ will again satisfy $\Sigma|b(n)|^2 < \infty$ and $b(n) = 0$ for $n \notin E$; therefore $g \sim \Sigma b(n)e^{inx} \in L^p$, hence $f \in (L^p)_0$, but $\Sigma|\widehat{f}(n)|^q = \Sigma|a(n)^q = \infty$.

A modification of this example can be used to give a negative answer to a problem posed by Helgason [**3**, p. 254]; Helgason noticed that for $p \leqslant 2$ the subspace of L^p invariant under multiplication of the Fourier coefficients by bounded sequences (i.e. L^2) was also invariant under arbitrary permutation of the Fourier coefficients and he posed the question of whether the same would be true for $p > 2$. To answer this question we start with a function $f \in L^2$ which satisfies $\widehat{f}(n) = 0$ for $n < 0$, we shall also suppose that $f \notin L^p$ (the existence of such a function is guaranteed by the existence of a translation invariant bounded projection from L^p to $H^p = \{f \in L^p : \widehat{f}(n) = 0,\ n < 0\}$ and the fact that $L^p \neq L^2$). Let now σ be a permutation of the integers onto the integers such that the set $E = \{\pm 2^n\}$ is mapped onto the nonnegative integers and let g be the L^2 function with Fourier coefficients $\widehat{g}(n) = \widehat{f}(\sigma(n))$, then, if $n \in E$, $\sigma(n) < 0$ and $\widehat{g}(n) = 0$. Thus, as we saw before, $g \in (L^p)_0$ but the permutation σ^{-1} maps g onto f which is not a member of L^p.

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