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OPERATIONAL FORMULAE FOR JACOBI AND OTHER POLYNOMIALS

Nota *) *di* R. P. SINGH (*Bhopal*)

1. INTRODUCTION. – In a recent paper [6] GOULD and HOPPER have given the operational relations

$$\mathfrak{D}^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}^{\gamma}(x, a, p) D^k, \tag{1.1}$$

$$x^n \mathfrak{D}^n = \prod_{j=0}^{n-1} (xD - p\gamma x^{\gamma} + \alpha - j), \tag{1.2}$$

$$\mathfrak{D} = D - p\gamma x^{\gamma-1} + \frac{\alpha}{x}, \quad D = \frac{d}{dx}, \tag{1.3}$$

and

$$H_n^{\gamma}(x, \alpha, p) = (-1)^n x^{-\alpha} e^{px^{\gamma}} D^n (x^{\alpha} e^{-px^{\gamma}}). \tag{1.4}$$

The relation (1.1) is an extension of Burchnall's [1] operational relation for Hermite polynomials

$$(D - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x) D^k, \tag{1.5}$$

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Indirizzo dell'A.: Dept. of Maths., Motilal Vigyan Mahavidyalaya, Bhopal, (M. P.), India.

and Carlitz's [2] formula for Laguerre polynomials

$$\prod_{j=1}^{n}(xD - x + \alpha + j) = n!\sum_{k=0}^{n}\frac{x^k}{k!}\,L_{n-k}^{\alpha+k}(x)D^k\,. \tag{1.6}$$

Chatterjea [3, 4, 5] has studied operational relations for generalized Bessel polynomials. His main formula is

$$\prod_{j=1}^{n}\{x^2D + (a+2j)x + b\} = \sum_{\gamma=0}^{n}\binom{n}{\gamma}b^{n-\gamma}x^{2\gamma}y_{n-\gamma}(x, a + \\ +2\gamma + 2, b)\cdot D^{\gamma}\,. \tag{1.7}$$

The object of this paper is to develope certain operational formulae for Jacobi and related polynomials and to study some properties of these polynomials derivable with the help of operational relations.

2. Operational formulae. – The Rodrigues' formula for Jacobi polynomials is

$$P_n^{(\alpha,\beta)}(x) = \\ = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}D^n\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}\,. \tag{2.1}$$

If f is any sufficiently differentiable function of x, we have

$$D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}\cdot f] = \\ = D^{n-1}[(1-x)^{\alpha+n-1}(1+x)^{\beta+n-1}\cdot \\ \cdot\{(1-x^2)D - (\alpha+\beta+2n)x + \beta - \alpha\}f]\,,$$

and repeating the process, we have

$$\prod_{j=1}^{n}\{(1-x^2)D - (\alpha+\beta+2j)x + \beta - \alpha\}\cdot f \\ = (1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}\cdot f]\,. \tag{2.2}$$

We shall prove here (2.2) by the method of induction. Obviously, for $n = 1$ the identity (2.2) holds good. Again, replacing f by

$\{(1-x^2)D-(\alpha+\beta+2n+2)x+\beta-\alpha\}f$ in the above identity, we have

$$\begin{aligned}&\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\\ &\qquad\cdot\{(1-x^2)D-(\alpha+\beta+2n+2)x+\beta-\alpha\}f\\ &\qquad=(1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}\\ &\qquad\cdot\{(1-x^2)D-(\alpha+\beta+2n+2)x+\beta-\alpha\}f],\end{aligned}$$

which immediately yields

$$\begin{aligned}&\prod_{j=1}^{n+1}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot f=\\ &=(1-x)^{-\alpha}(1+x)^{-\beta}D^{n+1}[(1-x)^{\alpha+n+1}(1+x)^{\beta+n+1}\cdot f].\end{aligned}$$

Again

$$\begin{aligned}&(1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}\cdot f]\\ &=(1-x)^{-\alpha}(1+x)^{-\beta}\sum_{k=0}^{n}\binom{n}{k}D^{n-k}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]D^kf\\ &=n!\sum_{k=0}^{n}\frac{(-2)^{n-k}}{k!}(1-x^2)^kP_{n-k}^{(\alpha+k,\beta+k)}(x)D^kf.\end{aligned}$$

Therefore, from (2.2), the operational formula for Jacobi polynomial is

$$\begin{aligned}&\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\\ &\qquad=\sum_{k=0}^{n}\frac{(-2)^{n-k}n!}{k!}(1-x^2)^kP_{n-k}^{(\alpha+k,\beta+k)}(x)D^k.\end{aligned}\tag{2.3}$$

In case $f=1$, we have from (2.3)

$$\begin{aligned}&\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot 1\\ &\qquad=(-2)^n n!P_n^{(\alpha,\beta)}(x).\end{aligned}\tag{2.4}$$

For $\alpha = \beta$, (2.4) yields

$$(2.5) \qquad \prod_{j=1}^{n} \{(1-x^2)D - (\alpha + j)x\} \cdot 1 = (-2)^n n! P_n^{(\alpha,\alpha)}(x)$$
$$= \frac{(-2)^n n!}{(1+2\alpha)_n n}(1+\alpha)_n C_n^{\alpha+\frac{1}{2}}(x),$$

and for $\alpha = \beta = 0$, we have

$$(2.6) \qquad \prod_{j=1}^{n} \{(1-x^2)D - 2jx\} \cdot 1 = (-2)^n n! P_n(x).$$

It may of interest to point out that Burchnall's relation for Hermite polynomials (1.5) is a particular case of (2.3). Indeed, for $\alpha = \beta = \lambda - \frac{1}{2}$, we have from (2.3)

$$(2.7) \qquad \prod_{j=1}^{n} \{(1-x^2)D - (2\lambda - 1 + 2j)x\}$$
$$= \sum_{k=0}^{n} \frac{(-2)^{n-k} n! \left(\lambda + k + \frac{1}{2}\right)_{n-k}}{k!\,(2\lambda + 2k)_{n-k}} (1-x^2)^k \cdot C_{n-k}^{\lambda+k}(x) D^k.$$

Replacing x by $x/\sqrt{\lambda}$, letting $\lambda \to \infty$ and using Toscano's [8] relation

$$(2.8) \qquad \lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} C_n^{\lambda}(x/\sqrt{\lambda}) = \frac{1}{n!} H_n(x),$$

we find that (2.7) ultimately reduces to (1.5).

3. Some applications of operational formulae. – Starting with (2.2) or (2.3), we easily obtain the following relations:

$$(3.1) \qquad \prod_{j=1}^{n} \{(1-x^2)D - (\alpha + \beta + 2j)x + \beta - \alpha\} \cdot x$$
$$= (-2)^n n!\, x P_n^{(\alpha,\beta)}(x) + (-2)^{n-1} n!\, (1-x^2) P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$(3.2)\qquad \prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot(1-x) = (-2)^n n!\,(1-x)P_n^{(\alpha+1,\beta)}(x),$$

$$(3.3)\qquad \prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot(1+x) = (-2)^n n!\,(1+x)P_n^{(\alpha,\beta+1)}(x)\,.$$

The relations (2.4) and (3.1) follow the associative law. Now by combining (2.4), (3.1), (3.2) and (3.3) we eassily obtain

$$(3.4)\qquad 2P_n^{(\alpha,\beta)}(x) = 2P_n^{(\alpha+1,\beta)}(x) - (1-x)P_{n-1}^{(\alpha+1,\beta+1)}(x)\,,$$

$$(3.5)\qquad 2P_n^{(\alpha,\beta)}(x) = 2P_n^{(\alpha,\beta+1)}(x) + (1-x)P_{n-1}^{(\alpha+1,\beta+1)}(x)\,,$$

$$(3.6)\qquad 2P_n^{(\alpha,\beta)}(x) = (1-x)P_n^{(\alpha+1,\beta)}(x) + (1+x)P_n^{(\alpha,\beta+1)}(x)\,.$$

Again, operating both sides of (2.4) by $\{(1-x^2)D-(\alpha+\beta)x+\beta-\alpha\}$, we have

$$(3.7)\qquad 2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(x) = \{(\alpha+\beta)x-\beta+\alpha\}\,P_n^{(\alpha,\beta)}(x) - (1-x^2)DP_n^{(\alpha,\beta)}(x)\,.$$

Further

$$\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot 1 = \prod_{j=1}^{n-1}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\} \cdot\{-(\alpha+\beta+2n)x+\beta-\alpha\}\,.$$

Using (2.3) and (2.4) the above relation finally yields

$$(3.8)\qquad 4nP_n^{(\alpha,\beta)}(x) + 2\{(\beta-\alpha)-(\alpha+\beta+2n)x\}\,P_{n-1}^{(\alpha,\beta)}(x) + (\alpha+\beta+2n)(1-x^2)P_{n-2}^{(\alpha+1,\beta+1)}(x) = 0\,.$$

Next, we observe from (2.4) that

$$(-2)^{n+m}(n+m)!\,P_{n+m}^{(\alpha,\beta)}(x) = \prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}$$

$$\cdot \prod_{k=1}^{m}\{(1-x^2)D-(\alpha+\beta+2n+2k)x+\beta-\alpha\}\cdot 1$$
$$= (-2)^m m!\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta+\alpha\}\cdot P_m^{(\alpha+n,\beta+n)}(x)$$
$$= \sum_{k=0}^{n}\frac{n!\,m!}{k!}(-2)^{n+m-k}(1-x^2)^k P_{n-k}^{(\alpha+k,\beta+k)}(x)D^k P_m^{(\alpha+n,\beta+n)}(x)\,.$$

Again, since [7]

$$D^k P_n^{(\alpha,\beta)}(x) = 2^{-k}(1+\alpha+\beta+n)_k P_{n-k}^{(\alpha+k,\beta+k)}(x)\,,$$

therefore we obtain

$$P_{n+m}^{(\alpha,\beta)}(x) = \frac{n!\,m!}{(n+m)!}\sum_{k=0}^{n}\frac{(-1)^k(1-x^2)^k}{k!\,2^{2k}}(\alpha+\beta + 2n+m+1)_k\cdot P_{n-k}^{(\alpha+k,\beta+k)}(x)P_{m-k}^{(\alpha+n+k,\beta+n+k)}(x)\,. \tag{3.9}$$

In particular case when $n=1$, $m>1$, we finally obtain from (3.9)

$$(1-x^2)D^2y+[\beta-\alpha-(\alpha+\beta+2)x]Dy + (m+1)(\alpha+\beta+m+2)y = 0\,, \tag{3.10}$$

where $y = P_{m+1}^{(\alpha,\beta)}(x)$, which is the differential equation for Jacobi polynomials.

4. Some special cases. – In relation (2.2), replacing f by e^{-x}, we have

$$\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot e^{-x}$$
$$= (1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}e^{-x}]$$
$$= (1-x)^{-\beta}(1+x)^{-}\beta\cdot e\cdot\sum_{k=0}^{n}\binom{n}{k}D^k(1-x)^{\alpha+n})D^{n-k}\}(1+x)^{\beta+n}e^{-1-x}\}.$$

since

$$L_n^{\alpha}(x) = \frac{x^{-\alpha}e^{x}}{n!}D^n[x^{\alpha+n}e^{-x}]\,,$$

where $L_n^{\alpha}(x)$ are laguerre polynomials, we have therefore

$$(4.1)\quad \prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot e^{-x}$$
$$=\sum_{k=0}^{n} n\,!\,(-1)^k\binom{\alpha+n}{k}(1-x)^{n-k}(1+x)^k e^{-x}L_{n-k}^{\beta+k}(1+x)\,.$$

Also we have

$$(4.2)\quad \prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}\cdot e^{-x}$$
$$=\sum_{k=0}^{n}(-2)^n\frac{n\,!}{k\,!}\left(\frac{1-x^2}{2}\right)^k e^{-x}P_{n-k}^{(\alpha+k,\beta+k)}(x)\,.$$

On equating (4.1) and (4.2) ,we have the identity

$$(4.3)\quad \sum_{k=0}^{n}(-1)^k\binom{\alpha+n}{k}(1-x)^{n-k}(1+x)^k L_{n-k}^{\beta+k}(x)$$
$$=\sum_{k=0}^{n}\frac{(-2)^n}{k\,!}\left(\frac{1-x^2}{2}\right)^k P_{n-k}^{(\alpha+k,\beta+k)}(x)\,.$$

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