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ON THE APPROXIMATE SOLUTIONS OF LINEAR  
EQUATIONS IN THE  $l_2$ -SPACE

*Nota \*) di RABINDRANATH SEN (a Calcutta)*

In the present paper we give a method of solving approximately linear equations in  $l_2$ -space. This method is an extension of Altman's method of approximate solution of linear algebraic equations (Altman, 1957). It is indicated, in this paper, that this method is immediately applicable to any linear equation in a separable Hilbert Space  $H$  and consequently to linear functional equations in  $L_2$ .

1. - Let us consider the  $l_2$ -space. Let  $K$  be a bounded linear transformation of a subspace  $X$  of  $l_2$  into a subspace  $Y$  of  $l_2$ . We suppose that  $K$  is one-to-one.

Consider the linear equation,

$$(1) \quad Kx = y(x \in X, y \in Y).$$

Let  $\{e_i\}$  be a complete orthonormal system. Since,  $K$  is one-to-one,  $\{a_i\}$  where  $a_i = Ke_i$  is linearly independent.

For if  $a_i$ 's are dependant then there exist constants  $\alpha_i$ 's not

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all zeros such that,

$$\begin{aligned} \Sigma \alpha_i a_i &= 0 \quad \text{or} \\ \text{or } \Sigma \alpha_i K e_i &= 0 \quad \text{or, } K \Sigma \alpha_i e_i = 0 \quad \text{since, } K \text{ is bounded} \\ &\therefore \Sigma \alpha_i e_i = 0 \end{aligned}$$

*i.e.*,  $e_i$ 's are not independent, which is a contradiction.

The sequence  $\{a_i\}$  is weakly convergent. For every  $\varphi \in l_2$ ,

$$\begin{aligned} |(a_m, \varphi) - (a_n, \varphi)| &= |(K e_m, \varphi) - (K e_n, \varphi)| \\ &= |(e_m, K^* \varphi) - (e_n, K^* \varphi)| \\ &\longrightarrow 0 \\ &\text{as } m, n \text{ both } \longrightarrow \infty. \end{aligned}$$

because  $\{e_i\}$  is weakly convergent to  $\theta$ .

Again, for every  $\varphi \in l_2$ ,

$$\begin{aligned} (a_i, \varphi) = (K e_i, \varphi) &= (e_i, K^* \varphi) \longrightarrow \theta \\ &\text{as } i \longrightarrow \infty. \end{aligned}$$

Therefore, for every  $\varphi \in l_2$ ,

$$(a_i, \varphi) \longrightarrow (\theta, \varphi) \quad \text{as } i \longrightarrow \infty.$$

and hence  $a_i \longrightarrow \theta$  weakly as  $i \longrightarrow \infty$ .

Define the infinite set of vectors as follows:

$$\begin{aligned} (2) \quad y_1 &= y - P_1 y, \quad y_2 = y_1 - P_2 y_1, \dots \\ y_n &= y_{n-1} - P_n y_{n-1}, \quad y_{2n} = y_{2n-1} - P_{2n} y_{2n-1} \end{aligned}$$

where  $P_i$  denotes the orthogonal projection on

$$a_i \quad (i = 1, 2, 3, \dots, n, \dots)$$

$$\begin{aligned} (3) \text{ Then, } \quad \|y_{k+1}\|^2 &= \|y_k\|^2 - \|P_{k+1} y_k\|^2. \\ \text{where } P_k y &= \frac{(y, K e_k)}{\|K e_k\|^2} K e_k, \end{aligned}$$

It follows from (3) that  $\{\|y_k\|^2\}$  is monotonously decreasing and bounded below and is therefore also convergent.

We shall next show that  $\{y_k\}$  ( $K = 1, 2, \dots, n, \dots$ ) is weakly convergent.

From (3), we have  $\{I_k\}$  is bounded, and hence contains a weakly convergent subsequence  $\{y_{k_p}\}$ .

Let,  $y_{k_p} \xrightarrow{\text{weakly}} v$  as  $p \longrightarrow \infty$ .

Now,  $y_{k_p+1} = y_{k_p} - \alpha_{k_p+1} a_{k_p+1}$ .

where  $\alpha_{k_p+1} = \frac{(y_{k_p}, a_{k_p+1})}{\|a_{k_p+1}\|^2}$

Let,  $\alpha_{i_p} = (y_{k_p}, a_i)$ .

Now, as  $i \longrightarrow \infty$   $\alpha_{i_p} \longrightarrow 0$ , for all  $I_{k_p}$ 's. ( $p = 1, 2, \dots$ ).

because  $a_i \xrightarrow{\text{weakly}} \theta$ .

Also,

$\alpha_{i_p} \longrightarrow (v, a_i)$  for all  $a_i$ 's.

as  $p \longrightarrow \infty$

because  $y_{k_p} \xrightarrow{\text{weakly}} v$

as  $p \longrightarrow \infty$ .

So,  $\alpha_{i_p} \longrightarrow 0$  as both  $i$  &  $p \longrightarrow \infty$ .

$\therefore \alpha_{k_p+1} \longrightarrow 0$  as  $p \longrightarrow \infty$ .

Hence,

$y_{k_p+1} = y_{k_p} - \alpha_{k_p+1} a_{k_p+1}$

$\xrightarrow{\text{weakly}} v$  as  $p \longrightarrow \infty$ .

because,  $a_{k_p+1} \xrightarrow{\text{weakly}} \theta$

as  $p \longrightarrow \infty$ .

Similarly, we can prove that,

$y_{k_p+i} \xrightarrow{\text{weakly}} v$  as  $p \longrightarrow \infty$ .

for all  $i$  (finite).

If,  $i \longrightarrow \infty$ ,  $p \longrightarrow \infty$  also.

$$y_{k,p+1} \xrightarrow{\text{weakly}} v \text{ as } i \longrightarrow \infty.$$

$\therefore \{y_i\}$  is weakly convergent

and converges to

$$v = y - \sum_{i=1}^{\infty} \alpha_i a_i = y^{(1)}. \tag{say}$$

Now, again we define the infinite set of vectors as follows:

$$y_{(1)}^{(1)} = y^{(1)} - \frac{(y^{(1)}, a_1)}{\|a_1\|^2} a_1, \quad y_{(2)}^{(1)} = y_{(1)}^{(1)} - \frac{(y_{(1)}^{(1)}, a_2)}{\|a_2\|^2} a_2,$$

$$(4) \quad y_{(i)}^{(1)} = y_{(i-1)}^{(1)} - \frac{(y_{(i-1)}^{(1)}, a_i)}{\|a_i\|^2} a_i, \dots (i = 1, 2, \dots, n, \dots).$$

We can prove similarly that  $\{y_{(i)}^{(1)}\} \xrightarrow{\text{weakly}} y^{(2)}$ .

We thus obtain the following infinite sequences:

$$y_{(j)}^{(i)} = y_{(j-1)}^{(i)} - \frac{(y_{(j-1)}^{(i)}, a_j)}{\|a_j\|^2} a_j, \dots \quad \begin{matrix} i = 0, 1, 2, \dots, n, \dots \\ j = 0, 1, 2, \dots, n, \dots \end{matrix}$$

with  $y_{(0)}^{(0)} = y$ ,  $y_{(j)}^{(0)} = y_{(j)} = y_j$  and  $y_{(0)}^{(i)} = y^{(i)}$ .

The above sequences can be written in the following array:

$$\begin{matrix} y & , & y_{(1)} & , & y_{(2)} & , & \dots & y_{(i)} & , & \dots \\ y^{(1)} & , & y_{(1)}^{(1)} & , & y_{(2)}^{(1)} & , & \dots & y_{(i)}^{(1)} & , & \dots \\ \vdots & & & & & & & & & \\ y^{(n)} & , & y_{(1)}^{(n)} & , & y_{(2)}^{(n)} & , & \dots & y_{(i)}^{(n)} & , & \dots \\ \vdots & & & & & & & & & \end{matrix}$$

Now, from (3) and (4),

$$(5) \quad (y_{(1)}, a_1) = 0, \quad (y_{(1)}^{(1)}, a_1) = 0, \dots \quad (y_{(i)}^{(i)}, a_1) = 0, \dots$$

Also

$$(y_{(2)}, a_2) = 0, \quad (y_{(2)}^{(1)}, a_2) = 0, \dots \quad (y_{(2)}^{(i)}, a_2) = 0, \dots$$

.....

$$(y_{(k)}, a_k) = 0, \quad (y_{(k)}^{(1)}, a_k) = 0, \dots \quad (y_{(k)}^{(i)}, a_k) = 0, \dots$$

Now,  $\{y_{(i)}^{(n)}\}$  is bounded and contains a weakly convergent subsequence  $\{y_{(i)}^{(n_p)}\}$ .

Now, let  $\{y_{(i)}^{(n_p)}\} \xrightarrow{\text{weakly}} v_1$  as  $p \longrightarrow \infty$

$$\therefore (y_{(1)}^{(n_p)}, a_1) = 0 \text{ \& since, } y_{(1)}^{(n_p)} \xrightarrow{\text{weakly}} v_1,$$

$$\therefore (v_1, a_1) = 0.$$

$$y_{(2)}^{(n_p)} = y_{(1)}^{(n_p)} - \frac{(y_{(1)}^{(n_p)}, a_2)}{\|a_2\|^2} a_2,$$

$$y_{(1)}^{(n_p)} - y_{(2)}^{(n_p)} = \frac{(y_{(1)}^{(n_p)}, a_2)}{\|a_2\|^2} a_2 \xrightarrow{\text{weakly}} \frac{(v_1, a_2)}{\|a_2\|^2} a_2.$$

$$\therefore y_{(2)}^{(n_p)} \xrightarrow{\text{weakly}} v_2 = v_1 - \frac{(v_1, a_2)}{\|a_2\|^2} a_2.$$

$$(6) \quad \|y_{(2)}^{(n_p)}\|^2 = \|y_{(1)}^{(n_p)}\|^2 - \|P_2 y_{(1)}^{(n_p)}\|^2.$$

Also,

$$\|y\| \geq \|y^{(1)}\| \geq \|y^{(2)}\| \geq \|y^{(3)}\| \geq \dots$$

Also

$$(7) \quad \|y^{(n_p)}\| \geq \|y_{(1)}^{(n_p)}\| \geq \|y_{(2)}^{(n_p)}\| \geq \dots \geq \|y^{(n_p+1)}\| \geq \\ \geq \|y_{(1)}^{(n_p+1)}\| \geq \|y_{(2)}^{(n_p+1)}\| \geq \dots$$

$\therefore \{\|y_{(i)}^{(n_p)}\|\}$  is a monotonic decreasing sequence and bounded below and converges to  $V_1$  (say).

Also,  $\{\|y_{(2)}^{(n_p)}\|\}$  is a monotonic decreasing sequence and bounded below and converges to  $V_2$  (say).

From (7),

$$V_1 \geq V_2 \geq V_1.$$

$$\therefore V_1 = V_2.$$

From (6), we have  $\{\|P_2 y_{(1)}^{(np)}\|^2\}$  is convergent and converges to zero.

$\therefore \{P_2 y_{(1)}^{(np)}\}$  convergers strongly to  $\theta$ .

$$\begin{aligned} \text{Lt}_{p \rightarrow \infty} y_{(2)}^{(np)} &= \text{Lt}_{p \rightarrow \infty} y_{(1)}^{(np)} - \text{Lt}_{p \rightarrow \infty} P_2 y_{(1)}^{(np)} && \text{(weak limit).} \\ &\therefore v_2 = v_1. \end{aligned}$$

Similarly, we can prove,  $v_{i+1} = v_i$ .

$$\therefore v_1 = v_2 = v_3 = \dots = v_i = \dots = v \quad \text{(say)}$$

Now,  $(v_1, a_1) = 0$ ,  $(v_2, a_2) = 0$ , ...,  $(v_i, a_i) = 0$  for all  $i$ .

Or,  $(v, a_i) = 0$  for all  $i$ .

$(K^*v, e_i) = 0$  for all  $i$ .

$$\therefore K^*v = \theta.$$

$\therefore K^*$  is bounded and has an inverse  $[\overline{a}l_2 = l_2]$ .

$$\therefore v = \theta.$$

Define,

$$\begin{aligned} \omega_1 &= \frac{(y, Ke_1)}{\|Ke_1\|^2} e_1, & \omega_2 &= \frac{(y_{(1)}, Ke_2)}{\|Ke_2\|^2} e_2, \dots \\ \omega_k &= \frac{(y_{(k-1)}, a_k)}{\|a_k\|^2} e_k, \dots \end{aligned} \quad (8)$$

and

$$\begin{aligned} \omega_1^{(i)} &= \frac{(y^{(i)}, Ke_1)}{\|Ke_1\|^2} e_1, & \omega_2^{(i)} &= \frac{(y_{(1)}^{(i)}, Ke_2)}{\|Ke_2\|^2} e_2, \dots \\ \omega_k^{(i)} &= \frac{(y_{(k-1)}^{(i)}, Ke_k)}{\|Ke_k\|^2} e_k. \end{aligned}$$

Define,

$$x_k^{(0)} = \omega_1 + \omega_2 + \dots + \omega_k.$$

$$x_{(k)}^{(i)} = \omega_1^{(i)} + \omega_2^{(i)} + \dots + \omega_k^{(i)}. \quad (9)$$

Now,

$$\begin{aligned} Kx_{(k)}^{(0)} &= K\omega_1 + K\omega_2 + \dots + K\omega_k \\ &= y - y_k. \end{aligned}$$

$$\therefore Kx_k^{(0)} \xrightarrow{\text{weakly}} y - y^{(1)} \text{ as } k \longrightarrow \infty .$$

$$Kx_k^{(1)} = K\omega_{(1)}^{(1)} + K\omega_{(2)}^{(1)} + \dots + K\omega_k^{(1)} =$$

$$= y^{(1)} - y_{(k)}^{(1)} \xrightarrow{\text{weakly}} y^{(1)} - y^{(2)}$$

$$\text{as } k \longrightarrow \infty .$$

$$\therefore K(x_k^{(0)} + x_k^{(1)} + x_k^{(2)} + \dots + x_k^{(i)}) =$$

$$= (y - y_{(k)}) + (y^{(1)} - y_{(k)}^{(1)}) + \dots + (y^{(i-1)} - y_{(k)}^{(i-1)}) .$$

$$\text{as } k \longrightarrow \infty \text{ R.H.S. } \xrightarrow{\text{weakly}} y - y^{(i)} .$$

$$\text{and as } i \longrightarrow \infty \quad y^{(i)} \xrightarrow{\text{weakly}} \theta .$$

$$(10) \quad KX_{ik} = K(x_k^{(0)} + x_k^{(1)} + \dots + x_k^{(i)}), \quad \text{where } X_{ik} = \sum_0^i x_k^{(i)} .$$

$$\text{R.H.S. } \xrightarrow{\text{weakly}} y \text{ as } k \longrightarrow \infty \text{ and then } i \longrightarrow \infty .$$

Hence,  $\lim_{i \rightarrow \infty} (\lim_{k \rightarrow \infty} X_{ik})$  is the solution of the given equation  $Kx = y$ .

Now,  $L_2$ -space is isomorphic and isometric to  $l_2$ -space. Hence, any linear equation in  $L_2$ -space can be reduced to an infinite dimensional matrix equation in  $l_2$  and hence can be solved by the above method.

REMARK 1:

For the error estimation we have the following recurrent formula obtained from (3).

$$\|y_{(k+1)}\|^2 = \|y_{(k)}\|^2 - \frac{|(y_{(k)}, a_{k+1})|^2}{\|a_{k+1}\|^2}$$

or,

$$(11) \quad \|y_{(k)}\|^2 - \|y_{(k+1)}\|^2 = \frac{|(y_{(k)}, a_{k+1})|^2}{\|a_{k+1}\|^2} .$$

We put an end to computations when the R.H.S. of (11) is sufficiently small and then we estimate the value of  $y^{(1)}$ .



Proceeding in this manner, we estimate the value of  $y^{(i)}$

$$(12) \quad \text{Now } y^{(i+1)} = y^{(i)} - \sum_{k=1}^{\infty} \alpha_k^{(i)} a_k,$$

where

$$\alpha_k^{(i)} = \frac{y_{k-1}^{(i)}, a_k}{\|a_k\|^2}$$

Also,

$$\|y^{(i+1)}\| \leq \|y^{(i)}\|.$$

So, we put an end to computations when  $\|y^{(i)}\|$  is sufficiently small.

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