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Some integrals involving products of Bessel and Legendre functions

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SOME INTEGRALS INVOLVING PRODUCTS OF BESSEL AND LEGENDRE FUNCTIONS

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Sunto. – The integral:

$$\int_0^c x^{\varrho-1}(c^2 - x^2)^{-\frac{1}{2}\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx ,$$

where $\mathrm{Re}\,(\varrho) > 0$, $\mathrm{Re}\,(\sigma) < 1$, is evaluated in terms of hypergeometric series and a number of particular cases thereof are discussed.

1. – Let c be real, non-zero and finite, $\mathrm{Re}\,(\varrho + \lambda + \mu) > 0$ and $\mathrm{Re}\,(\sigma) < 1$ so that by making use of the formula [3, p. 314]:

$$\int_0^1 x^{\lambda-1}(1 - x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx =$$

$$= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\lambda - \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 + \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)},$$

$$\mathrm{Re}\,(\lambda) > 0 , \quad \mathrm{Re}\,(\mu) < 1 ;$$

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we have:

$$\int_0^c x^{\varrho-1}(c^2-x^2)^{-\frac{1}{2}\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx =$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}\left(\frac{1}{2a}\right)^{\lambda+2r}\left(\frac{1}{2b}\right)^{\mu+2s}}{r!\,s!\,\Gamma(\lambda+r+1)\Gamma(\mu+s+1)}\cdot$$

$$\cdot\int_0^c x^{\varrho+\lambda+\mu+2r+2s-1}(c^2-x^2)^{-\frac{1}{2}\sigma} P_\nu^\sigma\left(\frac{x}{c}\right) dx = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\cdot$$

$$\cdot\frac{(-)^{r+s}2^{\sigma-1}c^{\varrho-\sigma}\Gamma\left\{\frac{1}{2}(\lambda+\mu+\varrho+2r+2s)\right\}}{r!\,s!\,\Gamma(\lambda+r+1)\Gamma(\mu+s+1)\Gamma\left\{\frac{1}{2}(\lambda+\mu-\nu+\varrho-\sigma+2r+2s+1)\right\}}\cdot$$

$$\cdot\frac{\Gamma\left\{\frac{1}{2}(\lambda+\mu+\varrho+2r+2s+1)\right\}\left(\frac{c}{2a}\right)^{\lambda+2r}\left(\frac{c}{2b}\right)^{\mu+2s}}{\Gamma\left\{\frac{1}{2}(\lambda+\mu+\nu+\varrho-\sigma+2r+2s+2)\right\}},$$

and therefore:

$$\text{(1.1)}\qquad \int_0^c x^{\varrho-1}(c^2-x^2)^{-\frac{1}{2}\sigma} J_\lambda\left(\frac{2x}{a}\right) J_\mu\left(\frac{2x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx =$$

$$=\frac{2^{\sigma-1}c^{\delta-\sigma}\Gamma\left(\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)}{a^\lambda b^\mu\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\delta-\nu-\sigma+1)\right\}\Gamma\left\{\frac{1}{2}(\delta+\nu-\sigma+2)\right\}}\cdot$$

$$\cdot F\left|\begin{matrix}\frac{1}{2}\delta & , & \frac{1}{2}\delta+\frac{1}{2} : & - ; & - ; \\ \frac{1}{2}(\delta-\nu-\sigma+1) & , & \frac{1}{2}(\delta+\nu-\sigma+2) : & \lambda+1 ; & \mu+1 ;\end{matrix}\right. -\frac{c^2}{a^2}, -\frac{c^2}{b^2}\Bigg|,$$

where $\delta = \lambda + \mu + \varrho$ and the notation for the double hypergeometric series is due to Burchnall and Chaundy [2, p. 112] in preference, for the sake of brevity, to that introduced by Kampę De Fériet [1, p. 150].

For $\sigma = 0$, (1.1) gives:

$$(1.2)\qquad \int_0^c x^{\varrho-1} J_\lambda\left(\frac{2x}{a}\right) J_\mu\left(\frac{2x}{b}\right) P_\nu\left(\frac{x}{c}\right) dx =$$

$$= \frac{c^\delta \Gamma\left(\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)}{2a^\lambda b^\mu \Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\delta-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta+\nu+2)\right\}} \cdot$$

$$\cdot F\left[\begin{matrix} \frac{1}{2}\delta, & \frac{1}{2}\delta+\frac{1}{2} & : & - & ; & - & ; \\ \frac{1}{2}(\delta-\nu+1), & \frac{1}{2}(\delta+\nu+2) & : & \lambda+1 & ; & \mu+1 & ; \end{matrix}\; -\frac{c^2}{a^2}, -\frac{c^2}{b^2}\right],$$

where, as before, $\text{Re}(\delta) > 0$.

2. When $a = b$ the double series on the right of (1.1) is equal to:

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\delta\right)_k \left(\frac{1}{2}\delta+\frac{1}{2}\right)_k \left(-\frac{c^2}{a^2}\right)^k}{k!\left(\frac{1}{2}\delta-\frac{1}{2}\nu-\frac{1}{2}\sigma+\frac{1}{2}\right)_k \left(\frac{1}{2}\delta+\frac{1}{2}\nu-\frac{1}{2}\sigma+1\right)_k (\mu+1)_k} \cdot$$

$$\cdot \sum_{r=0}^{k} \frac{(-k)_r(-\nu-k)_r}{r!\,(\lambda+1)_r},$$

and by employing Vandermonde's theorem to sum the inner series this simplifies as a ${}_4F_5$. Therefore, a special case of (1.1) is:

$$(2.1)\qquad \int_0^c x^{\varrho-1}(c^2-x^2)^{-\frac{1}{2}\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx =$$

$$= \frac{2^{\sigma-1}c^{\delta-\sigma}\Gamma\left(\frac{1}{2}\,\delta\right)\Gamma\left(\frac{1}{2}\,\delta+\frac{1}{2}\right)}{a^{\lambda+\mu}\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}\,(\delta-\nu-\sigma+1)\right\}\Gamma\left\{\frac{1}{2}\,(\delta+\nu-\sigma+2)\right\}}\,\cdot$$

$$\cdot\, {}_4F_5\left[\begin{matrix}\frac{1}{2}\,\delta\,, & \frac{1}{2}\,\delta+\frac{1}{2}\,, & \frac{1}{2}\,(\lambda+\mu+1)\,, & \frac{1}{2}\,(\lambda+\mu+2)\,; \\ \frac{1}{2}\,(\delta-\nu-\sigma+1), & \frac{1}{2}(\delta+\nu-\sigma+2), & \lambda+1,\ \mu+1, & \lambda+\mu+1;\end{matrix}\; -\frac{c^2}{a^2}\right],$$

where Re $(\delta) > 0$, Re $(\sigma) < 1$, and in a similar way (1.2) gives:

$$\text{(2.2)}\qquad \int_0^c x^{\varrho-1}J_\lambda\left(\frac{x}{a}\right)J_\mu\left(\frac{x}{a}\right)P_\nu\left(\frac{x}{c}\right)dx =$$

$$= \frac{c^{\delta}\Gamma\left(\frac{1}{2}\,\delta\right)\Gamma\left(\frac{1}{2}\,\delta+\frac{1}{2}\right)}{2a^{\lambda+\mu}\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}\,(\delta-\nu+1)\right\}\Gamma\left\{\frac{1}{2}\,(\delta+\nu+2)\right\}}\,\cdot$$

$$\cdot\, {}_4F_5\left[\begin{matrix}\frac{1}{2}\,\delta\,, & \frac{1}{2}\,\delta+\frac{1}{2}\,, & \frac{1}{2}\,(\lambda+\mu+1)\,, & \frac{1}{2}\,(\lambda+\mu+2)\,; \\ \frac{1}{2}\,(\delta-\nu+1)\,, & \frac{1}{2}\,(\delta+\nu+2)\,, & \lambda+1\,,\ \mu+1\,, & \lambda+\mu+1\,;\end{matrix}\; -\frac{c^2}{a^2}\right]$$

valid when Re $(\delta) > 0$.

If in (2.2) we let $a = 1$, $\varrho = \lambda$ and assume ν to be integral ($= n$, say), we get Bose's formula [3, p. 337]. In a similar way when $\nu = 0$, (2.1) leads to Bailey's integral [3, p. 338].

3. Following the method illustrated in § 1 we also have:

$$(3.1)\qquad \int_0^a x^{\varrho-1}(a^2-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu\left(\frac{x}{a}\right) J_\lambda\left(\frac{2x}{b}\right) dx =$$

$$= \frac{2^{\mu-1}a^{\varrho+\lambda-\mu}\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\lambda\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\lambda+\frac{1}{2}\right)}{b^\lambda\Gamma(\lambda+1)\Gamma\left\{\frac{1}{2}(\varrho+\lambda-\mu-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho+\lambda-\mu+\nu+2)\right\}}\cdot$$

$$\cdot\, {}_2F_3\left[\begin{matrix}\frac{1}{2}\varrho+\frac{1}{2}\lambda, & \frac{1}{2}\varrho+\frac{1}{2}\lambda+\frac{1}{2} & & ; \\ \frac{1}{2}(\varrho+\lambda-\mu-\nu+1), & \frac{1}{2}(\varrho+\lambda-\mu+\nu+2), & \lambda+1 & ;\end{matrix}\; -\frac{a^2}{b^2}\right],$$

valid if Re $(\varrho+\lambda)>0$ and Re $(\mu)<1$, and the corresponding formula for $\mu=0$ is:

$$(3.2)\qquad \int_0^a x^{\varrho-1} P_\nu\left(\frac{x}{a}\right) J_\lambda\left(\frac{2x}{b}\right) dx =$$

$$= \frac{a^{\varrho+\lambda}\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\lambda\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\lambda+\frac{1}{2}\right)}{2b^\lambda\Gamma(\lambda+1)\Gamma\left\{\frac{1}{2}(\varrho+\lambda-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho+\lambda+\nu+2)\right\}}\cdot$$

$$\cdot\, {}_2F_3\left[\begin{matrix}\frac{1}{2}\varrho+\frac{1}{2}\lambda, & \frac{1}{2}\varrho+\frac{1}{2}\lambda+\frac{1}{2} & & ; \\ \frac{1}{2}(\varrho+\lambda-\nu+1), & \frac{1}{2}(\varrho+\lambda+\nu+2), & \lambda+1 & ;\end{matrix}\; -\frac{a^2}{b^2}\right],$$

provided that Re $(\varrho+\lambda)>0$.

When $b=2$ and ν is an integer the last formula reduces to an integral evaluated by Bose [3, p. 337], and if in (3.1) we set $\varrho=\frac{3}{2}-\mu$, $\lambda=\nu+\frac{1}{2}$, the ${}_2F_3$ can be expressed as a product of two Bessel functions [4, p. 147]; for $b=2$ we then have the known formula [3, p. 337], namely:

$$\int_0^a x^{\frac{1}{2}-\mu}(a^2 - x^2)^{-\frac{1}{2}\mu} P_\nu^\mu\left(\frac{x}{a}\right) J_{\nu+\frac{1}{2}}(x)dx = \tag{3.3}$$

$$= \left(\frac{1}{2}\pi\right)^{\frac{1}{2}} a^{1-\mu} J_{\frac{1}{2}-\mu}\left(\frac{1}{2}a\right) J_{\nu+\frac{1}{2}}\left(\frac{1}{2}a\right) ,$$

$$\operatorname{Re}(\mu - \nu) < 2 , \quad \operatorname{Re}(\mu) < 1 .$$

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