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## SOME INTEGRALS INVOLVING PRODUCTS OF BESSEL AND LEGENDRE FUNCTIONS

*Nota \**) di H. M. SRIVASTAVA (a Jodhpur, INDIA)

*Sunto.* - The integral:

$$\int_0^c x^{\rho-1} (c^2 - x^2)^{-\frac{1}{2}\sigma} J_\lambda \left( \frac{x}{a} \right) J_\mu \left( \frac{x}{b} \right) P_\nu^\sigma \left( \frac{x}{c} \right) dx,$$

where  $\operatorname{Re}(\rho) > 0$ ,  $\operatorname{Re}(\sigma) < 1$ , is evaluated in terms of hypergeometric series and a number of particular cases thereof are discussed.

**1.** - Let  $c$  be real, non-zero and finite,  $\operatorname{Re}(\rho + \lambda + \mu) > 0$  and  $\operatorname{Re}(\sigma) < 1$  so that by making use of the formula [3, p. 314]:

$$\begin{aligned} & \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = \\ &= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\lambda - \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 + \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)}, \\ & \operatorname{Re}(\lambda) > 0, \quad \operatorname{Re}(\mu) < 1; \end{aligned}$$

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we have:

$$\begin{aligned}
 & \int_0^c x^{\varrho-1} (c^2 - x^2)^{-\frac{1}{2}\sigma} J_\lambda \left( \frac{x}{a} \right) J_\mu \left( \frac{x}{b} \right) P_\nu^\sigma \left( \frac{x}{c} \right) dx = \\
 & = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^{r+s} \left( \frac{1}{2a} \right)^{\lambda+2r} \left( \frac{1}{2b} \right)^{\mu+2s}}{r! s! \Gamma(\lambda + r + 1) \Gamma(\mu + s + 1)} \cdot \\
 & \cdot \int_0^c x^{\varrho+\lambda+\mu+2r+2s-1} (c^2 - x^2)^{-\frac{1}{2}\sigma} P_\nu^\sigma \left( \frac{x}{c} \right) dx = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \cdot \\
 & \frac{(-)^{r+s} 2^{\sigma-1} c^{\varrho-\sigma} \Gamma \left\{ \frac{1}{2} (\lambda + \mu + \varrho + 2r + 2s) \right\}}{r! s! \Gamma(\lambda+r+1) \Gamma(\mu+s+1) \Gamma \left\{ \frac{1}{2} (\lambda + \mu - \nu + \varrho - \sigma + 2r + 2s + 1) \right\}} \cdot \\
 & \frac{\Gamma \left\{ \frac{1}{2} (\lambda + \mu + \varrho + 2r + 2s + 1) \right\} \left( \frac{c}{2a} \right)^{\lambda+2r} \left( \frac{c}{2b} \right)^{\mu+2s}}{\Gamma \left\{ \frac{1}{2} (\lambda + \mu + \nu + \varrho - \sigma + 2r + 2s + 2) \right\}},
 \end{aligned}$$

and therefore:

$$\begin{aligned}
 (1.1) \quad & \int_0^c x^{\varrho-1} (c^2 - x^2)^{-\frac{1}{2}\sigma} J_\lambda \left( \frac{2x}{a} \right) J_\mu \left( \frac{2x}{b} \right) P_\nu^\sigma \left( \frac{x}{c} \right) dx = \\
 & = \frac{2^{\sigma-1} c^{\delta-\sigma} \Gamma \left( \frac{1}{2} \delta \right) \Gamma \left( \frac{1}{2} \delta + \frac{1}{2} \right)}{a^\lambda b^\mu \Gamma(\lambda + 1) \Gamma(\mu + 1) \Gamma \left\{ \frac{1}{2} (\delta - \nu - \sigma + 1) \right\} \Gamma \left\{ \frac{1}{2} (\delta + \nu - \sigma + 2) \right\}} \cdot F \left[ \begin{array}{c} \frac{1}{2} \delta \quad , \quad \frac{1}{2} \delta + \frac{1}{2} : - ; - ; - \\ \frac{1}{2} (\delta - \nu - \sigma + 1) , \quad \frac{1}{2} (\delta + \nu - \sigma + 2) : \lambda + 1 ; \mu + 1 ; - \end{array} \right. \\
 & \left. - \frac{c^2}{a^2} , - \frac{c^2}{b^2} \right],
 \end{aligned}$$

where  $\delta = \lambda + \mu + \varrho$  and the notation for the double hypergeometric series is due to Burchnell and Chaundy [2, p. 112] in preference, for the sake of brevity, to that introduced by Kampé De Fériet [1, p. 150].

For  $\sigma = 0$ , (1.1) gives:

$$\begin{aligned}
 (1.2) \quad & \int_0^c x^{\varrho-1} J_\lambda\left(\frac{2x}{a}\right) J_\mu\left(\frac{2x}{b}\right) P_\nu\left(\frac{x}{c}\right) dx = \\
 & \frac{e^\delta \Gamma\left(\frac{1}{2} \delta\right) \Gamma\left(\frac{1}{2} \delta + \frac{1}{2}\right)}{2a^\lambda b^\mu \Gamma(\lambda + 1) \Gamma(\mu + 1) \Gamma\left\{\frac{1}{2}(\delta - \nu + 1)\right\} \Gamma\left\{\frac{1}{2}(\delta + \nu + 2)\right\}} \\
 & \cdot F \left[ \begin{matrix} \frac{1}{2} \delta, \frac{1}{2} \delta + \frac{1}{2} & : & - & ; & - & ; \\ \frac{1}{2}(\delta - \nu + 1), \frac{1}{2}(\delta + \nu + 2) & : & \lambda + 1 & ; & \mu + 1 & ; & -\frac{c^2}{a^2}, -\frac{c^2}{b^2} \end{matrix} \right],
 \end{aligned}$$

where, as before,  $\text{Re}(\delta) > 0$ .

2. When  $a = b$  the double series on the right of (1.1) is equal to:

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} \delta\right)_k \left(\frac{1}{2} \delta + \frac{1}{2}\right)_k \left(-\frac{c^2}{a^2}\right)^k}{k! \left(\frac{1}{2} \delta - \frac{1}{2} \nu - \frac{1}{2} \sigma + \frac{1}{2}\right)_k \left(\frac{1}{2} \delta + \frac{1}{2} \nu - \frac{1}{2} \sigma + 1\right)_k (\mu + 1)_k} \\
 & \cdot \sum_{r=0}^k \frac{(-k)_r (-\nu - k)_r}{r! (\lambda + 1)_r},
 \end{aligned}$$

and by employing Vandermonde's theorem to sum the inner series this simplifies as a  ${}_4F_5$ . Therefore, a special case of (1.1) is:

$$(2.1) \quad \int_0^c x^{\varrho-1} (c^2 - x^2)^{-\frac{1}{2}\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) P_\nu\left(\frac{x}{c}\right) dx =$$

$$\begin{aligned}
 & \frac{2^{\sigma-1} e^{\delta-\sigma} \Gamma\left(\frac{1}{2} \delta\right) \Gamma\left(\frac{1}{2} \delta + \frac{1}{2}\right)}{a^{\lambda+\mu} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma\left\{\frac{1}{2}(\delta-\nu-\sigma+1)\right\} \Gamma\left\{\frac{1}{2}(\delta+\nu-\sigma+2)\right\}} \\
 & \cdot {}_4F_5 \left[ \begin{array}{c} \frac{1}{2} \delta, \quad \frac{1}{2} \delta + \frac{1}{2}, \quad \frac{1}{2}(\lambda+\mu+1), \quad \frac{1}{2}(\lambda+\mu+2); \\ \frac{1}{2}(\delta-\nu-\sigma+1), \quad \frac{1}{2}(\delta+\nu-\sigma+2), \quad \lambda+1, \quad \mu+1, \quad \lambda+\mu+1; \end{array} \right. \\
 & \qquad \qquad \qquad \left. - \frac{c^2}{a^2} \right],
 \end{aligned}$$

where  $\text{Re}(\delta) > 0$ ,  $\text{Re}(\sigma) < 1$ , and in a similar way (1.2) gives:

$$\begin{aligned}
 (2.2) \quad & \int_0^c x^{2-1} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) P_\nu\left(\frac{x}{c}\right) dx = \\
 & \frac{c^\delta \Gamma\left(\frac{1}{2} \delta\right) \Gamma\left(\frac{1}{2} \delta + \frac{1}{2}\right)}{2 a^{\lambda+\mu} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma\left\{\frac{1}{2}(\delta-\nu+1)\right\} \Gamma\left\{\frac{1}{2}(\delta+\nu+2)\right\}} \\
 & \cdot {}_4F_5 \left[ \begin{array}{c} \frac{1}{2} \delta, \quad \frac{1}{2} \delta + \frac{1}{2}, \quad \frac{1}{2}(\lambda+\mu+1), \quad \frac{1}{2}(\lambda+\mu+2); \\ \frac{1}{2}(\delta-\nu+1), \quad \frac{1}{2}(\delta+\nu+2), \quad \lambda+1, \quad \mu+1, \quad \lambda+\mu+1; \end{array} \right. \\
 & \qquad \qquad \qquad \left. - \frac{c^2}{a^2} \right]
 \end{aligned}$$

valid when  $\text{Re}(\delta) > 0$ .

If in (2.2) we let  $a = 1$ ,  $\rho = \lambda$  and assume  $\nu$  to be integral ( $= n$ , say), we get Bose's formula [3, p. 337]. In a similar way when  $\nu = 0$ , (2.1) leads to Bailey's integral [3, p. 338].

3. Following the method illustrated in § 1 we also have:

$$\begin{aligned}
 (3.1) \quad & \int_0^a x^{\varrho-1} (a^2 - x^2)^{-\frac{1}{2}\mu} P_\nu^\mu \left( \frac{x}{a} \right) J_\lambda \left( \frac{2x}{b} \right) dx = \\
 & = \frac{2^{\mu-1} a^{\varrho+\lambda-\mu} \Gamma \left( \frac{1}{2} \varrho + \frac{1}{2} \lambda \right) \Gamma \left( \frac{1}{2} \varrho + \frac{1}{2} \lambda + \frac{1}{2} \right)}{b^\lambda \Gamma(\lambda + 1) \Gamma \left\{ \frac{1}{2} (\varrho + \lambda - \mu - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\varrho + \lambda - \mu + \nu + 2) \right\}} \\
 & \cdot {}_2F_3 \left[ \begin{matrix} \frac{1}{2} \varrho + \frac{1}{2} \lambda, & \frac{1}{2} \varrho + \frac{1}{2} \lambda + \frac{1}{2} & ; & -\frac{a^2}{b^2} \\ \frac{1}{2} (\varrho + \lambda - \mu - \nu + 1), & \frac{1}{2} (\varrho + \lambda - \mu + \nu + 2), & \lambda + 1 & ; \end{matrix} \right],
 \end{aligned}$$

valid if  $\operatorname{Re}(\varrho + \lambda) > 0$  and  $\operatorname{Re}(\mu) < 1$ , and the corresponding formula for  $\mu = 0$  is:

$$\begin{aligned}
 (3.2) \quad & \int_0^a x^{\varrho-1} P_\nu \left( \frac{x}{a} \right) J_\lambda \left( \frac{2x}{b} \right) dx = \\
 & = \frac{a^{\varrho+\lambda} \Gamma \left( \frac{1}{2} \varrho + \frac{1}{2} \lambda \right) \Gamma \left( \frac{1}{2} \varrho + \frac{1}{2} \lambda + \frac{1}{2} \right)}{2b^\lambda \Gamma(\lambda + 1) \Gamma \left\{ \frac{1}{2} (\varrho + \lambda - \nu + 1) \right\} \Gamma \left\{ \frac{1}{2} (\varrho + \lambda + \nu + 2) \right\}} \\
 & \cdot {}_2F_3 \left[ \begin{matrix} \frac{1}{2} \varrho + \frac{1}{2} \lambda, & \frac{1}{2} \varrho + \frac{1}{2} \lambda + \frac{1}{2} & ; & -\frac{a^2}{b^2} \\ \frac{1}{2} (\varrho + \lambda - \nu + 1), & \frac{1}{2} (\varrho + \lambda + \nu + 2), & \lambda + 1 & ; \end{matrix} \right],
 \end{aligned}$$

provided that  $\operatorname{Re}(\varrho + \lambda) > 0$ .

When  $b = 2$  and  $\nu$  is an integer the last formula reduces to an integral evaluated by Bose [3, p. 337], and if in (3.1) we set  $\varrho = \frac{3}{2} - \mu$ ,  $\lambda = \nu + \frac{1}{2}$ , the  ${}_2F_3$  can be expressed as a product of two Bessel functions [4, p. 147]; for  $b = 2$  we then have the

known formula [3, p. 337], namely:

$$(3.3) \quad \int_0^a x^{\frac{1}{2}-\mu} (a^2 - x^2)^{-\frac{1}{2}\mu} P_\nu^\mu \left( \frac{x}{a} \right) J_{\nu+\frac{1}{2}}(x) dx =$$

$$= \left( \frac{1}{2} \pi \right)^{\frac{1}{2}} a^{1-\mu} J_{\frac{1}{2}-\mu} \left( \frac{1}{2} a \right) J_{\nu+\frac{1}{2}} \left( \frac{1}{2} a \right),$$

$$\operatorname{Re}(\mu - \nu) < 2, \quad \operatorname{Re}(\mu) < 1.$$

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