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A SUMMATION THEOREM FOR DOUBLE HYPERGEOMETRIC SERIES

by L. CARLITZ *)

1. In contrast with the situation for ordinary hypergeometric series, not many summation formulas are known for double series of hypergeometric type. The writer [2], [3] has proved the following results.

$$\sum_{s=0}^{m}\sum_{r=0}^{n}\frac{(-m)_r(-n)_s(\alpha)_{r+s}(\beta)_r(\beta')_s}{r!\,s!\,(\gamma)_{r+s}(\delta)_r(\delta')_s} \tag{1.}$$

$$=\frac{(\beta+\beta'-\alpha)_{m+n}(\beta')_m(\beta)_n}{(\beta+\beta')_{m+n}(\beta'-\alpha)_m(\beta-\alpha)_n},$$

provided

$$\left\{\begin{aligned}\gamma+\delta&=\alpha+\beta-m+1\\ \gamma+\delta'&=\alpha+\beta'-n+1\\ \gamma&=\beta+\beta';\end{aligned}\right. \tag{1.2}$$

$$\sum_{r+s\leqslant n}\frac{(-n)_{r+s}(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r!\,s!\,(\gamma)_{r+s}(\delta)_r(\delta')_s} \tag{1.3}$$

$$=\frac{(\beta)_n(\beta')_n(\gamma-\alpha-\alpha')_n}{(\gamma)_n(\beta-\alpha')_n(\beta'-\alpha)_n},$$

provided

$$\left\{\begin{aligned}\gamma+\delta&=\alpha+\beta-n+1\\ \gamma+\delta'&=\alpha'+\beta'-n+1\\ \gamma&=\beta+\beta'.\end{aligned}\right. \tag{1.4}$$

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In the present note we consider the sum

$$(1.5)\qquad S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c)_m (c')_n}{m!\, n!\, (b)_m (b)_n (c + c')_{m+n}}$$

which is absolutely convergent for

$$(1.6)\qquad \begin{cases} R(c + c' - b) \geqslant 0 \\ R(b - a - c) > 0 \\ R(b - a' - c') > 0 . \end{cases}$$

We shall show that

$$(1.7)\qquad S = \frac{\Gamma(c - a')\Gamma(c' - a)\Gamma(b)\Gamma(b - a - a')\Gamma(c + c')}{\Gamma(c + c' - a - a')\Gamma(b - a)\Gamma(b - a')\Gamma(c)\Gamma(c')} .$$

This result was orginally obtained by using a reduction formula [1, p. 81] for the Appell function F_2. However we shall give a direct proof below.

2. By Vandermonde's theorem

$$\frac{(b)_{m+n}}{(b)_m (b)_n} = \sum_{k=0}^{\min(m,n)} \frac{(-m)_k (-n)_k}{k!\, (b)_k} .$$

Thus (1.5) becomes

$$S = \sum^{\infty} \sum^{\infty} \frac{(a)_m (a')_n (c)_m (c')_n}{m!\, n!\, (c + c')_{m+n}} \sum_{k=0}^{\min(m,n)} \frac{(-m)_k (-n)_k}{k!\, (b)_k}$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k (a')_k (c)_k (c')_k}{k!\, (b)_k (c + c')_{2k}} \sum_{m=0}^{\infty} \frac{(a + k)_m (c + k)_m}{m!\, (c + c' + 2k)_m} \cdot \sum_{n=0}^{\infty} \frac{(a' + k)_n (c' + k)_n}{n!\, (c + c' + 2k + m)_m} .$$

By Gauss's theorem the sum on the extreme right is equal to

$$\frac{\Gamma(c + c' + 2k + m)\Gamma(c - a' + m)}{\Gamma(c + c' - a' + k + m)\Gamma(c - k - m)} =$$

$$= \frac{\Gamma(c + c')\Gamma(c + a')}{\Gamma(c + c' - a')\Gamma(c)} \frac{(c + c')_{2k+m} (c - a')_m}{(c + c' - a')_{k+m} (c)_{k+m}} .$$

It follows that

$$S = \frac{\Gamma(c+c')\Gamma(c-a')}{\Gamma(c+c'-a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c)_k(c')_k}{k!\,(b)_k(c+c')_{2k}}$$

$$\cdot \sum_{m=0}^{\infty} \frac{(a+k)_m(c+k)_m}{m!\,(c+c'+2k)_m} \frac{(c+c')_{2k+m}(c-a')_m}{(c+c'-a')_{k+m}(c)_{k+m}}$$

$$= \frac{\Gamma(c+c')\Gamma(c-a')}{\Gamma(c+c'-a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c')_k}{k!\,(b)_k(c+c'-a')_k} \cdot \sum_{m=0}^{\infty} \frac{(a+k)_m c-a')_m}{m!\,(c+c'-a'+k)_m}.$$

Applying Gauss's theorem to the inner sum, we get

$$S = \frac{\Gamma(c+c')\Gamma(c-a')}{\Gamma(c+c'-a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c')_k}{k!\,(b)_k(c+c'-a')_k} \frac{\Gamma(c+c'-a'+k)\Gamma(c'-a)}{\Gamma(c+c'-a-a')\Gamma(c'+k)}$$

$$= \frac{\Gamma(c+c')\Gamma(c-a')\Gamma(c'-a)}{\Gamma(c)\Gamma(c')\Gamma(c+c'-a-a')} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k}{k!\,(b)_k}$$

$$= \frac{\Gamma(c+c')\Gamma(c-a')\Gamma(c'-a)}{\Gamma(c)\Gamma(c')\Gamma(c+c'-a-a_{\prime})} \frac{\Gamma(b)\Gamma(b-a-a')}{\Gamma(b-a)\Gamma(b-a')}$$

by another application of Gauss's theorem. This evidently completes the proof of (1.7).

3. If we take $a = -m$, $a' = -n$, where m, n are nonnegative integers, (1.7) reduces to

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-m)_r(-n)_s(b)_{r+s}(c)_r(c')_s}{r!\,s!\,(b)_r(b)_s(c+c')_{r+s}} = \frac{(b)_{m+n}(c')_m(c)_n}{(b)_m(b)_n(c+c')_{m+n}}. \tag{3.1}$$

If we put

$$u_{m,n}(b;\,c,\,c') = \frac{(b)_{m+n}(c)_m(c')_n}{(b)_m(b)_n(c+c')_{m+n}},$$

we may write (3.1) in the form

$$\sum_{s=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} \binom{m}{r} \binom{n}{s} u_{r,s}(b;\,c,\,c') = u_{m,n}(b;\,c',\,c). \tag{3.2}$$

It also follows from (3.1) that

$$\sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_{m+n}(c')_m(c)_n}{m!\,n!\,(b)_m(b)_n(c+c')_{m+n}}\, x^m y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n}{m!\,n!}\, x^m y^n \sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(-m)_r(-n)_s(b)_{r+s}(c)_r(c')_s}{r!\,s!\,(b)_r(b)_s(c+c')_{r+s}}$$

$$= \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(a)_r(a')_s(b)_{r+s}(c)_r(c')_s}{r!\,s!\,(b)_r(b)_s(c+c')_{r+s}}\, x^r y^s \sum_{m=0}^{\infty} \frac{(a+r)_m}{m!}\, x^m \sum_{n=0}^{\infty} \frac{(a'+s)_n}{n!}\, y^n$$

$$= \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(a)_r(a')_s(b)_{r+s}(c)_r(c')_s}{r!\,s!\,(b)_r(b)_s(c+c')_{r+s}}\, \frac{x^r}{(1-x)^{a+r}}\, \frac{y^s}{1-y^{a'+s}}\,.$$

Hence if we put

$$F(a,\ a';\ b;\ c,\ c';\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_{m+n}(c)_m(c')_n}{m!\,n!\,(b)_m(b)_n(c+c')_{m+n}}\, x^m y^n$$

it is clear that

(3.3) $$F(a,\ a';\ b;\ c',\ c;\ x,\ y) =$$
$$= (1-x)^{-a}(1-y)^{-a'} F\left(a,\ a';\ b;\ c,\ c';\ \frac{x}{1-x},\ \frac{y}{1-y}\right).$$

REFERENCES

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