

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

L. CARLITZ

**A summation theorem for double hypergeometric series**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 37 (1967), p. 230-233

[http://www.numdam.org/item?id=RSMUP\\_1967\\_\\_37\\_\\_230\\_0](http://www.numdam.org/item?id=RSMUP_1967__37__230_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1967, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## A SUMMATION THEOREM FOR DOUBLE HYPERGEOMETRIC SERIES

by L. CARLITZ \*)

**1.** In contrast with the situation for ordinary hypergeometric series, not many summation formulas are known for double series of hypergeometric type. The writer [2], [3] has proved the following results.

$$(1.) \quad \sum_{s=0}^m \sum_{r=0}^n \frac{(-m)_r (-n)_s (\alpha)_{r+s} (\beta)_r (\beta')_s}{r! s! (\gamma)_{r+s} (\delta)_r (\delta')_s} \\ = \frac{(\beta + \beta' - \alpha)_{m+n} (\beta')_m (\beta)_n}{(\beta + \beta')_{m+n} (\beta' - \alpha)_m (\beta - \alpha)_n},$$

provided

$$(1.2) \quad \begin{cases} \gamma + \delta = \alpha + \beta - m + 1 \\ \gamma + \delta' = \alpha + \beta' - n + 1 \\ \gamma = \beta + \beta'; \end{cases}$$

$$(1.3) \quad \sum_{r+s \leq n} \frac{(-n)_{r+s} (\alpha)_r (\alpha')_s (\beta)_r (\beta')_s}{r! s! (\gamma)_{r+s} (\delta)_r (\delta')_s} \\ = \frac{(\beta)_n (\beta')_n (\gamma - \alpha - \alpha')_n}{(\gamma)_n (\beta - \alpha')_n (\beta' - \alpha)_n},$$

provided

$$(1.4) \quad \begin{cases} \gamma + \delta = \alpha + \beta - n + 1 \\ \gamma + \delta' = \alpha' + \beta' - n + 1 \\ \gamma = \beta + \beta'. \end{cases}$$

---

\*) Supported in part by NSF grant GP-5174.

Indirizzo dell'A.: Department of Mathematics, Duke University, Durham, North Carolina, USA.

In the present note we consider the sum

$$(1.5) \quad S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c)_m (c')_n}{m! n! (b)_m (b)_n (c + c')_{m+n}}$$

which is absolutely convergent for

$$(1.6) \quad \begin{cases} R(c + c' - b) \geq 0 \\ R(b - a - c) > 0 \\ R(b - a' - c') > 0. \end{cases}$$

We shall show that

$$(1.7) \quad S = \frac{\Gamma(c - a') \Gamma(c' - a) \Gamma(b) \Gamma(b - a - a') \Gamma(c + c')}{\Gamma(c + c' - a - a') \Gamma(b - a) \Gamma(b - a') \Gamma(c) \Gamma(c')}.$$

This result was originally obtained by using a reduction formula [1, p. 81] for the Appell function  $F_2$ . However we shall give a direct proof below.

## 2. By Vandermonde's theorem

$$\frac{(b)_{m+n}}{(b)_m (b)_n} = \sum_{k=0}^{\min(m,n)} \frac{(-m)_k (-n)_k}{k! (b)_k}.$$

Thus (1.5) becomes

$$\begin{aligned} S &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (c)_m (c')_n}{m! n! (c + c')_{m+n}} \sum_{k=0}^{\min(m,n)} \frac{(-m)_k (-n)_k}{k! (b)_k} \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (a')_k (c)_k (c')_k}{k! (b)_k (c + c')_{2k}} \sum_{m=0}^{\infty} \frac{(a + k)_m (c + k)_m}{m! (c + c' + 2k)_m} \sum_{n=0}^{\infty} \frac{(a' + k)_n (c' + k)_n}{n! (c + c' + 2k + m)_m}. \end{aligned}$$

By Gauss's theorem the sum on the extreme right is equal to

$$\begin{aligned} &\frac{\Gamma(c + c' + 2k + m) \Gamma(c - a' + m)}{\Gamma(c + c' - a' + k + m) \Gamma(c - k - m)} = \\ &= \frac{\Gamma(c + c') \Gamma(c + a')}{\Gamma(c + c' - a') \Gamma(c)} \frac{(c + c')_{2k+m} (c - a')_m}{(c + c' - a')_{k+m} (c)_{k+m}}. \end{aligned}$$

It follows that

$$\begin{aligned} S &= \frac{\Gamma(c+c')\Gamma(c-a')}{\Gamma(c+c'-a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c)_k(c')_k}{k!(b)_k(c+c')_{2k}} \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{(a+k)_m(c+k)_m}{m!(c+c'+2k)_m} \frac{(c+c')_{2k+m}(c-a')_m}{(c+c'-a')_{k+m}(c)_{k+m}} \\ &= \frac{\Gamma(c+c')\Gamma(c-a')}{\Gamma(c+c'-a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c')_k}{k!(b)_k(c+c'-a')_k} \cdot \sum_{m=0}^{\infty} \frac{(a+k)_m(c-a')_m}{m!(c+c'-a'+k)_m}. \end{aligned}$$

Applying Gauss's theorem to the inner sum, we get

$$\begin{aligned} S &= \frac{\Gamma(c+c')\Gamma(c-a')}{\Gamma(c+c'-a')\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k(c')_k}{k!(b)_k(c+c'-a')_k} \frac{\Gamma(c+c'-a'+k)\Gamma(c'-a)}{\Gamma(c+c'-a-a')\Gamma(c'+k)} \\ &= \frac{\Gamma(c+c')\Gamma(c-a')\Gamma(c'-a)}{\Gamma(c)\Gamma(c')\Gamma(c+c'-a-a')} \sum_{k=0}^{\infty} \frac{(a)_k(a')_k}{k!(b)_k} \\ &= \frac{\Gamma(c+c')\Gamma(c-a')\Gamma(c'-a)}{\Gamma(c)\Gamma(c')\Gamma(c+c'-a-a')} \frac{\Gamma(b)\Gamma(b-a-a')}{\Gamma(b-a)\Gamma(b-a')} \end{aligned}$$

by another application of Gauss's theorem. This evidently completes the proof of (1.7).

**3.** If we take  $a = -m$ ,  $a' = -n$ , where  $m, n$  are nonnegative integers, (1.7) reduces to

$$(3.1) \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-m)_r(-n)_s(b)_{r+s}(c)_r(c')_s}{r!s!(b)_r(b)_s(c+c')_{r+s}} = \frac{(b)_{m+n}(c)_m(c)_n}{(b)_m(b)_n(c+c')_{m+n}}.$$

If we put

$$u_{m,n}(b; c, c') = \frac{(b)_{m+n}(c)_m(c')_n}{(b)_m(b)_n(c+c')_{m+n}},$$

we may write (3.1) in the form

$$(3.2) \quad \sum_{s=0}^m \sum_{r=0}^n (-1)^{r+s} \binom{m}{r} \binom{n}{s} u_{r,s}(b; c, c') = u_{m,n}(b; c', c).$$

It also follows from (3.1) that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c')_m (c)_n}{m! n! (b)_m (b)_n (c+c')_{m+n}} x^m y^n \\ &= \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n}{m! n!} x^m y^n \sum_{r=0}^m \sum_{s=0}^n \frac{(-m)_r (-n)_s (b)_{r+s} (c)_r (c')_s}{r! s! (b)_r (b)_s (c+c')_{r+s}} \\ &= \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(a)_r (a')_s (b)_{r+s} (c)_r (c')_s}{r! s! (b)_r (b)_s (c+c')_{r+s}} x^r y^s \sum_{m=0}^{\infty} \frac{(a+r)_m}{m!} x^m \sum_{n=0}^{\infty} \frac{(a'+s)_n}{n!} y^n \\ &= \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(a)_r (a')_s (b)_{r+s} (c)_r (c')_s}{r! s! (b)_r (b)_s (c+c')_{r+s}} \frac{x^r}{(1-x)^{a+r}} \frac{y^s}{1-y^{a'+s}}. \end{aligned}$$

Hence if we put

$$F(a, a'; b; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_{m+n} (c)_m (c')_n}{m! n! (b)_m (b)_n (c+c')_{m+n}} x^m y^n$$

it is clear that

$$\begin{aligned} (3.3) \quad F(a, a'; b; c, c'; x, y) &= \\ &= (1-x)^{-a} (1-y)^{-a'} F\left(a, a'; b; c, c'; \frac{x}{1-x}, \frac{y}{1-y}\right). \end{aligned}$$

### REFERENCES

- [1] BAILEY W. N.: *Generalized hypergeometric series*, Cambridge, 1935.
- [2] CARLITZ L.: *A Saalschützian theorem for double series*, Journal of the London Mathematical Society, vol. 38 (1963), pp. 415-418.
- [3] CARLITZ L.: *Another Saalschützian theorem for double series*, Rendiconti del Seminario Matematico della Università di Padova, vol. 34 (1964), pp. 220-203.

Manoscritto pervenuto in redazione il 14 maggio 1966.