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STRUCTURE THEORY FOR GEOMETRIC LATTICES

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1. Introduction.

A geometric lattice (Birkhoff [1], and in Jónsson [5], a matroid lattice) is a lattice which is complete, atomistic, continuous, and semimodular.

A sublattice of a geometric lattice need not be geometric. Consequently, any, categorical analysis of geometric lattices considered as algebras with two operators will most likely be inconclusive.

It is possible, however, to define a geometric lattice as a set L, together with an operator sup (*supremum or join*), defined on arbitrary subsets of L and taking values in L, and with a binary relation $\downarrow$ (*covers, or is equal to*). In writing the axioms, it is convenient to write $0 = \sup \Phi$, $x \vee y = \sup \{x, y\}$, and $x \leq y$ if and only if $x \vee y = y$. Two of the axioms may be taken to be

α) $y \downarrow x$ if and only if $x \leq y$ and $x < z \leq y$ implies $z = y$.

β) $\forall x < y \quad \exists p \downarrow 0 \cdot \ni \cdot x < x \vee p \leq y$.

Axiom α expresses the connection between sup and $\downarrow$, while axiom β indicates that the atoms separate the lattice elements.

In this way, geometric lattices form an axiomatic model class [2] of relational structures. A substructure P of a geometric lattice Q

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is also a geometric lattice if axioms α and β apply in P. A relation-homomorphic image Q of a geometric lattice P is also a geometric lattice if axiom α holds in Q.

In section 2, we show how mappings preserving join and cover correspond to the strong maps of geometries introduced by Higgs [4]. Images of geometric lattices are studied in section 3. The kernel of such a map is shown to be a closure with the exchange property, acting on a geometric lattice. We are indebted to Professor G.-C. Rota for his recommendation of this approach.

2. Strong maps.

We write $y \downarrow x$ if an element y covers or is equal to an element x in a lattice. A function f from a lattice P to a lattice Q is *cover-preserving* if and only if $y \downarrow x$ implies $f(y) \downarrow f(x)$, for all elements x, y in P.

PROPOSITION 1. Any lattice-epimorphism is cover-preserving.

Proof. Assume $y \downarrow x$ in P. For any element c such that $f(x) \leq \leq c \leq f(y)$, choose a preimage $z \in P$. Then $c = (f(z) \wedge f(y)) \vee f(x) = = f((z \wedge y) \vee x)$. Since $x \leq (z \wedge y) \vee x \leq y$. and $y \downarrow x$, $(z \wedge y) \vee x$ is equal either to x or to y. Thus c is equal either to $f(x)$ or to $f(y)$, and $f(y) \downarrow f(x)$. ∎

A *join-homomorphism* from a geometric lattice P into a geometric lattice Q is any function $f\colon P \rightarrow Q$ such that $f(\sup X) = = \sup f(X)$ for every subset $X \subseteq P$. Such a join-homomorphism f determines and is determined by the restriction $f \mid A$ of f to the set A of atoms of P.

PROPOSITION 2. A join-homomorphism f from a geometric lattice P into a geometric lattice Q is cover-preserving if and only if the image of each atom of P is either 0 or else an atom of Q.

Proof: If f is cover-preserving, and $p \downarrow 0$ in P, then $f(p) \downarrow f(0)$ in Q. But $f(0) = f(\sup \Phi) = \sup f(\Phi) = \sup \Phi = 0$. Conversely, if $p \downarrow 0$

in P implies $f(p) \downarrow 0$ in Q, and if $y \downarrow x$ in P, choose an atom $p \in P$ such that $x \vee p = y$. Then $f(p) \downarrow 0$ in Q, and $f(y) = f(x \vee p) = f(x) \vee \vee f(p)$, so $f(y) \downarrow f(x)$ in Q, by semimodularity. ▮

A join-homomorphism from a lattice P to a lattice Q is *non-singular* if and only if $f(x) = 0$ implies $x = 0$. A *strong map* from a geometric lattice P into a geometric lattice Q is any non-singular cover-preserving join-homomorphism from P to Q. Higgs [4] coined the term « strong map » for those functions f from a geometry G_1 to a geometry G_2 such that $f(\overline{X}) \subseteq \overline{f(X)}$, for each subset $X \subseteq G_1$. In other words, f is a « strong map » if and only if the inverse image of each G_2-closed set is G_1-closed. Proposition 3, below, justifies our usage of the term.

If A is the set of atoms of a geometric lattice P, then $\overline{X} = \{p \in A\,;\ p \leq \sup X\}$ defines a closure operator with the exchange property (MacLane [6]), and a geometry, in the sense of Higgs [3]. This geometry we shall denote $G(P)$.

Proposition 3. If f is a non-singular strong map from a geometric lattice P, with atom set A, into a geometric lattice Q, then $f \mid A$ is a strong map from the geometry $G(P)$ into the geometry $G(Q)$. Conversely, if g is a strong map from $G(P)$ into $G(Q)$, then $g = f \mid A$ for a unique non-singular strong map $f\colon\ P \longrightarrow Q$.

Proof: A strong map f carries atoms of P into atoms of Q, so $f \mid A$ is a function from $G(P)$ into $G(Q)$. If $p \in \overline{X}$, for a subset $X \subseteq G(P)$, then $p \leq \sup X$, and $f(p) \leq f(\sup X) = \sup\ f(X)$. Thus $f(p) \in \overline{f(X)}$, and $f(\overline{X}) \subseteq \overline{f(X)}$.

Conversely, if g is a strong map from $G(P)$ into $G(Q)$, and if f is a join-preserving function from P into Q which extends g, f must satisfy, for each $x \in P$, $f(x) = f(\sup\{p\,;\, p$ an atom, $p \leq x\})$, because P is atomistic, and must satisfy

$$f(x) = \sup\{g(p)\,;\, p \text{ an atom, } p \leq x\} \tag{1}$$

because f preserves join. The assumption that g is a strong map is required for the following proof that f, defined by equation (1), is join-preserving. If Y is a subset of the lattice P,

$$\begin{aligned} f(\sup Y) &= \sup \{g(p);\ p \text{ an atom},\ p \leq \sup Y\} \\ &\geq \sup \{g(p);\ p \text{ an atom},\ p \leq y \text{ for some } y \in Y\} \\ &= \sup_{y \in Y} \sup \{g(p);\ p \text{ an atom},\ p \leq y\} \\ &= \sup f(Y). \end{aligned}$$

If p is an atom beneath sup Y, p is in the closure of the set of atoms $\{q;\ q \leq y$ for some $y \in Y\}$, $g(p)$ is in the closure of the set of atoms $\{g(q);\ q \leq y$ for some $y \in Y\}$ because g is a strong map, and $g(p) \leq \sup f(Y)$. Thus $f(\sup Y) = \sup f(Y)$. ∎

The function f, defined by equation (1), is non-singular, and is cover-preserving, because P is atomistic, f is join-preserving, and Q is semimodular.

Non-singularity, join-preservation, and cover-preservation are properties preserved under composition of functions. On each geometric lattice P, the identity function is a strong map. Therefore geometric lattices and strong maps are the objects aud morphisms respectively, of an abstract category, which we shall denote G.

PROPOSITION 4. A substructure P of a geometric lattice Q is a geometric lattice if axioms α and β hold in P.

Proof: If P is such a substructure of Q, then P is a subset of Q, closed with respect to arbitrary join. Thus P is a complete lattice. $y \downarrow x$ in P if and only if $y \downarrow x$ in Q, and the axiom for definition of $\downarrow$ in terms of sup is satisfied. $0 = \sup \Phi$ is an element of P, so the atoms of P are precisely those atoms of Q contained in the subset P. Since axiom β holds in P, P is atomistic. Since the atoms of P are a subset of the atoms of Q, P, being atomistic, is continuous. If $p \downarrow 0$ in P, and $x \in P$, then $x \vee p$, the supremum in Q, is an element of P, and $x \vee p \downarrow x$ in Q implies $x \vee p \downarrow x$ in P. Thus P is semimodular. It is clear that the injection map preserves join and cover, and is non-singular. ∎

COROLLARY TO PROPOSITION 4. A subset P of a geometric lattice Q is a substructure of Q if and only if P is the set of arbitrary joins of subsets of some subset of the atoms of Q. ∎

In the following section, we investigate the dual notion: images of geometric lattices.

3. Images.

Given any strong map $f\colon P \longrightarrow Q$ in the category G of geometric lattices, the image of P in Q is also geometric. The operator J on P defined by $J(x) = \sup \{y\,;\, f(y) = f(x)\}$ is a finitistic closure operator with the exchange property. If R is the natural map from J-closed elements of P into the lattice P/J of all J-closed elements of P, then $RJ\colon P \longrightarrow P/J$ is a strong map, and $P/J \simeq \operatorname{Im} f$. Thus any strong map $f\colon P \longrightarrow Q$ may be factored $f = f_3 f_2 f_1$, where f_1 is the strong map from P onto P/f, f_2 is an isomorphism of P/f with $\operatorname{Im} f$, and f_3 is a one-one strong map from $\operatorname{Im} f$ into Q. These facts are proven below.

PROPOSITION 5. *If f is a strong map from a geometric lattice P into a complete, continuous lattice Q, then $\operatorname{Im} f$, the image of P in Q, is also geometric.*

Proof: Since f is a join-homomorphism, $\operatorname{Im} f$ is closed with respect to join, and is a complete lattice, with order induced by that on Q. If $y \in \operatorname{Im} f$, choose a preimage x of y, and express x as a join of atoms in P. Then y is the join of the images of those atoms, and $\operatorname{Im} f$ is atomistic. If an atom p is beneath the supremum of a set X of atoms of $\operatorname{Im} f$, consider p and the atoms in X as elements of Q. Since Q is continuous, there is a finite subset $X' \subseteq X$ such that $p \leq \sup X'$. But $p \leq \sup X'$ in $\operatorname{Im} f$, so $\operatorname{Im} f$, being atomistic, is continuous [5]. If q is an atom of $\operatorname{Im} f$, and $q \not\lesseqgtr y$ for an element $y \in \operatorname{Im} f$, choose preimages x of y and z of q. For each atom $p \in P$ such that $p \leq z$, either $f(p) = 0$ or $f(p) = q$. Since $q = \sup \{f(p)\,;\, p \leq z\}$, $f(p) = q$ for some atom $p \in P$. Then $p \downarrow 0$, $p \vee x \downarrow x$, $q \vee y = f(p) \vee f(x) = f(p \vee x) \downarrow f(x) = y$, and $\operatorname{Im} f$ is semimodular [5]. ▮

A *join-congruence* on a complete lattice L is an equivalence relation $\backsim$ on L such that for any subset $X \subseteq L$ and any function $h\colon L \longrightarrow L$ dominated by $\backsim$ (ie: $x \backsim h(x)$ for all $x \in L$), $\sup X \backsim \sup h(X)$.

LEMMA TO PROPOSITION 6. If $\backsim$ is a join-congruence on a complete lattice L, then the operator J defined by $J(x) = \sup \{y ; y \backsim x\}$ is a closure operator.

Proof: Since $x \backsim x$, $x \leq J(x)$. If $x \leq y$, and $z \backsim x$, then $y \vee z \backsim y \vee x = y$, and $z \leq y \vee z \leq J(y)$. Thus $J(x) \leq J(z)$. $J(x) = \sup \{y ; y \backsim x\} \backsim \sup \{x\} = x$, so $z \backsim J(x)$ if and only if $z \backsim x$, and $JJ(x) = J(x)$. ▮

A closure operator acting on a set, ie: on the Boolean algebra of all subsets of that set, has a lattice of closed subsets which is geometric if the closure is finitary and has the exchange property [3]. These two properties may be rephrased for closure operators on more general lattices. A closure J on a complete lattice L is *finitary* if and only if, for each directed subset $X \subseteq L$, $J(\sup X) = \sup J(X)$.

PROPOSITION 6. [2] A closure J on a complete atomistic continuous lattice L is finitary if and only if, for every atom $p \in L$ and every element $x \in L$, such that $p \leq J(x)$, there exists a finite set Y of atoms beneath x, such that $p \leq J(\sup Y)$.

Proof: Assume J is finitary, p an atom, and $p \leq J(x)$ for some $x \in L$. Let X be the set of joins of finite sets of atoms beneath x. X is directed, so $J(x) = J(\sup X) = \sup J(X)$. $J(X)$ is also directed, in a continuous lattice, so $p \leq \sup J(X)$ implies $p \leq J(y)$ for some $y \in X$. Conversely, assume X is a directed subset of L. Then $J(\sup X) \geq \sup J(X)$. Let p be any atom beneath $J(\sup X)$. Select a finite set Y of atoms beneath $\sup X$, such that $p \leq J(\sup Y)$. For each atom $q \in Y$, select an element x_q above q in the directed set X. Then choose an element $x \in X$ such that $x_q \leq x$ for all $q \in Y$. $q \leq J(x) \leq \sup J(X)$, so $J(\sup X) = \sup J(X)$. ▮

A closure J, on a complete atomistic lattice L, has the *exchange property* if and only if, for any atoms $p, q \in L$ and any element $x \in L$, $p \not\leq J(x)$ and $p \leq J(x \vee q)$ imply $q \leq J(x \vee p)$.

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[2]) cf. Cohn [2], Theorem II.1.2: A closure system on a set is algebraic if and only if it is inductive.

PROPOSITION 7. If f is a strong map from a geometric lattice P into a geometric lattice Q, the operator J defined on P by

$$J(x) = \sup \{y ; f(y) = f(x)\}$$

is a finitary closure operator with the exchange property.

Proof: Consider the equivalence relation $\backsim$ defined by $x \backsim y$ if and only if $f(x) = f(y)$. If $h : P \rightarrow P$ is any function dominated by $\backsim$, and if X is any subset of P, $f(\sup h(X)) = \sup f(h(X)) = \sup f(X) = f(\sup X)$, so $\sup X \backsim \sup h(X)$, and $\backsim$ is a join-congruence. By the above lemma, J is a closure eperator on P. If p is an atom of P, and $p \leq J(x)$ for some $x \in P$, $f(p) \downarrow 0$ in the finitistic lattice Q, and $f(x) = \sup f(X)$, where $X = \{q ; q \downarrow 0, q \leq x$ in $P\}$. We may select a finite subset $X' \subseteq X$ such that $f(p) \leq \leq \sup f(X')$. Then $p \leq J(\sup X')$ because $p \leq p \vee \sup X'$, and $f(p \vee \vee \sup X',) = f(p) \vee \sup f(X') = \sup f(X') = f(\sup X')$. Thus J is finitistic. Note that $x \leq J(y)$ if and only if $f(x) \leq f(y)$. If p and q are atoms of P, and x is an element of P such that $p \not\leq J(x)$, $p \leq \leq J(x \vee q)$, then $f(x) \vee f(p)$ covers $f(x)$. Since $f(p) \leq f(x \vee q) = = f(x) \vee f(q)$, and $f(x) \vee f(q) \downarrow f(x)$, we have $f(x \vee p) = f(x \vee q)$, $f(q) \leq f(x \wedge p)$, and $q \leq J(x \vee p)$. ▮

LEMMA TO PROPOSITION 8. If J is a closure operator on a lattice L, and if R is the natural injection of J-closed elements of L into the lattice L/J of all J-closed elements of L, then the composition $RJ : L \rightarrow L/J$ is a join-homomorphism.

Proof: If X is a subset of L, $\sup RJ(X)$ is the image in L/J of the least closed element of L lying above $J(x)$, for all $x \in X$, ie: the least closed element lying above x, for all $x \in X$, ie: $J(\sup X)$. Thus $\sup RJ(X) = RI(\sup X)$. ▮

PROPOSITION 8. If J is a finitistic closure operator with the exchange property on a geometric lattice P, then the lattice P/J is geometric. If $J(\Phi) = \Phi$, and if R is the natural injection of the J-closed elements of P into P/J, then RJ, is a strong map.

Proof: Let y be any element of P/J, and X any directed set in P/J. By proposition 6, $\sup\limits_{P/J} X = J(\sup\limits_{P} X) = \sup\limits_{P} J(X) = \sup\limits_{P} X$. Thus $y \wedge \sup\limits_{P} X = \sup\limits_{P} y \wedge X \leq \sup\limits_{P/J} y \wedge X \leq y \wedge \sup\limits_{P/J} X = y \wedge \sup\limits_{P} X$, and $\sup\limits_{P/J} y \wedge X = y \wedge \sup\limits_{P/J} X$, so the complete lattice P/J is continuous. By the above lemma, RJ is a join-homomorphism. If $J(\Phi) = \Phi$, RJ is non-singular.

Assume $y \downarrow x$ in P, and assume z is closed, with $J(x) < z \leq J(y)$. Choosean atom p such that $x \vee p = y$, and let q be any atom such that $J(x) < J(x) \vee q \leq z$. Then $q \not\leq J(x)$, $q \leq J(x \vee p)$, so $p \leq J(x \vee q) \leq J(z) = z$. Thus $x \vee p \leq z$, and $J(y) = z$, so RJ is cover-preserving. By proposition 4, P/J, the image of a geometric lattice in a complete continuous lattice, is geometric. ▮

PROPOSITION 9. Any map $f: P \longrightarrow Q$ in the category G of geometric lattices and strong maps has a factorization $f = f_3 f_2 f_1$ in G, where f_1 is onto, f_2 is an isomorphism, and f_3 is one-one.

Proof: Let J be the closure determined by f. Then the natural map $f_1 = RJ: P \longrightarrow P/J$ is onto. $f_2 = fR^{-1}$, cut down to Im f, is clearly one-one, onto, and order- preserving. Since the statement $x \leq J(y) \Longleftrightarrow f(x) \leq f(y)$ holds, in particular, for closed elements of P, f_2 is an order isomorphism, and therefore a strong map. f_3 is then the natural embedding of Im f into Q. Since the order and cover relations on Im f are those induced by Q, f_3 is a strong map. ▮

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