Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 38 (1967), p. 23-26
<http://www.numdam.org/item?id=RSMUP_1967__38__23_0>

GEOMETRIC DUALITY

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Duality for geometries (see [4], [3]) may be expressed in terms of complementation of subsets, together with negation of the dependence relation:

$$e\,\delta^* (X - e) \Longleftrightarrow e\,\overline{\delta}\,(\subseteq X - e). \tag{1}$$

The dependence relations δ and δ^* give rise to closure operators J and J^* with the exchange property. J and J^* may be considered to act on the Boolean algebra B of all subsets of G, and on the dual lattice $\widetilde{B}$, respectively. Then

$$J^* (\widetilde{x}) = J^* (\widetilde{y}) \Longleftrightarrow J(x) \neq J(y) \tag{2}$$

for all pairs x, y of subsets of G, such that y covers x in B.

Closure operators with the exchange property also occur as the kernels of strong maps [2] from one geometric lattice to another. This suggests a more general form of duality for geometries. Indeed, we shall prove that if J is a closure satisfying appropriate exchange and finiteness properties on a geometric lattice P, and if the dual lattice $\widetilde{P}$ is also geometric, then condition (2), above, determines uniquely a closure operator J^* on $\widetilde{P}$, satisfying the same exchange and finiteness conditions. The condition on the lattice P is satisfied, for example, if P is a complemented modular lattice of finite height [1].

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The relationship of geometric duality holding between the lattices P/J and $\widetilde{P}/J^*$ is more general than that obtaining in the theories of Whitney [4]. It coincides with the duality of Whitney if P is a finite Boolean algebra.

Under the same condition on the lattice P, namely that $\widetilde{P}$ also be geometric, we prove that a closure satisfying finiteness conditions has a dual closure defined by (2) if and only if it has the exchange property.

An element x in a geometric lattice P is *cofinite* if and only if $x < x \vee p$ for only finitely many atoms p in P. A closure J on P is *cofinitary* if and only if $y \downarrow x$ and $J(x) \neq J(y)$ imply the existence of a cofinite element z such that $x \leq z$ and $J(z) \neq J(y \vee z)$.

PROPOSITION 1. If a lattice P and its lattice dual $\widetilde{P}$ are both geometric, and if J is a finitary and cofinitary closure with the exchange property on P, then there is a unique closure J^* on $\widetilde{P}$ satisfying condition (2), and $\widetilde{P}/J^*$ is geometric.

Proof: For each element $y \in P$, let $T(y) = \inf \{x ; y \downarrow x, \text{ and } y = x \text{ or } J(y) \neq J(x)\}$. If J^* is any closure on $\widetilde{P}$ satisfying condition (2), then $y \downarrow x$ implies $\widetilde{x} \leq J^*(\widetilde{y}) \Longleftrightarrow J^*(\widetilde{x}) = J^*(\widetilde{y}) \Longleftrightarrow J(x) \neq J(y)$. Since the lattice P is complemented modular and coatomistic, the interval $[0, y]$ is coatomistic, and $J^*(\widetilde{y}) = \widetilde{T}(y)$. We prove that J^*, thus defined, is a closure operator with the required properties. $y \geq T(y)$ implies $\widetilde{y} \leq J^*(\widetilde{y})$. Assume $z \leq y$ and $y \downarrow x$. Then $T(y) \leq$ $\leq x \Longleftrightarrow J(x) < J(y)$. If $J(x) < J(y)$, then $J(x \wedge z) \leq J(x)$, so $J(x \wedge z) < J(z)$, and $T(z) \leq x \wedge z$. Thus $T(z) \leq T(y)$, and $\widetilde{y} \leq \widetilde{z}$ implies $J^*(\widetilde{y}) \leq J^*(\widetilde{z})$. Assume that for some element $y \in P$, there exists an element z such that $T(y) \downarrow z$, and such that $J(z) < J(T(y))$. Choose a cofinite element x such that $z \leq x$ and $J(x) < J(x \vee T(y))$. Then the interval $[x \wedge y, y]$ is finite. Let w be a maximal element of $[x \wedge y, y]$ such that $J(w) < J(w \vee T(y))$. If $w \vee T(y) \neq y$, choose an element u covering w such that $u \vee T(y)$ covers $w \vee T(y)$. Since $w \vee T(y)$

and $u \vee T(y)$ are in the interval $[T(y), y]$, $J(w \vee T(y)) < J(u \vee T(y))$, $J(w) < J(u)$, and, by the exchange property, $J(u) < J(u \vee T(y))$. This contradicts the maximal property of w, so $w \vee T(y) = y$, and $T(y) \leq \leq w < y$, by the definition of T. This contradicts the definition of w, so $T(y) \downarrow w \Longrightarrow J(w) = J(y)$, and $TT(y) = T(y)$. Thus $J^*(J^*(\tilde{y})) = J^*(\tilde{y})$, and J^* is a closure. J^* is finitary because J is cofinitary. If elements x and y cover $x \wedge y$ in P, and are thus covered by $x \vee y$, and if $J^*(\widetilde{x \vee y}) < J^*(\tilde{x}) = J^*(\widetilde{x \wedge y})$, then $J(x \wedge y) < J(x) = J(x \vee y)$. If, moreover, $J^*(\widetilde{x \vee y}) < J^*(\tilde{z})$, then $J(y) = J(x \vee y)$, $J(y) \neq J(x \wedge y)$, and $J^*(\tilde{y}) = J^*(\widetilde{x \wedge y})$. Thus J^* has the exchange property, and $\tilde{P}/J^*$ is a geometric lattice. ▮

As an example of duality relative to a complemented modular lattice, consider the seven-point projective plane mapped into a five-point line in such a way that one line j is mapped to a point. The empty set, the line j, the four points off j, and the plane are closed relative to this strong map. Only $\tilde{j}$ and $\tilde{\Phi}$ are closed relative to the dual closure on the dual plane, and $\tilde{j}$ is the dual-closure of the empty subset of the dual plane.

A partial converse to proposition 1 is available, which characterizes closures with the exchange property as those closures which have duals.

PROPOSITION 2. If a lattice P and its lattice dual $\tilde{P}$ are both geometric, if J is a finitary and cofinitary closure on P, and if $\tilde{T}$ is a closure on $\tilde{P}$, where $T(y) = \inf\{x ; y \downarrow x$, and $y = x$ or $J(y) \neq \neq J(x)\}$, then J has the exchange property.

Proof: Assume x and y cover $x \wedge y$, so $x \vee y$ covers x and y. Assume further that $J(x \wedge y) < J(x) = J(x \vee y)$ and $J(x \wedge y) < J(y)$. If $J(y) < J\, x \vee y)$, then $T(x \vee y) \leq T(y)$. Since $J(x \wedge y) < J(y)$, $T(y) \leq T(x \wedge y)$. If T is a closure, then $T(x \vee y) = T(x \wedge y) = T(x)$, contradicting $J(x) = J(x \vee y)$. Thus $J(y) = J(x \vee y)$, and J has the exchange property. ▮

Added in proof: The following, provided by D. A. Higgs, and printed here with his permission, defines the scope of the preceding duality theory. It is known that *every modular geometric lattice is a direct join* (cartesian product) *of projective geometries.* We have considered, above, geometric lattices L whose dual lattices $\tilde{L}$ are continuous. Under this assumption, Higgs proves that the projective geometries involved in the above direct join decomposition must be of *finite height.* The essential result is as follows.

PROPOSITION 3. (D. A. Higgs) A projective geometry L of infinite height cannot be dual continuous.

Proof.: Let $\{p_i \,;\ i = 0, 1, \dots\}$ be an independent enumerably infinite set of atoms of L, where L is geometric, modular, and every element of rank 2 covers at least 3 atoms. Let r_n bs a third atom covered by $p_n \vee p_{n+1}$, $n = 0, 1, \dots$ Let $a = \sup_i r_i$ and $x_i = \sup_{j \geq i} p_j$. Then $\inf_i x_i = 0$, because each atom beneath x_0 is dependent upon a unique minimal (finite) subset of $\{p_i\}$. Thus $a \vee \inf_i x_i = a \vee 0 = a$, while $\inf_i (a \vee x_i) = \inf_i x_0 = x_0 > a$. Thus L is not dual-continuous. ∎

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Manoscritto pervenuto in redazione il 25 luglio 1966.