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# Some generating functions for associated Legendre functions

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# SOME GENERATING FUNCTIONS FOR ASSOCIATED LEGENDRE FUNCTIONS

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1. INTRODUCTION. This paper will present a couple of generating functions for associated Legendre Functions defined [1, p. 122] by

$$(1.1) \qquad P_n^m(x) = \frac{1}{\Gamma(1-m)} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} {}_2F_1\left[\begin{matrix} -n, n+1 ; \\ 1-m ; \end{matrix} \frac{1}{2} - \frac{1}{2}x\right]$$

$$(|1-x| < 2)$$

and

$$(1.2) \qquad Q_n^m(x) = e^{mi\pi}\, 2^{-1-n}\, \pi^{1/2} \frac{p(n+m+1)}{p\left(n+\frac{3}{2}\right)} x^{-n-m-1} (x^2-1)^{\frac{1}{2}m} \times$$

$$\times\, {}_2F_1\left[\begin{matrix} \frac{1}{2}n + \frac{1}{2}m + 1, \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2} ; \\ n + \frac{3}{2} ; \end{matrix} x^{-2}\right] \quad (|x| > 1)$$

where the functions $P_n^m(x)$ and $Q_n^m(x)$ are known as the Legendre functions of the first and second kind, respectively.

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The generating functions developed are:

$$(1.3)\quad \sum_{n=0}^{\infty} e^{im\pi}\, Q_n^m(x)\, t^n = \frac{\Gamma(-m)\,\Gamma(1+m)}{2}\left[\left(\frac{x-1}{x+1}\right)^m - 1\right] \varrho^{-1-m} \times$$

$$\times \left[\frac{1}{2}(x+1)\,t\right]^{\frac{1}{2}m} \cdot P_m^m\left(\frac{1-t}{\varrho}\right)$$

where

$$\varrho = (1 - 2\,x\,t + t^2)^{\frac{1}{2}}$$

and with the restriction that $m$ is not zero or positive or negative integer

$$(1.4)\quad \sum_{n=0}^{\infty} \frac{(1-m)_n}{n\,!}\; {}_2F_1\left[\begin{matrix}-n, 1\,; \\ 1-m\,;\end{matrix}\; y\right] P_n^m(x)\, t^n =$$

$$= \mu^{-1-m}\left[-\frac{1}{2}\,ty\,(x - t + \varrho)\right]^{\frac{1}{2}m} P_m^m\left(\frac{\varrho + ty}{\mu}\right)$$

where

$$\mu = [1 - 2xt\,(1-y) + t^2\,(1-y)^2]^{\frac{1}{2}}$$

and

$$\varrho = (1 - 2\,xt + t^2)^{\frac{1}{2}}$$

$$(1.5)\quad \sum_{n=0}^{\infty} \frac{(1-m)_n}{n\,!}\; {}_1F_2\left[\begin{matrix}-n\,; \\ 1+m, 1-m\,;\end{matrix}\; y\right] P_n^m(x)\, t^n =$$

$$= \frac{\varrho^{-1}}{\Gamma(1+m)}\, e^{\frac{1}{2}mi\pi}\, J_{-m}\left\{\left[\frac{2\,ty\,(x+t-\varrho)}{\varrho^2}\right]^{\frac{1}{2}}\right\} \times I_m\left\{\left[\frac{-2\,ty\,(x-t+\varrho)}{\varrho^2}\right]^{\frac{1}{2}}\right\}$$

where

$$\varrho = (1 - 2\,xt + t^2)^{\frac{1}{2}}$$

and with the restriction that $m$ is not an integer.

2. In this section we enlist some results for ready reference:

$$(2.1) \qquad F_4[a, b ; c, d ; x, y] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{n+k} (b)_{n+k}}{k!\, n!\, (c)_k (d)_n} x^k y^n$$

which is formula (9) in [1; p. 224]

$$(2.2) \quad F_4\left[\begin{matrix} \alpha, \beta ; \\ 1+\alpha-\beta, \beta ; \end{matrix} \frac{-u}{(1-u)(1-v)}, \frac{-v}{(1-u)(1-v)}\right] =$$

$$= (1-v)^{\alpha} {}_2F_1\left[\begin{matrix} \alpha, & \beta ; \\ 1+\alpha-\beta ; \end{matrix} \frac{-u(1-v)}{1-u}\right]$$

which is formula (8) in [1; p. 238]

$$(2.3) \qquad {}_2F_1\left[\begin{matrix} a, b ; x \\ c ; \end{matrix}\right] = (1-x)^{-b} {}_2F_1\left[\begin{matrix} c-a, b ; \\ . c ; \end{matrix} \frac{x}{x-1}\right]$$

which is formula (22) in [1; p. 64]

$$(2.4) \qquad \varrho^{-1-n} P_n^m \left(\frac{x-t}{\varrho}\right) = \sum_{k=0}^{\infty} \binom{n-m+k}{k} P_{n+k}^m (x)\, t^k$$

which is formula (2) in [2; p. 264]

$$(2.5) \qquad P_n^{(\alpha, -\alpha)}(x) = \Gamma(1+\alpha) \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}\alpha} \frac{(1+\alpha)_n^*}{n!} P_n^{-\alpha}(x)$$

which is result (8) in [4].

3. Proof of (1.3). In the proof, I shall use the expansion of $Q_n^m(x)$ in terms of $\frac{x-1}{x+1}$ as [1; p. 132]

$$e^{-im\pi} Q_n^m(x) = 2^{-1-n} \Gamma(m) (x+1)^{n+\frac{1}{2}m} (x-1)^{-\frac{1}{2}m} {}_2F_1\left(-n, -n-m; 1-m; \frac{x-1}{x+1}\right) +$$

$$(3.1)\quad + 2^{-1-n}\,\frac{\Gamma(1+m+n)\,\Gamma(-m)}{\Gamma(1+n-m)}\,(x+1)^{-\frac{1}{2}m+n}\,(x+1)^{\frac{1}{2}m}\times$$

$$\times\,{}_2F_1\left(-n,\,-n+m\,;1+m\,;\frac{x-1}{x+1}\right).$$

$= A + B$ (say)

Now

$$A = 2^{-1-n}\,\Gamma(m)(x+1)^{n+\frac{1}{2}m}(x-1)^{-\frac{1}{2}m}\,{}_2F_1\left(-n,\,-n-m;1-m;\frac{x-1}{x+1}\right)$$

$$=\frac{\Gamma(m)}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\sum_{k=0}^{n}\frac{(1)_n\,(1+m)_n}{k\,!\,(n-k)\,!\,(1-m)_k\,(1+m)_{n-k}}\times$$

$$\times\left[\frac{1}{2}(x-1)\right]^k\left[\frac{1}{2}(x+1)\right]^{n-k}.$$

Multiplying both sides by $t^n$ and summing as indicated, we have

$$\sum_{n=0}^{\infty}At^n=\frac{\Gamma(m)}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1)_n\,(1+m)_n}{k\,!\,(n-k)\,!\,(1-m)_k\,(1+m)_{n-k}}\times$$

$$\times\left[\frac{1}{2}(x-1)\,t\right]^k\left[\frac{1}{2}(x+1)\,t\right]^{n-k}$$

$$=\frac{\Gamma(m)}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1)_{n+k}\,(1+m)_{n+k}}{k\,!\,n\,!\,(1-m)_k\,(1+m)_n}\times$$

$$\times\left[\frac{1}{2}(x+1)\,t\right]^k\left[\frac{1}{2}(x+1)\,t\right]^n$$

and using the result (2.1)

$$(3.2)\qquad=\frac{\Gamma(m)}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}F_4\left[\begin{matrix}1, & 1+m\,;\\ 1-m, & 1+m\,;\end{matrix}\;\frac{1}{2}(x-1)\,t,\,\frac{1}{2}(x+1)\,t\right].$$

The hypergeometric function of two arguments in (3.2) can be represented as the product of two functions of unit argument.

Taking

$$\frac{-u}{(1-u)(1-v)} = \frac{1}{2}(x-1)\,t\,; \qquad \frac{-v}{(1-u)(1-v)} = \frac{1}{2}(x+1)\,t$$

i.e $$u = 1 - \frac{2}{1+t+\varrho}\,; \quad v = 1 - \frac{2}{1-t+\varrho}$$

where

$$\varrho = (1 - 2\,xt + t^2)^{\frac{1}{2}}$$

and using the results (2.2), (2.3) and definition of $P_n^m(x)$ as (1.1) we get

$$\sum_{n=0}^{\infty} At^n = \frac{\Gamma(m)}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \cdot \frac{2}{1-t+\varrho} \cdot \left[1 - \frac{1-t-\varrho}{1-t+\varrho}\right]^{-1-m} \times$$

$$\times\, {}_2F_1\left[\begin{matrix}-m,\, 1+m\,; \\ 1-m\,;\end{matrix}\ \frac{1}{2} - \frac{1}{2}\left(\frac{1-t}{\varrho}\right)\right]$$

$$\text{(3.3)} \qquad = \frac{\Gamma(m)\,\Gamma(1-m)}{2}\,\varrho^{-1-m}\left[\frac{1}{2}(x+1)\,t\right]^{\frac{1}{2}m} P_m^m\left(\frac{1-t}{\varrho}\right).$$

Again

$$B = \frac{\Gamma(1+m+n)\,\Gamma(-m)}{\Gamma(1+n-m)}\,2^{-1-n}(x+1)^{-\frac{1}{2}m+n}(x-1)^{\frac{1}{2}m}\,{}_2F_1\left[\begin{matrix}-n,\,-n+m\,; \\ 1+m\,;\end{matrix}\ \frac{x-1}{x+1}\right]$$

$$= \frac{\Gamma(1+m)\,\Gamma(-m)}{2\,\Gamma(1-m)}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\frac{(1+m)_n}{(1-m)_n}\sum_{k=0}^{n}\frac{(1)_n\,(1-m)_n}{k!\,(n-k)!\,(1+m)_k\,(1-m)_{n-k}} \times$$

$$\times\left[\frac{1}{2}(x-1)\right]^k\left[\frac{1}{2}(x+1)\right]^{n-k}.$$

Multiplying both sides by $t^n$ and summing as indicated, we have

$$\sum_{n=0}^{\infty} Bt^n = \frac{\Gamma(1+m)\,\Gamma(-m)}{2\,\Gamma(1-m)} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1)_n\,(1+m)_n}{k!\,(n-k)!\,(1+m)_k\,(1-m)_{n-k}} \times$$

$$\times \left[\frac{1}{2}(x-1)\,t\right]^k \left[\frac{1}{2}(x+1)\,t\right]^{n-k}$$

$$= \frac{\Gamma(1+m)\,\Gamma(-m)}{2\Gamma(1+m)} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(1)_{n+k}\,(1+m)_{n+k}}{k!\;n!\;(1+m)_k\;(1-m)_n} \times$$

$$\times \left[\frac{1}{2}(x-1)\,t\right]^k \left[\frac{1}{2}(x+1)\,t\right]^n .$$

and using the result (2.1)

$$(3.4) = \frac{\Gamma(1+m)\Gamma(-m)}{2\,\Gamma(1-m)} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} F_4\left[\begin{matrix}1, & 1+m\,; \\ 1-m, & 1+m\,;\end{matrix}\;\frac{1}{2}(x-1)\,t, \frac{1}{2}(x+1)\,t\right].$$

Proceeding as above and using resulte (2.2), (2.3) and (1.1) we get

$$(3.5)\ \sum_{n=0}^{\infty} Bt^n = \frac{p(1+m)\,p(-m)}{2} \left(\frac{x-1}{x+1}\right)^m \varrho^{-1-m} \left[\frac{1}{2}(1+1)\,t\right]^{\frac{1}{2}m} P_m^m\left(\frac{1-t}{\varrho}\right).$$

Hence combining (3.3) and (3.5) we have

$$\sum_{n=0}^{\infty} e^{-im\pi}\, Q_n^m(x)\, t^n = \frac{p(1+m)\,p(-m)}{2} \left[\left(\frac{x-1}{x+1}\right)^m - 1\right] \cdot \varrho^{-1-m} \times$$

$$\times \left[\frac{1}{2}(x+1)\,t\right]^{\frac{1}{2}m} P_m^m\left(\frac{1-t}{\varrho}\right)$$

where $P_m^m\left(\frac{1-t}{\varrho}\right)$ is Legendre function in which degree and order are equal.

4. PROOF OF (1.4). In the proof, I shall use the generating function for Legendre polynomial as [3]

$$(4.1) \qquad \varrho^{-1-n}\left[\frac{1}{2}t(x+1)\right]^{\frac{1}{2}m} P_m^m\left(\frac{1-t}{\varrho}\right) = \sum_{n=0}^{\infty} P_n^m(x)\, t^n$$

where

$$\varrho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

Putting

$$x = \frac{X-T}{G}, \qquad t = \frac{-TY}{G}$$

where

$$G = (1 - 2XT + T^2)^{\frac{1}{2}}$$

(4.1) becomes

$$[1 - 2XT(1-Y) + T^2(1-Y)]^{-\frac{1}{2}-\frac{1}{2}m}\left[-\frac{1}{2}TY(X-T+G)\right]^{\frac{1}{2}m} \times$$

$$\times P_m^m\left\{\frac{G+TY}{[1 - 2XT(1-Y) + T^2(1-Y)^2]^{\frac{1}{2}}}\right\}$$

$$= \sum_{n=0}^{\infty} G^{-1-n} P_n^m\left(\frac{X-T}{G}\right)(-TY)^n$$

with (2.4), the right-hand side of above becomes

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-m+k}{k} P_{n+k}^m(X)(-TY)^n T^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n-m}{k} P_n^m(X)(-TY)^{n-k} T^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n-m}{n-k} P_n^m(X) T^n (-Y)^k$$

$$= \sum_{n=0}^{\infty}\frac{(1-m)_n}{n!}\, {}_2F_1\left[\begin{matrix}-n, 1;\\ 1-m;\end{matrix}\; Y\right] P_n^m(X)\, T^n.$$

which completes the proof of (1.4)

5. **Proof of** (1.5) In the proof, I shall use a generating function for Jacobi Polynomial defined as [2; p. 262]

$$(5.1)\qquad \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)\, t^n}{(1+\alpha)_n\,(1+\beta)_n} =$$

$$= \Gamma(1+\alpha)\,\Gamma(1+\beta)\left(\frac{1}{2}t\right)^{-\frac{1}{2}\alpha-\frac{1}{2}\beta}(1-x)^{-\frac{1}{2}\alpha}(1+x)^{-\frac{1}{2}\beta}\times$$

$$\times J_\alpha\{[2t(1-x]^{\frac{1}{2}}\}\, I_\beta\{[2t(1+x)]^{\frac{1}{2}}\}.$$

Putting $\beta = -\alpha$ and using (2.5), we get

$$(5.2)\qquad \sum_{n=0}^{\infty} \frac{P_n^{-\alpha}(x)\, t^n}{n!\,(1+\alpha)_n} =$$

$$= (-1)^{\frac{1}{2}\alpha}\,\Gamma(1-\alpha)\, J_\alpha\{[2t(1-x)]^{\frac{1}{2}}\}\, I_\beta\{[2t(1+x)]^{\frac{1}{2}}\}$$

Taking

$$x = \frac{X-T}{G}, \quad t = \frac{-TY}{G} \text{ and } m = -\alpha$$

where

$$G = (1 - 2XT + T^2)^{\frac{1}{2}}$$

(5.2) becomes

$$\frac{e^{\frac{1}{2}im\pi}}{\Gamma(1+m)} J_{-m}\left\{\left[\frac{2TY(X+T-G)}{G^2}\right]^{\frac{1}{2}}\right\} I_m\left\{\left[\frac{-2TY(G+X-T)}{G^2}\right]^{\frac{1}{2}}\right\}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!\,(1+m)_n} \cdot P_n^m\left(\frac{X-T}{G}\right)\left(\frac{-TY}{G}\right)^n.$$

With (2.4), right-hand side of above becomes

$$= G \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{n!\,(1+m)_n}\binom{n-m+k}{k} P_{n+k}^m(X)\, T^{n+k}(-Y)^n$$

$$= G \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!\,(1+m)_{n-k}} \binom{n-m}{k} P_n^m(X)\, T^n (-Y)^{n-k}$$

$$= G \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!\,(1+m)_k} \binom{n-m}{n-k} P_n^m(X)\, T^n (-Y)^k$$

$$= G \sum_{n=0}^{\infty} \frac{(1-m)_n}{n!}\, {}_1F_2\left[\begin{matrix} -n\,; \\ 1+m,\, 1-m\,; \end{matrix}\; Y\right] P_n^m(X)\, T^n$$

which proves (1.5).

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