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CERTAIN SUMS OF DOUBLE HYPERGEOMETRIC SERIES

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1. Let Kampé de Fériét function [1] in a modified notation be defined as

$$F_{e, \nu}^{\lambda, \mu} \left(\begin{matrix} a_\lambda : b_\mu, b'_\mu \\ c_e : d_\nu, d'_\nu \end{matrix} ; \right) = \sum_{m, n=0}^{\infty} \frac{(a_\lambda)_{m+n} (b_\mu)_m (b'_\mu)_n}{(c_e)_{m+n} (d_\nu)_m (d'_\nu)_n m! n!}$$

where $\lambda + \mu \leq 1 + \rho + \nu$ and a_λ stands for the sequence $a_1, a_2, \dots, a_\lambda$; and $(a)_m = a(a+1) \dots (a+m-1)$, $(a)_0 = 1$.

Carlitz [3-6] has given four sums of $F_{1,1}^{1,2}$ and a sum of $F_{0,2}^{1,2}$ under different sets of condition. Jain [7] has given a sum of $F_{1,1}^{0,3}$ viz.

$$(1.1) \quad F_{1,1}^{0,3} \left(\begin{matrix} - : c - a, a ; c - b, b ; -m, -n ; \\ c : 1 - a - b + c - m, 1 + a + b - c - n ; \end{matrix} \right) \\ = \frac{(a)_m (c - a)_n (b)_m (c - b)_n}{(c)_{m+n} (a + b - c)_m (c - a - b)_n}$$

where $a + b - c$ is not an integer.

Recently Srivastava ¹⁾ [9] studied the sum of $F_{1,1}^{1,2}$ in a more systematic way and gave a number of results.

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The object of this paper is to study the sum of $F_{1,1}^{0,3}$, in detail.

$$\begin{aligned}
 2. \quad F &= F_{1,1}^{0,3} \left(- : a, a' ; b, b' ; -m, -n ; \right) \\
 &= \sum_{p, q=0} \frac{(a)_p (a')_q (b)_p (q')_q (-m)_p (-n)_q}{(\bar{d})_{p+q} (c)_p (c')_q p! q!} \\
 &= \sum \frac{(a)_p (b)_p (-m)_p}{(\bar{d})_p (c)_p p!} {}_3F_2 \left[\begin{matrix} a', b', -n \\ d+p, c' \end{matrix} \right].
 \end{aligned}$$

Now using the transformation [2]

$$(2.1) \quad {}_3F_2 \left[\begin{matrix} a, b, -m \\ \bar{d}, c \end{matrix} \right] = \frac{(c-b)_m}{(c)_m} {}_3F_2 \left[\begin{matrix} d-a, b, -m \\ \bar{d}, 1+b-c-m \end{matrix} \right]$$

we get

$$\begin{aligned}
 (2.2) \quad F &= \frac{(c'-b')_n}{(c')_n} \sum_{p=0}^m \frac{(a)_p (b)_p (-m)_p}{(\bar{d})_p (c)_p p!} {}_3F_2 \left[\begin{matrix} d-a'+p, b', -n \\ \bar{d}+p, 1+b'-c'-n \end{matrix} \right] \\
 &= \frac{(c'-b')_n}{(c')_n} \sum_{q=0}^n \frac{(d-a')_q (b')_q (-n)_q}{(d)_q (1+b'-c'-n)_q q!} {}_4F_3 \left[\begin{matrix} d-a'+q, a, b, -m \\ \bar{d}+q, c, d-a' \end{matrix} \right].
 \end{aligned}$$

Choosing

$$(2.3) \quad d = a + a'$$

and

$$(2.4) \quad d + c = 1 + a + b - m$$

${}_4F_3$ on the right reduces to ${}_3F_2$ Saalschützian.

Using Saalschützian theorem [8], we get

$$F = \frac{(d-a)_m (d-b)_m (c'-b')_n}{(d)_m (d-a-b)_m (c')_n} {}_4F_3 \left[\begin{matrix} a, b', d-b+m, -n \\ \bar{d}+m, 1+b'-c'-n, d-b \end{matrix} \right].$$

This ${}_4F_3$ reduces to ${}_2F_1$ under six different sets of conditions and the resulting ${}_2F_1$ can be summed by Gauss theorem.

Thus assuming the conditions (2.3) & (2.4) are satisfied, we get

$$(2.5) \quad F = \frac{(d-a)_m (d-b)_m (a)_n (b)_n}{(d)_{m+n} (d-a-b)_m (a+b-d)_n}$$

if $d = b + b'$ and $a = 1 + b' - c' - n$.

which is the same as (1.1)

$$(2.6) \quad = \frac{(d-a)_{m+n} (d-b)_{m+n}}{(d)_{m+n} (d-a-b)_m (m)_n}$$

if $b' = d - b$ and $c' = 1 - n - m$.

$$(2.7) \quad = \frac{(d-b)_m (c')_{m+n} m!}{(d)_m (1+b)_m (c')_m (c')_n}$$

if $d = a - m$ and $b' = d - b$.

$$(2.8) \quad = \frac{(d-a)_m (c' - b')_m (c')_{m+n}}{(d)_m (c)_m (c')_m (c')_n}$$

if $d = a - m$ and $a - b + m = 1 + b' - c' - n$.

$$(2.9) \quad = \frac{(d-a)_m (d-b)_m (a)_n m!}{(d)_m (d-a-b)_m (c')_n (d-b)_n (m-n)!}$$

if $m \geq n$ } if $a = 1 + b' - c' - n$
and
 if $m < n$ } $d = b' - m$

$= 0$.

$$(2.10) \quad = \frac{(d-a)_m (d-b)_m (b' - b)_n (d-a-b)_n}{(d)_m (d-a-b)_m (-b)_n (d-b)_n}$$

if $d = b' - m$ and $c' = 1 + b - n$.

provided b is not a positive integer $< n$.

Now using (2.1) in the right hand side of (2.2), assuming $d = b + b'$ and changing the order of summation, we get

$$(2.11) \quad F = \frac{(b')_n (b + c' - a')_n}{(c')_n (d)_n} \sum_{q=0}^n \frac{(c' - b')_q (b)_q (-n)_q}{(1 - b' - n)_q (b + c' - a')_q q!} \cdot {}_4F_3 \left[\begin{matrix} b + c' - a' + n, b + q, a, -m \\ c, d + n, b + c' - a' + q \end{matrix} \right].$$

This ${}_4F_3$ is Saalschützian if (2.4) is satisfied.

If $c = b + c' - a' + n$, ${}_4F_3$ reduces to ${}_3F_2$ which can be summed by Saalschützian theorem and we get

$$F = \frac{(b + c' - a' - a)_m (b')_n (b + c' - a')_n (c' - a')_m}{(b + c' - a')_m (c')_n (d)_n (c' - a' - a)_m} \cdot {}_3F_2 \left[\begin{matrix} c' - b', b, -n \\ c - n + m, c - a - n \end{matrix} \right].$$

If further $c = b + n - m$ it reduces to ${}_2F_1$ and is summed by Gauss theorem. Hence

$$(2.12) \quad F = \frac{(1 + a - b)_{m-n} (d - b)_n (d - a - a')_m m!}{(1 - b)_{m-n} (d)_n (a' - m)_n (1 + a)_m}.$$

If $b' = d - b, d + c = 1 + a + b - m, c' = a' - m, c = b + n - m$.

Similarly if we take $d = a - n$ in (2.11) and assume condition (2.4) is satisfied, then summing ${}_3F_2$ by Saalschütz's theorem we get

$$F = \frac{(d - b)_n (b + c' - a')_n (c' - a')_m (-n)_m}{(c')_n (d)_n (-b - n)_m (b + c' - a')_m} \cdot {}_3F_2 \left[\begin{matrix} c' - b', b, -n + m, \\ 1 - b' - n, b + c' - a' + m \end{matrix} \right].$$

This ${}_3F_2$ reduces to ${}_2F_1$ if either $a' = d + m$ or $a' = c' + m$,

which can be summed by Gauss theorem. Thus

$$(2.13) \quad F = \frac{(b')_m (c' - b')_{n-m} (c' - a')_m (1 + b)_n n!}{(c')_n (d)_m (1 + b)_{n-m} (n - m)!} \quad \text{if } n \geq m$$

$$= 0 \quad \text{if } n < m$$

provided $d = b + b', d + c = 1 + a + b - m, a = d + n, a' = d + m$ and

$$(2.14) \quad F = \frac{(b')_m (b)_{n+1} m! n!}{(1 - a')_m (d)_n (n - m)! (b + n - m)} \quad \text{if } n \geq m$$

$$= 0 \quad \text{if } n < m$$

provided $d = b + b', d + c = 1 + a + b - m, a = d + n, a' = c' + m$.

3. In this section, we consider that case of $F_{1,1}^{0,3}$ in which one parameter in the numerator cancell one parameter in the denominator.

$$f = F_{1,1}^{0,3} \left(\begin{matrix} - : a, a'; b, b'; c, c' \\ d : e, a' \end{matrix} ; \right) = \sum_{p=0}^{\infty} \frac{(a)_p (b)_p (c)_p}{(d)_p (e)_p p!} {}_2F_1 \left[\begin{matrix} b', c' \\ d + p \end{matrix} \right]$$

which by Gauss theorem is

$$(3.1) \quad = \frac{\Gamma(d - b' - c') \Gamma(d)}{\Gamma(d - b') \Gamma(d - c')} {}_4F_3 \left[\begin{matrix} a, b, c, d - b' - c' \\ e, d - b', d - c' \end{matrix} \right]$$

choosing $d = b + b', {}_4F_3$ in (3.1) reduces to ${}_3F_2$.

Thus

$$f = \frac{\Gamma(b - c') \Gamma(d)}{\Gamma(b) \Gamma(d - c')} {}_3F_2 \left[\begin{matrix} a, c, b - c' \\ e, d - c' \end{matrix} \right].$$

It can be summed by Gauss theorem if $d = c + c'$ or $e = b - c'$.

Thus

$$(3.2) \quad f = \frac{\Gamma(b - c') \Gamma(d) \Gamma(e) \Gamma(e - a - b - c')}{\Gamma(b) \Gamma(c) \Gamma(e - a) \Gamma(e - b + c')}$$

if $d = b + b' = c + c'$.

$$(3.3) \quad = \frac{\Gamma(e)\Gamma(d)\Gamma(d-a-c-c')}{\Gamma(b)\Gamma(d-a-c')\Gamma(d-c-c')} \quad \text{if } b = d - b = e + c'.$$

Using Saalschütz's theorem, we get

$$(3.4) \quad = \frac{\Gamma(b-c')\Gamma(d)\Gamma(b'-c)\Gamma(b'-a)\Gamma(d-a-c-c')}{\Gamma(b)\Gamma(b'-a-c)\Gamma(d-a-c')\Gamma(d-c-c')\Gamma(b')}$$

provided $d = b + b'$, $d + e = 1 + a + b + c$ and a or c is a negative integer.

Using Dixon theorem, we get

$$(3.5) \quad = \frac{\Gamma(b-c')\Gamma(d)\Gamma\left(1 + \frac{1}{2}a\right)\Gamma(e)\Gamma\left(1 + \frac{1}{2}a - b - c + c'\right)}{\Gamma(b)\Gamma(1+a)\Gamma\left(e - \frac{1}{2}a\right)\Gamma\left(d - c' - \frac{1}{2}a\right)\Gamma(1+a-b-c+c')}$$

provided when $d = b + b'$ and

$$\text{either } e = 1 + a - c, d = 1 + a - b + 2c'$$

$$\text{or } e = 1 + a - b + c', d = 1 + a - c + c'.$$

The above ${}_3F_2$ can be summed by Watson theorem in four different sets of conditions. Sum under one set of condition is

$$(3.6) \quad f =$$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b-c')\Gamma(d)\Gamma\left(c + \frac{1}{2}\right)\Gamma(1-d+c+c')}{\Gamma(b)\Gamma\left(\frac{1}{2} + \frac{1}{2}a\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}b - \frac{1}{2}c'\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}a + c\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}b + \frac{1}{2}c' + c\right)}$$

$$\text{if } d = b + b', e = 2c, 2d = 1 + a + b + c'.$$

The ${}_3F_2$ can also be summed by Whipples theorem in two different sets of conditions. Sum in one case is

(3.7) =

$$\frac{\pi 2^{1-2c} \Gamma(b - c') \Gamma(d) \Gamma(e)}{\Gamma(b) \Gamma\left(\frac{1}{2}a + \frac{1}{2}d - \frac{1}{2}c'\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}c\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}d - c'\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}c'\right)}$$

if $d = b + b', a + b - c' = 1$, and $d + e = 1 + 2c + c'$.

Taking $c = d - b' - c'$, ${}_4F_3$ in (3.1) reduces to ${}_3F_2$ which can be summed by Saalschützs, Dixon, Whipples theorem, hence giving at least four different sums of $F_{1,1}^{0,3}$.

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