

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 42 (1969), p. 201-207

http://www.numdam.org/item?id=RSMUP_1969__42__201_0

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ON MULTIPLICITY FUNCTIONS AND LEBESGUE AREA *)

by D. J. SCHAEFER

1. Introduction.

Let $T : Q \rightarrow E_3$ be a continuous transformation from the unit square Q in the uv -plane into Euclidean 3-space E_3 . Many writers have been concerned with the problem of finding formulas which express the Lebesgue area $A(T)$ in terms of multiplicity functions. This paper will show relationships between some of the results of Mickle [3] and Federer [2], and will present, for the case $A(T) < \infty$, a modified definition of significant maximal model continua (s.m.m.c.'s) (see Mickle [3]) which is more natural if one is interested in the tangential properties of the Frechet surface defined by T .

2. Plane transformations.

2.1. Throughout section 2, T will denote a plane transformation. Let $T : \Delta \rightarrow E_2$ be a continuous, bounded transformation from $\Delta \subset Q$ into E_2 , where Δ is connected and open relative to Q . We write $T : (B, A) \rightarrow (D, C)$ when $A \subset B$, $C \subset D$, $T(A) \subset C$, and $T(B) \subset D$. For $y \in E_2$ and $r > 0$, let $C(y, r) = \{z : z \in E_2, |z - y| < r\}$. Let $\mathbf{C}A$ denote the complement of set A . It is easily shown that if $y \in T(\Delta)$ and V is a component of $T^{-1}[C(y, r)]$, then $T : (\mathbf{C}\mathbf{I}_\Delta V, \mathbf{B}_\Delta V) \rightarrow (E_2, \mathbf{C}C(y, r))$, where $\mathbf{C}\mathbf{I}_\Delta V$ denotes the closure of V relative to Δ , and $\mathbf{B}_\Delta V$ the boundary of V relative to Δ .

*) Research supported in part by Aerospace Research Laboratories, Wright-Patterson AFB, Ohio.

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Then (see [1]) T induces a homomorphism h_T on the 2-dimensional Čech cohomology groups with integer coefficients and based on locally finite coverings:

$$h_T : K^2[E_2, \mathbf{CC}(y, r)] \rightarrow K^2[\mathbf{C1}_\Delta V, \mathbf{B}_\Delta V].$$

Define subsets of E_2 for $0 < r < 1$ as follows. $A(y, r) = \{z : |z - y| \leq 1/r\}$, $B(y, r) = \{z : r \leq |z - y| \leq 1/r\}$, and $U(y, r) = \{z : |z - y| > 1/r\}$.

By the excision theorem [1, p. 243], the following isomorphism holds.

$$K^2[E_2, \mathbf{CC}(y, r)] \approx K^2[A(y, r), B(y, r)].$$

Suppose V is a 2-manifold whose closure relative to Δ is compact. Then $\mathbf{C1}_\Delta V = \mathbf{C1}V$ and $\mathbf{B}_\Delta V = \mathbf{B}V$. If $K^{*3}[A(y, r), B(y, r)]$ and $K^*[\mathbf{C1}V, \mathbf{B}V]$ denote the cohomology groups for the indicated pairs as defined in [5], we have the following isomorphisms (see [5, pp. 63-64], [1, pp. 253-254]).

$$K^2[A(y, r), B(y, r)] \approx K^{*3}[A(y, r), B(y, r)]$$

$$K^2[\mathbf{C1}V, \mathbf{B}V] \approx K^{*3}[\mathbf{C1}V, \mathbf{B}V].$$

2.2. Let $F(r)$ be the family of components of $T^{-1}[C(y, r)]$. Let $V \in F(r)$ have compact closure relative to Δ . Let $D(T, r, V)$ and $\mu(y, T, V)$ be as defined in [2] and [5] respectively. From 2.1 it follows that $D(T, r, V) = |\mu(y, T, V)|$. Let (see [2])

$$M(T, \Delta, y) = \lim_{r \rightarrow 0} \sum D(T, r, V) \quad (\text{sum over } V \in F(r)),$$

where $F(r)$ is the collection of components of $T^{-1}[C(y, r)]$. We use e.m.m.c. as the abbreviation for essential maximal model continua as defined in [4].

2.3. LEMMA. $M(T, \Delta, y) \geq 1$ implies that y is the image of an e.m.m.c. under T .

PROOF. From the definition of $M(T, \Delta, y)$ and the relation between D and μ , there is an r such that $0 < r < 1$ and a component V_0 of

$T^{-1}[C(y, r)]$ such that $\mu(y, T, V_0) \neq 0$. Such a V_0 is a domain with closure in Δ , and $T^{-1}(y) \cap V_0 = T^{-1}(y) \cap \mathbf{CIV}_0$. Let $0 < r' < r$ and let $\Omega_{r'}$ be the collection of components of $T^{-1}[C(y, r')]$ that lie in V_0 .

Then

$$T^{-1}(y) \cap [\cup V'] = T^{-1}(y) \cap V_0 = T^{-1}(y) \cap \mathbf{CIV}_0. \quad (V' \in \Omega_{r'})$$

Therefore the class $\Omega_{r'}$ is (y, T, V_0) complete, i.e. $\mathbf{CIV}_0 \cap T^{-1}(y) \subset \cup V'$, the union taken over $V' \in \Omega_{r'}$. By [5, p. 126 theorem 3], $\mu(y, T, V_0) = \sum \mu(y, T, V')$, the sum taken over $V' \in \Omega_{r'}$. Furthermore, $\mathbf{CIV}'_0 \subset V_0$ because $\mathbf{CIV}'_0 \subset T^{-1}[\mathbf{CIC}(y, r')] \subset T^{-1}[C(y, r)]$ and V'_0 and its closure lie in the same component of $T^{-1}(C(y, r))$. Therefore V'_0 is an indicator region of T and y is the image an e.m.m.c. [5, p. 165].

3. i-fold essential and significant maximum model continua.

3.1. Let Q be the unit square in E_2 and $T : Q \rightarrow E_3$ denote a continuous transformation. Let $T = lm$ be a monotone-light factorization of T and denote the middle-space by \mathbf{M} . For a point $x \in E_3$ and a maximum model continuum (m.m.c.) $\gamma \subset T^{-1}(x)$, let $\Delta(\gamma, r)$ denote the component of $T^{-1}[S(x, r)]$ which contains γ , where $S(x, r)$ is the open sphere in E_3 with center x , radius r . Let $A[T, \Delta(\gamma, r)]$ denote the Lebesgue area of $T | \Delta(\gamma, r)$. Let $a \in \mathbf{M}$ be such that $a = m(\gamma)$. Then we also denote $\Delta(\gamma, r)$ by $\Delta(a, r)$. Let $L_2^*(T, a)$, $L_{*2}(T, a)$, and $E_2(T, a)$ be as defined in [2]. We define sets as follows.

$$Z_2 = \{z : z \in Q, L_2^*(T, mz) = L_{*2}(T, mz) = E_2(T, mz) = 0\},$$

$$Z_1 = \{z : z \in Q, L_2^*(T, mz) = L_{*2}(T, mz) = E_2(T, mz) = 1\},$$

$$Z_3 = Q - Z_1 \cup Z_2.$$

Denoting the Hausdorff 2-measure in \mathbf{M} by H_T^2 , the Hausdorff 2-measure in E_3 by H^2 , and number of m.m.c.'s having non-empty intersections with $T^{-1}(x) \cap Z_1$ by $N^*[x, T, Z_1]$, we can state of the following.

3.2. THEOREM. If $A(T) < \infty$, then $A(T) = \int N^*[x, T, Z_1] dH^2$.

PROOF. From [2, 8.17] we have

$$(1) \quad A(T) = \int \sigma(x) dH^2,$$

where $\sigma(x) = \sum L_2^*(T, a)$, the sum $a \in M$ such that $l(a) = x$. We will show that

$$(2) \quad \sigma(x) = N^*[x, T, Z_1] \text{ for } H^2 \text{ a.e. } x \in E_3.$$

Note that Z_1 is the union of m.m.c.'s under T and let γ be any m.m.c. in $T^{-1}(x) \cap Z_1$. Letting $a = m(\gamma)$, we have $l(a) = x$ and $L_2^*(T, a) = 1$. Hence

$$(3) \quad \sigma(x) \geq N^*[x, T, Z_1].$$

Suppose inequality (3) to be strict. Then there is an $a \in M$ such that $l(a) = x$, $L_2^*(T, a) > 0$ and $a \notin mZ_1$. Therefore $a \in m(Z_3)$ and $x \in T(Z_3)$. But [2, 8.16] gives $H^2_T[m(Z_3)] = 0$ under our assumptions. Since $H^2_T[m(Z_3)] \geq H^2[T(Z_3)]$, the latter value is zero. Therefore strict inequality in (3) holds only on a set of H^2 -measure zero. (1) and (2) imply the theorem.

3.3. In [3], Mickle makes the following definitions. Γ denotes the collection of H^2 -measurable sets of E_3 . Let U denote the unit sphere in E_3 . $\pi_p : E_3 \rightarrow E_2$ is the projection of E_3 onto the plane normal to the direction determined by $p \in U$. Let $\Gamma_p = \{E : E \in \Gamma, L_2\pi_p(E) = 0\}$ where L_2 is the Lebesgue exterior planar measure. For each $E \in \Gamma$ define

$$H_p(E) = \inf H^2(E - E_p) \quad (E_p \in \Gamma_p).$$

If $E \in \Gamma$, $p \in U$, and m and n are positive integers,

$$G_{nm}(E, p) = \{x : H_p[E \cap S(x, r)] > \pi r^2/n \text{ for some } r, 0 < r < 1/m\}.$$

Define

$$D^*(T, \mathbf{O}) = \cup_n \cap_m \cup_p G_{nm}[T(\mathbf{O} \cap \mathbf{E}_p), p], \quad (n, m = 1, 2, \dots; p \in U)$$

where \mathbf{O} is an open set in the uv -plane and \mathbf{E}_p is the union of e.m.m.c.'s under $\pi_p T : Q \rightarrow E_2$. Let Ω denote the class of sets in the uv -plane.

An m.m.c. γ under T is called a significant m.m.c. (s.m.m.c.) if and only if for every open set $\mathbf{O} \in \Omega$ such that $\gamma \subset \mathbf{O}$ we have $T(\gamma) \in D^*(T, \mathbf{O})$. The set $\mathbf{S} = \mathbf{S}(T)$ is defined the union of all s.m.m.c.'s under T .

3.4. We make the following modification.

Define

$$D^\#(T, \mathbf{O}) = \cup_p \cup_n \cap_m G_{nm}[T(\mathbf{O} \cap \mathbf{E}_p), p] \quad (n, m = 1, 2, \dots; p \in U)$$

Let $\mathbf{S}^\# = \mathbf{S}^\#(T)$ be the union of all m.m.c.'s γ under T such that for each $\mathbf{O} \in \Omega$ such that $\gamma \subset \mathbf{O}$ we have $T(\gamma) \in D^\#(T, \mathbf{O})$. It is clear from the definition that $\mathbf{S}^\# \subset \mathbf{S}$, and that with $\mathbf{S}^\#$ we single out particular, though not unique, planes.

3.5. LEMMA. Let $T : Q \rightarrow E_3$ be a continuous transformation. Let Z_1 and $\mathbf{S}^\#$ be as defined in 3.1 and 3.3. Then $Z_1 \subset \mathbf{S}^\#$.

PROOF. Let γ be an m.m.c. under T in Z_1 and let $x = T(\gamma)$. Then there is a $p \in U$ such that

$$(1) \quad \lim_{r \rightarrow 0} L_2\{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z) \geq 1\} / \pi r^2 = 1.$$

By lemma 2.3 each z such that $M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z) \geq 1$ is the image of an e.m.m.c. γ_z under $\pi_p T \mid \Delta(\gamma, z)$. γ_z is also an e.m.m.c. under $\pi_p T : Q \rightarrow E_2$ (see [4]), so $z \in \pi_p T(\Delta(\gamma, r) \cap E_p)$. Therefore

$$(2) \quad \{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z) \geq 1\} \subset \pi_p T(\Delta(\gamma, r) \cap E_p).$$

The middle space \mathbf{M} is a separable metric space. Let $\{a_i\}$ be a countable dense set in \mathbf{M} and $\mathbf{S}(a, r)$ the open sphere in \mathbf{M} with center a , radius r . Define $\mathbf{O}_{ij} = m^{-1}\mathbf{S}(a_i, 1/j)$ for $i, j = 1, 2, \dots$. Suppose $\gamma \subset \mathbf{O}_{ij}$. Then $m(\gamma) = a \in \mathbf{S}(a_i, 1/j)$. Let r_1 be small enough that $\mathbf{S}(a, 2r_1) \subset \mathbf{S}(a_i, 1/j)$. Then for $0 < r \leq r_1$, $m[\Delta(\gamma, r)] \subset \mathbf{S}(a, 2r) \subset \mathbf{S}(a_i, 1/j)$ and

$$(3) \quad \Delta(\gamma, r) \subset \mathbf{O}_{ij}.$$

From (1) there exists an $r_2 > 0$ such that

$$(4) \quad L_2\{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z) \geq 1\} / \pi r^2 \geq \frac{1}{2} \text{ for } 0 < r \leq r_2.$$

Observe that $T(\Delta(\gamma, r)) \subset S(x, r)$ so from (3) we have

$$(5) \quad \begin{aligned} \pi_p[T(\mathbf{O}_{ij} \cap \mathbf{E}_p) \cap S(x, r)] &\supset \pi_p T(\Delta(\gamma, r) \cap \mathbf{E}_p), \\ 0 < r \leq r_0 &= \min(r_1, r_2). \end{aligned}$$

(5), (2), and (4) imply that if $0 < r \leq r_0$,

$$\begin{aligned} H_p[T(\mathbf{O}_{ij} \cap \mathbf{E}_p) \cap S(x, r)] / \pi r^2 &\geq L_2\{\pi_p[T(\mathbf{O}_{ij} \cap \mathbf{E}_p) \cap S(x, r)]\} / \pi r^2 \\ &\geq L_2[\pi_p T(\Delta(\gamma, r) \cap \mathbf{E}_p)] / \pi r^2 \\ &\geq L_2\{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z) \geq 1\} / \pi r^2 \geq \frac{1}{2}. \end{aligned}$$

Therefore,

$$\limsup_{r \rightarrow 0} H_p[T(\mathbf{O}_{ij} \cap \mathbf{E}_p) \cap S(x, r)] / \pi r^2 > 0,$$

which implies $T(\gamma) \in D_p[T(\mathbf{O}_{ij} \cap \mathbf{E}_p)]$ and hence $\gamma \subset \mathbf{S}^\#$. γ was an arbitrary m.m.c. in Z_1 , hence $Z_1 \subset \mathbf{S}^\#$.

3.6. THEOREM. Let $T : Q \rightarrow E_3$ be a continuous transformation. Then $T(Z_1) \subset T(\mathbf{S}^\#) \subset T(\mathbf{S})$ and if $A(T) < \infty$, then

$$H^2[T(Z_1)] = H^2[T(\mathbf{S}^\#)] = H^2[T(\mathbf{S})].$$

PROOF. $T(Z_1) \subset T(\mathbf{S}^\#)$ from lemma 3.5, and $T(\mathbf{S}^\#) \subset T(\mathbf{S})$ from the definitions of $\mathbf{S}^\#$ and \mathbf{S} in 3.3 and 3.4. Observe that $T(\mathbf{S}) = T(\mathbf{S} - Z_1) \cup T(Z_1)$, so the theorem follows when it is shown that $H^2[T(\mathbf{S} - Z_1)] = 0$. Since $\mathbf{S} - Z_1$ and Z_1 are disjoint unions of m.m.c.'s,

$$N^*[x, T, \mathbf{S}] = N^*[x, T, \mathbf{S} - Z_1] + N^*[x, T, Z_1], \quad x \in E_3.$$

Therefore

$$(1) \quad \int N^*[x, T, \mathbf{S}] dH^2 = \int N^*[x, T, \mathbf{S} - Z_1] dH^2 + \int N^*[x, T, Z_1] dH^2.$$

But $A(T) < \infty$, so theorem 3.2, [3] and (1) imply that $0 = \int N^*[x, T, \mathbf{S} - Z_1] dH^2$. Since $x \in T(\mathbf{S} - Z_1)$ implies $N^*[x, T, \mathbf{S} - Z_1] \geq 1$, it follows that $0 = H^2[T(\mathbf{S} - Z_1)]$ and the theorem is proved.

3.7. Remark. By arguments essentially the same as those of Mickle [3] one can show that $\mathbf{S}^\#$ satisfies invariance under Frechet equivalence and that whenever $N^*[x, T, \mathbf{S}^\#]$ is measurable, $A(T) = \int N^*[x, T, \mathbf{S}^\#] dH^2$. In fact, when $A(T) < \infty$, $\mathbf{S}^\# = \mathbf{S}$. Measurability has not been established in case $A(T) = \infty$.

3.8. Remark. In view of theorem 3.4, one can apply the results on approximate tangential planes in [6] to the sets $T(\mathbf{S}^\#)$ and $T(Z_1)$.

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Manoscritto pervenuto in redazione il 30 settembre 1968.