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A PROBLEM IN RAYLEIGH-TAYLOR INSTABILITY

R. E. L. TURNER *)

1. In this paper we will study an initial — boundary value problem for a particular linearized Navier — Stokes equation. Such an equation arises in the study of Rayleigh-Taylor instability of a stratified fluid (cf. [1], p. 430), i.e., wherein the density differs from one « layer » to another in the fluid. The situation we treat in the following: a viscous incompressible fluid is confined to the region between two stationary infinite plates which we take to be parallel and at heights $z=0$ and $z=1$ in an (x, y, z) coordinate system. There is a constant external gravitational force in the negative z direction which, by choosing appropriate units, we may represent by the vector $G=(0, 0, -1)$. The equations governing the evolution are:

$$
(1.1) \qquad \begin{cases} \rho \dfrac{\partial V}{\partial t} + \rho \mathrm{V} \cdot \operatorname{grad} V = - \operatorname{grad} p + \mu \nabla^2 V + \rho G \\ \dfrac{\partial t}{\partial \rho} + V \cdot \operatorname{grad} \rho = 0 \\ \operatorname{div} V = 0 \end{cases}
$$

where ρ is the density, p is the pressure, and $V=(u, v, w)$ is the velocity vector. Again, we choose units so that the coefficient of viscosity $\mu=1$.

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In addition, we impose the « viscous » boundary condition that $V=0$ for $z=0$ and $z=1$. One particular solution of the steady state problem is

$$(1.2)\qquad \begin{cases} V=V_0\equiv(0,\ 0,\ 0) \\ \rho=\rho_0(z) \\ p=p_0\equiv -\displaystyle\int_0^z \rho_0(s)ds \end{cases}$$

where ρ_0 is a $C^1[0,\ 1]$ function. While we will treat the special case $\rho_0(z)=1+z$, if ρ_0 is a sufficiently smooth function with ρ_0 and $D_z\rho_0$ positive on $[0,\ 1]$, the same type of analysis goes through using weighted L^2 spaces and suitable changes of variables.

If we linearize the equations (1.1) about the solution (1.2) we arrive at a set of equations for the first order variations V, ρ, and p:

$$(1.3)\qquad \begin{cases} (1+z)\dfrac{\partial V}{\partial t}=-\operatorname{grad} p+\nabla^2 V+\rho G \\ \dfrac{\partial \rho}{\partial t}=-w \\ \operatorname{div} V=0. \end{cases}$$

As has been shown in [2], the problem described above is unstable in the sense that (1.3) has solutions which vanish at the boundary and which grow exponentially. In what follows we show that there are enough solutions of (1.3) to permit a unique eigenfunction expansion solution of a certain initial-boundary value problem for (1.3). In addition to the condition that V vanish at the boundary, we specific (more precisely in section 3)

$$(1.4)\qquad \begin{cases} V\,|_{t=0}=\tilde{V},\ \operatorname{div}\tilde{V}=0 \\ \rho\,|_{t=0}=\tilde{\rho} \end{cases}$$

where we assume, in addition, that V, ρ, p, $\tilde{V}$, and $\tilde{\rho}$ are all periodic in x with period 2π and independent of y, and that the velocities have a y-component zero. A similar problem in more general domains will be treated in a separate paper.

2. We let Ω denote the region: $-\infty<x<\infty$, $0\leq z\leq 1$ and Ω_1, the region: $0\leq x\leq 2\pi$, $0\leq z\leq 1$. Let $\mathfrak{D}(\Omega, 2\pi)$ denote the space of periodic functions on Ω, which, when considered as functions on the cylinder $S^1\times[0, 1]$ under the map $(x, z)\rightarrow(e^{ix}, z)$, are C^∞ and have compact support. If $\mathfrak{D}(\Omega, 2\pi)$ is given the usual inductive limit topology (cf. [3], Part. I), then the dual $\mathfrak{D}'(\Omega, 2\pi)$ is the space of periodic distributions on Ω. By $H^m(\Omega, 2\pi)$ we denote the space of those distributions $f\in\mathfrak{D}'(\Omega, 2\pi)$ which have all derivatives through m^{th} order in $L^2(\Omega_1)$. For $f(x, z)$ in $H^m(\Omega, 2\pi)$ we define the norm

$$\| f \|_m=\left(\sum_{j+l\leq m}\int_{\Omega_1}|D_x^jD_z^lf|^2dxdz\right)^{1/2}.$$

which makes $H^m(\Omega, 2\pi)$ a Hilbert space. The corresponding inner product is denoted by $(\cdot, \cdot)_m$. By $H^m[0, 1]$ we denote the Sobolev space of functions $f(z)$, $z\in[0, 1]$, having m derivatives in L^2 with the usual H^m norm, again denoted $\| f(z) \|_m$. We will also use $\| f \|$ to denote $\| f \|_0$ and (f, g) for $(f, g)_0$.

LEMMA 2.1 The space $H^m(\Omega, 2\pi)$ $(m=0, 1, 2, ...)$ consist of those f representable as a Fourier series $f(x, z)=\sum f_k(z)^{ikx}$ with $f_k\in H^m[0, 1]$ and satisfying

$$\sum_{k=-\infty}^{\infty}\sum_{p+q\leq m}\| k^pD_z^qf_k(z)\,^2\|<\infty. \tag{2.1}$$

The sum in (2.1) is equal to $\| f \|_m^2$.

PROOF. Suppose $f(x, z)\in H^m(\Omega, 2\pi)$. Using Fubini's theorem one sees that $f_k(z)=(2\pi)^{-1/2}\int_0^{2\pi} f(x, z)e^{-ikx}dx$ is measurable in z with $\| f_k \|\leq$ $\leq\| f(x, z) \|$. In fact

$$\sum_{k=-\infty}^{\infty}\| f_k \|^2\leq\| f \|, \tag{2.2}$$

from Bessel's inequality, and since linear combinations of functions of the form $\varphi(z)e^{ikx}$, $\varphi(z)\in C_0^\infty[0, 1]$, are dense in $L^2(\Omega_1)$, $f=\sum f_k(z)e^{ikx}$ with the series converging unconditionally in $L^2(\Omega_1)$. It follows that there is equality in (2.2). Since f is in $H^m(\Omega, 2\pi)$, for $p+q\leq m$, there exists a unique $f^{p,q} \in L^2(\Omega_1)$ for which

$$(f^{p,q}, \varphi) = (-1)^{p+q}(f, D_x^p D_z^q \varphi)$$

for any $\varphi \subset \mathfrak{D}(\Omega, 2\pi)$. Letting $\varphi = \varphi(z)e^{inx}$ with $\varphi(z) \in C_0^\infty[0, 1]$ we obtain

$$\int_0^1 f_n^{p,q}(z)\varphi(z)dz = (-1)^{p+q} \int_0^1 f_n(z)(in)^p D_z^p \varphi(z)dz$$

and hence, $f_n(z) \in H^m[0, 1]$ for each n with $(in)^p D_z^q f_n = f^{p,q}$ and $\| D_x^p D_z^q f \|_0^2 = \Sigma \| n^p D_z^q f_n^n \|^2$. Then $\| f \|_m^2$ is just the expression in (2.1). The reverse inclusion proceeds similarly.

By $H_j^m(\Omega, 2\pi)$ we denote the space of those $f \in H^m(\Omega, 2\pi)$ for which $D_z^q f = 0$ at $z = 0, 1$ (i.e. has zero trace; cf [4], p. 38) for $0 \leq q < j \leq m$ (letting $D_z^0 f = f$). It follows from theorem (3.10) of [4] that the spaces $H_j^m(\Omega, 2\pi)$ are closed subspaces of $H^m(\Omega, 2\pi)$.

We similarly define $H_j^m[0, 1]$ to be the subspace of functions $f(z) \in H^m[0, 1]$ *with* $f, Df, \ldots, D^{j-1}f$ vanishing at $z = 0$ and $z = 1$.

Lemma 2.2 For $m = 1, 2, \ldots, H_j^m(\Omega, 2\pi)$ consists of those $f = \Sigma f_k e^{ikx} \in H^m(\Omega, 2\pi)$ having $D_z^n f_k(0) = D_z^n f_k(1) = 0$ for all k and for $n = 0, 1, 2, \ldots, j-1$.

Proof. Since each $f_k(z)$ is continuous by Sobolev's imbedding theorem, the trace of f at $z = a$ is just $\Sigma f_k(a)e^{ik}$ (cf. [4], p. 38), which is 0 in L^2 if and only if $f_k(a) = 0$ for all k. A similar argument holds for derivatives.

We now define and derive some properties of operators that will be used in Section 3. We let $\mathfrak{D}(L_k) = H_2^4[0, 1]$ and for $w(z) \in \mathfrak{D}(L_k)$ define $L_k w = (D^2 - k^2)^2 w$ where here and in what follows D stands for D_z. Further, we let $\mathfrak{D}(M_k) = H_1^2[0, 1]$ and define $M_k w = (-D(1+z)D + k^2)w$. It is known (cf. [5], § 18) that L_k and M_k so defined are selfadjoint in $L^2[0, 1]$ for each k, are positive, and have compact inverses. In fact, for $f \in H$, $\| f \|^2 \leq \pi^2 \int_0^1 | Df |^2 dz \leq \pi^4 \int | D^2 f |^2 dz$ so

$$(L_k f, f) \geq \left(\frac{1}{\pi^2} + k^2 \right)^2 \| f \|^2. \tag{2.3}$$

LEMMA 2.3. There exists a constant $\eta_1 > 1$ such that for $f = \Sigma f_k e^{ikx}$ in $H_2^4(\Omega, 2\pi)$

$$\eta_1^{-1} \| f \|_4^2 \leq \Sigma_k \| L_k f_k \|^2 \leq \eta_1 \| f \|_4^2.$$

PROOF. We have

$$\| L_k f_k \|^2 = \| D^4 f_k \|^2 - 4k^2 Re(D^4 f_k, D^2 f_k) + 6k^4 \| D^2 f_k \|^2 + 4k^6 \| D f_k \|^2 + k^8 \| f_k \|^2$$

after reduction using integration by parts. Using

$$4k^2 \, | (D^4 f_k, D^2 f_k) | \leq \frac{2}{3} \| D^4 f_k \|^2 + 6k^4 \| D^2 f_k \|^2$$

it follows that

$$\| L_k f_k \|^2 \geq \frac{1}{3} \| D^4 f_k \|^2 + 4k^6 \| D f_k \|^2 + k^8 \| f_k \|^2.$$

Then since $(kD^3 f_k, kD^3 f_k) \leq \| D^4 f_k \|^2 + \| k^2 D^2 f_k \|^2$ and $(k^2 D^2 f_k, k^2 D^2 f_k) \leq \| D^4 f_k \|^2 + k^4 \| f_k \|^2$, it follows from Lemma 2.1 that $\eta_1^{-1} \| f \|_4^2 \leq \Sigma \| L_k f_k \|^2$ for η_1 sufficiently large. The remaining inequality is straightforward.

LEMMA 2.4. There exists a constant $\eta_2 > 1$ such that for $f = \Sigma f_k e^{ikx}$ in $H_2^2(\Omega, 2\pi)$,

$$\eta_2^{-1} \| f \|_2^2 \leq \Sigma \| L_k^{1/2} f_k \|^2 \leq \eta_2 \| f \|_2^2.$$

PROOF. The result follows using $\| L_k^{1/2} f_k \|^2 = (L_k f_k, f_k) = ((D^2 - k^2) f_k, (D^2 - k^2) f_k) = \| D^2 f_k \|^2 + 2k^2 \| D f_k \|^2 + k^4 \| f_k \|^2$ for $f_k \in H_2^4[0, 1]$ since $H_2^2[0, 1] = \mathfrak{D}(L_k^{1/2})$ is just the closure of $\mathfrak{D}(L_k)$ in the norm $(L_k f, f)^{1/2}$ (cf. [2], p. 153 and [6], Lemma 2).

LEMMA 2.5. For $k = \pm 1, \pm 2, \ldots$, $L_k^{-1/2} M_k L_k^{-1/2}$ acting in $\mathfrak{D}(L_k^{1/2})$ has a bounded selfadjoint closure (denoted by the same symbol) with

$$\| L_k^{-1/2} M_k L_k^{-1/2} \| \leq 2k^{-2} \tag{2.4}$$

PROOF. It follows from an interpolation theorem of E. Heinz (cf. [2]) that $M_k^{1/2} L_k^{-1/2}$ has a bounded closure and hence that $L_k^{-1/2} M_k L_k^{-1/2} = (M_{1/2}^k L_k^{-1/2})^* M_k^{1/2} L_k^{-1/2}$ is selfadjoint and bounded. For $f \in \mathfrak{D}(L_k)$ we have

$$\begin{aligned} \| M_k^{1/2} f \|^2 = (M_k f, f) &= \int_0^1 (-D(1+z)D + k^2) f \bar{f} dz \\ &= \int_0^1 ((1+z) \mid Df \mid^2 + k^2 \mid f \mid^2) \mathrm{dz} \\ &\leq \int_0^1 2(\mid Df \mid^2 + k^2 \mid f \mid^2) dz \\ &= 2 \int_0^1 (-D^2 + k^2) f \cdot \bar{f} dz \\ &\leq 2 \| (-D^2 + k^2) f \| \cdot \| f \| \\ &= 2 \| L_k^{1/2} f \| \cdot \| f \| \\ &\leq k^{-2} \| L_k^{1/2} f \|^2 + k^2 \| f \|^2 \\ &\leq 2k^{-2} \| L_{1/2}^k f \|^2. \end{aligned}$$

If follows by closure that

$$\| M_k^{1/2} f \| \leq \sqrt{2} k^{-1} \| L_k^{1/2} f \|$$

for $f \in \mathfrak{D}(L_k^{1/2})$, yielding (2.4).

LEMMA 2.6. For $w \in H_2^4[0, 1]$, $\| M_k w \| \leq \sqrt{10} k^{-2} \| L_k w \|$.

PROOF. We can write $M_k L_k^{-1} = M_k L_k^{-1/2} L_k^{-1/2}$ and from the identity in the proof of Lemma 2.4 we see that $\| M_k f \|^2 \leq 8 \| f \|_2^2 + 2k^4 \| f \|^2 \leq$ $\leq 10 \| L_k^{1/2} f \|^2$. Then since $\| L_k^{-1/2} \| \leq k^{-2}$ by inequality (2.3), the Lemma follows.

LEMMA 2.7. Let $P_i (i=1, 2)$ be positive selfadjoint operators with domains $\mathfrak{D}(P_1) = \mathfrak{D}(P_2)$ in a Hilbert space $\mathfrak{h}$. If there exists a constant

$\eta > 1$ such that

$$\eta^{-1}(P_2 f,\ f) \leq (P_1 f,\ f) \leq \eta(P_2 f,\ f)$$

for $f \in \mathfrak{D}(P_1)$, then $\mathfrak{D}(P_1^{1/2}) = \mathfrak{D}(P_2^{1/2})$.

PROOF. As noted above, $\mathfrak{D}(P_i^{1/2})$ is just the closure of $\mathfrak{D}(P_i)$ in the norm $(P_i f,\ f)^{1/2}$.

LEMMA 2.8 Let P and Q be positive selfadjoint operators in a Hilbert space $\mathfrak{h}$ and suppose $\mathfrak{D}(Q) \supset \mathfrak{D}(P^{1/2})$. If $P^{-1/2}QP^{-1/2}$ has a basis of eigenvectors (in particular, if P^{-1} is compact), then the problem $Pv = \lambda Qv$ has a set of eigenvalues λ_j and corresponding eigenvectors v_j forming an orthonormal basis in the Hilbert space $\mathfrak{D}(P^{1/2})$ where for $v,\ v' \in \mathfrak{D}(P^{1/2})$ the inner product is given by $\langle v,\ v' \rangle = (P^{1/2}v,\ P^{1/2}v')$, the latter inner product being that in $\mathfrak{h}$.

PROOF. The operator $P^{-1/2}QP^{-1/2}$ is selfadjoint and hence, under the hypotheses of the lemma, has an orthonormal basis of eigenvectors g_i with eigenvalues $\mu^i > 0$. One sees that each g^i is in $\mathfrak{D}(P^{1/2})$ and setting $P^{-1/2}g_i = v_i$ we have $v_i \in \mathfrak{D}(P)$ while $Pv_i = \lambda_i Qv_i$ with $\lambda_i = \mu_i^{-1}$. The expansion theorem in $\mathfrak{D}(P^{1/2})$ follovs easily from the $\{g_i\}$-expansion.

One calls a set of vectors $\varphi_n (n = 1,\ 2,\ \ldots)$ in a Hilbert space $\mathfrak{h}$, an unconditional basis if and only if each $f \in \mathfrak{h}$ has a unique unconditionally convergent expansion $f = \Sigma\, a_n \varphi_n$. We say φ_n spans $\mathfrak{h}$ if and only if $(\varphi_n,\ f) = 0$ for all n implies $f = 0$.

LEMMA 2.9. Let $\{e_n\}$ $(n > 0)$ be an orthonormal basis in $\mathfrak{h}$ and let $\{\varphi_n\}$ $(n = 0)$ be a set of vectors which span $\mathfrak{h}$. If $\Sigma_n \| e_n - \varphi_n \|^2 < \infty$, then $\{\varphi_n\}$ is an unconditional basis for $\mathfrak{h}$.

PROOF. Let n_0 be chosen so that $\Sigma_{n > n_0} \| e_n - \varphi_n \|^2 < 1$. Then if we let $\tilde{\varphi}_n = e_n$ for $1 \leq n \leq n_0$ and $\tilde{\varphi}_n = \varphi_n$ for $n > n_0$, we obtain $\sum_{n>0} \| e_n - \tilde{\Phi}_n \|^2 < 1$ and a theorem of Paley-Wiener (cf. [7], p. 208) insures that $\tilde{\varphi}_n$ is an unconditional basis. Then $\{\tilde{\varphi}_n\}$ $(n > n_0)$ is an unconditional basis of its closed linear span S which has codimension n_0 in λ. Let

S_0 be the linear span $\{\varphi_1, \ldots, \varphi_n\}$. If $S_0 \cap S = \{0\}$ then it is easy to see that $\{\varphi_n\}$ $(n>0)$ si an unconditional basis of $\mathfrak{h}$. If there were a vector $\varphi \neq 0$ in $S \cap S_0$, then the closed linear span of $\{\varphi_n\}$ $(n>0$ would have positive codimension in $\mathfrak{h}$, contradicting the assumption that $\{\varphi_n\}$ $(n>0)$ spans $\mathfrak{h}$

3. Existence and uniqueness. By $S(\Omega, 2\pi)$ we denote those vectors $V=(u(x, z), w(x, z))$ in $H_1^2(\Omega, 2\pi)\times H_1^2(\Omega, 2\pi)$ which are solenoidal; i.e., satisfy $D_x u + D_z w = 0$. Since $w(x, z) = \Sigma\, w_k(z) e^{ikx}$ and $u(x, z) = \Sigma\, u_k(z) e^{ikx}$ with u_k, $w_k \in H\,[0, 1]$ and since V is solenoidal, we see that $iku_k = -D_z w_k$ for all k. Thus w determines u except for the term $u_0(z)$, which can be any element of $H_1^2[0, 1]$, while $w_0(z) \equiv 0$. The space $S(\Omega, 2\pi)$ becomes a Hilbert space when given the inner product

$$\langle V, V' \rangle_S = \langle u, u' \rangle_2 + \langle w, w' \rangle_2$$

where $V=(u, w)$ and $V'=(u', w')$.

We let E denote the Hilbert space of equivalence classes of triples (V, ρ, p) in $S(\Omega, 2\pi)\times L^2(\Omega, 2\pi)\times H^1(\Omega, 2\pi)$ with (V_1, ρ_1, p_1) equivalent to (V_2, ρ_2, p_2) if and only $V_1 = V_2$, $\rho_1 = \rho_2$ and $p_1 - p_2 = p$, p being independent of x and z. The norm in E is

$$\| (V, \rho, p) \|_E^2 = \| V \|_S^2 + \| \rho \|^2 + \| D_x p \|^2 + \| D_z p \|^2.$$

THEOREM. Let $\tilde{V} = (\tilde{u}, \tilde{w}) \in S(\Omega, 2\pi)$ and $\tilde{\rho} \in L^2(\Omega, 2\pi)$ be given. Then for any $T>0$ there exists a unique triple $(V(x, z, t), \rho(x, z, t)$ $p(x, z, t))$ satisfying:

1) $(V, \rho, p) \in E$ for each $0 < t \leq T$.

2) The map $t \to (V, \rho, p) \in E$ is strongly differentiable for $0 < t < T$.

3) (V, ρ, p) satisfies equation (1.3) in the sense of $L^2(\Omega, 2\pi)$.

4) $\lim\limits_{t \searrow 0} (V(x, z, t), \rho(x, z, t)) = (\tilde{V}, \tilde{\rho})$ in the sense of $H^1(\Omega, 2\pi)\times H^1(\Omega, 2\pi)\times L^2(\Omega, 2\pi)$ for $(V, \rho) = (u, w, \rho)$.

PROOF. We write equation (1.3) in the form

$$
\left\{
\begin{array}{l}
(1+z)\dfrac{\partial u}{\partial t}=-D_x p+(D_x^2+D_z^2)u \\[2ex]
(1+z)\dfrac{\partial w}{\partial t}=-D_z p+(D_x^2+D_z^2)w-\rho \\[2ex]
\dfrac{\partial \rho}{\partial t}=-w \\[2ex]
D_x u+D_z w=0
\end{array}
\right.
\tag{3.1}
$$

and look for a solution:

$$
\left\{
\begin{array}{l}
u=\Sigma\,\widehat{u}_k(z,\ t)e^{ikx} \\
w=\Sigma\,\widehat{w}_k(z,\ t)e^{ikx} \\
\rho=\Sigma\,\widehat{\rho}_k(z,\ t)e^{ikx} \\
p=\Sigma\,\widehat{p}_k(z,\ t)e^{ikx}
\end{array}
\right.
\tag{3.2}
$$

with initial data

$$
\left\{
\begin{array}{l}
\tilde{u}=\Sigma\,\tilde{u}_k(z)e^{ikx} \\
\tilde{w}=\Sigma\,\tilde{w}_k(z)e^{ikx} \\
\tilde{\rho}=\Sigma\,\tilde{\rho}_k(z)e^{ikx}
\end{array}
\right.
\tag{3.3}
$$

Substituting the expressions (3.2) in equations (3.1) and equating Fourier coefficients we obtain

$$
\left\{
\begin{array}{l}
(1+z)\dfrac{\partial \widehat{u}_k}{\partial t}=-ik\widehat{p}_k+(D^2-k^2)\widehat{u}_k \\[2ex]
(1+z)\dfrac{\partial \widehat{w}_k}{\partial t}=-D\widehat{p}_k+(D^2-k^2)\widehat{w}_k-\widehat{\rho}_k \\[2ex]
\dfrac{\partial \widehat{\rho}_k}{\partial t}=-\widehat{w}_k \\[2ex]
ik\widehat{u}_k=-D\widehat{w}_k\,.
\end{array}
\right.
\tag{3.4}
$$

Suppose that $k \neq 0$ (we will return to the case $k=0$ later). Applying D to the first equation of (3.4) we can use the first and second equations to eliminate $\widehat{p}_k$. Then using the last equation to eliminate $\widehat{u}_k$ we arrive at

$$
(3.5) \qquad \begin{cases} \dfrac{\partial}{\partial t}(D(1+z)D-k^2)\widehat{w}_k=(D^2-k^2)^2\widehat{w}_k+k^2\widehat{\rho}_k \\ \dfrac{\partial}{\partial t}\widehat{\rho}_k=-\widehat{w}_k\,. \end{cases}
$$

Since both $\widehat{w}_k$ and $\widehat{u}_k$ must vanish at the boundary we see from the equation $ik\widehat{u}_k=-D\widehat{w}_k$ that we will want $\widehat{w}_k \in H_2^4[0, 1]$. Letting Φ_k denote the pair $(\widehat{w}_k, \widehat{\rho}_k)$ and defining the operator matrices

$$
P_k=\begin{bmatrix} L_k & k^2 \\ k^2 & 0 \end{bmatrix}; \qquad Q_k=\begin{bmatrix} M_k & 0 \\ 0 & k^2 \end{bmatrix},
$$

where L_k and M_k were defined in section 2, we can write (3.5) as

$$
(3.6) \qquad \left(P_k+\frac{\partial}{\partial t}Q_k\right)\Phi_k=0.
$$

Here $\mathfrak{D}(P_k)=\mathfrak{D}(L_k)\times L^2$ and $\mathfrak{D}(Q_k)=\mathfrak{D}(M_k)\times L^2$. It is easy to check that P_k and Q_k are selfadjoint in $L^2\times L^2$. If we seek a solution of the form $\Phi_k(z, t)=\Phi_k(z)e^{\lambda t}$, then (3.6) reduces to the relative eigenvalue problem

$$
(3.7) \qquad (P_k+\lambda Q_k)\Phi_k=0.
$$

We will show that (3.7) provides a basis of eigenvectors for $\mathfrak{D}(L_k^{1/2})\times L^2$.

We let

$$
P'_k=\begin{bmatrix} L_k & 0 \\ 0 & I \end{bmatrix}
$$

and $\tilde{P}_k=P_k+2k^{-2}Q_k$, with $\mathfrak{D}(P'_k)=\mathfrak{D}(\tilde{P}_k)=\mathfrak{D}(P_k)$. Both P'_k and $\tilde{P}_k$ are selfadjoint, and we note that replacing P_k by $\tilde{P}_k$ in (3.7) does not change the eigenfunctions. Using the inequality

$$2k^2 |(w, \rho)| \leq \frac{2}{3} k^4 \| w \|^2 + \frac{3}{2} \| \rho \|^2$$

one sees easily that

$$\frac{1}{3}(P'_k\Phi, \Phi) \leq (\tilde{P}_k\Phi, \Phi) \leq 6(P'_k\Phi, \Phi) \tag{3.8}$$

where the inner product of $\Phi=(w, \rho)$ and $\Phi'=(w', \rho')$ is $(w, w')+(\rho, \rho')$. It follows from (3.8) that $\tilde{P}_k$ is positive and, using Lemma 2.7, that $\mathfrak{D}(\tilde{P}_k^{1/2})=\mathfrak{D}((P'_k)^{1/2})=\mathfrak{D}(L_k^{1/2})\times L^2$. As a corollary

$$\| Q_k\tilde{P}_k^{-1/2} \| \leq \sqrt{10}k^2 \tag{3.9}$$

as a map in $L^2\times L^2$, recalling the proof of Lemma 2.6.

In addition we have

$$\| P_k\Phi \|^2 \geq \eta_3(k^{-4} \| L_k w \|^2 + \| \rho \|^2) \tag{3.10}$$

for $\Phi\in\mathfrak{D}(L_k)\times L^2$ and a constant η_3 independent of k.

To show this we write

$$\begin{aligned}
\| \tilde{P}_k\Phi \|^2 = {} & \| (L_k+2k^{-2}M_k)w+k^2\rho \|^2 + \| k^2 w+\rho \|^2 \\
= {} & \| L_k w \|^2 + \| 2k^{-2}M_k w \|^2 + \| k^2\rho \|^2 \\
& +2Re(L_k w, 2k^{-2}M_k w)+2Re(L_k w, k^2\rho) \\
& +2Re(2k^{-2}M_k w, k^2\rho)+ \| k^2 w \|^2 \\
& + \| \rho \|^2+2Re(k^2 w, \rho) \\
\geq {} & \| L_k w \|^2 + \| 2k^{-2}M_k w \|^2 + \| k^2\rho \|^2 \\
& -(4k^4+2)^{-1} \| L_k w \|^2-(4k^4+2) \| 2k^{-2}M_k w \|^2 \\
& -(1+1/2k^{-4})^{-1} \| L_k w \|^2-(1+1/2^{-4}) \| k^2\rho \|^2 \\
& -8 \| 2M_k w \|^2-1/8 \| \rho \|^2 \\
& + \| k^2 w \|^2+ \| \rho \|^2-8 \| k^2 w \|^2-1/8 \| \rho \|^2 \\
\geq {} & \eta_3(k^{-4} \| L_k w \|^2+ \| \rho \|^2
\end{aligned}$$

for all sufficiently large k, using $\| w \|^2 \geq k^{-8} \| L_k w \|^2$ and Lemma 2.6.

Since $\mathfrak{D}(\tilde{P})=\mathfrak{D}(L_k)\times L^2$ one sees from the closed graph theorem that for each k there is an $\eta_3=\eta_3(k)$ for which (3.10) holds. We can then choose η_3 sufficiently small so that it serves for all k. From (3.9) it follows immediately that

$$(3.11) \qquad \| P'_k\tilde{P}_k^{-1} \| \leq \eta_3^{-1}k^2$$

as a map in $L^2\times L^2$.

If we eliminate ρ from the pair of equations equivalent to (3.7) we obtain

$$(3.12) \qquad L_k w+\lambda M_k w=\frac{1}{\lambda}k^2 w.$$

Equation (3.12), in turn, is equivalent to

$$(3.13) \qquad A_k v-\lambda^2 B_k v-\lambda v=0$$

where $A_k=k^2L_k^{-1}$ and $B_k=L_k^{-1/2}M_kL_k^{-1/2}$ (cf. [2]). It is shown in [2] that (3.13) has a discrete « spectrum » consisting of a sequence of positive eigenvalues decreasing to zero and a sequence of negative eigenvalues decreasing to $-\infty$. The positive ones lie in the interval $[0, \| A_k \|]$ and the negative ones, in $(-\infty, -\| B_k \|^{-1})$. With the aid of inequality (2.3) and Lemma 2.5 we see that, for each k, the eigenvalues of (3.13) lie in $(-\infty, -k^2/2] \cup (0, k^{-2})$. Since the eigenvalues of L_k^{-1} converge to zero as $0(n^{-4})n=1, 2, \ldots$ ([5], § 4), A_k is in the compact operator class C_r (cf. [8], p. 1088) for any $r>1/4$. As such, from Theorem 2.24 of [9] we may conclude that the eigenvectors of (3.13) corresponding to positive eigenvalues form an unconditional basis in $L^2[0, 1]$ and hence, applying $L_k^{-1/2}$ to these vectors, that the corresponding eigenvectors of (3.12) form unconditional basis in $\mathfrak{D}(L_k^{1/2})$.

For each eigenvector v_{kn} of (3.13) we obtain eigenvectors $w_{kn}= =L_k^{-1/2}v_{kn}$ of (3.12) and $\Phi_{kn}=(w_{kn}, \lambda_{kn}^{-1}w_{kn})$ of (3.7). For each k, we let λ_{kn} run through the positive eigenvectors of (3.13) for $n=1, 2, \ldots$ (in decreasing order) and through the negative ones for $n=-1, -2, \ldots$.

Let $\mathfrak{L}\in\mathfrak{D}(\tilde{P}_k^{1/2})$ be the closed linear span of $\{\Phi_{kn}\}$ $(n\neq 0)$ where closure is with respect to the norm $\| P_k^{1/2}\Phi \|$ and the eigenvectors are normalized so that $(\tilde{P}_k\Phi_{kn}, \Phi_{kj})=\delta_{nj}$, *the* Kronecker delta. Let

$$\sigma = \inf_{\substack{\Phi\in\mathfrak{D}(\tilde{P}_k^{1/2})\\ \Phi\perp\mathfrak{L}}} \frac{(\tilde{P}_k^{1/2}\Phi,\ \tilde{P}_k^{1/2}\Phi)}{(Q_k\Phi,\ \Phi)} \tag{3.14}$$

where $(w, \rho)=\Phi\perp\mathfrak{L}$ means

$$(w,\ M_k w_{kn})+k^2(\rho,\ \lambda_{kn}^{-1}w_{kn})=0,\ n\neq 0. \tag{3.15}$$

If $\Phi\perp\mathfrak{L}$ then w determines ρ and in fact

$$\|\rho\| \leq C_k \|M_k^{1/2}w\| \tag{3.16}$$

for some constant C_k. To show this we require the result that the (normalized) set $\{w_{kn}\}$ $(n>0)$ provides an unconditional basis for $L^2[0, 1]$. Since the eigenvalues of L_k^{-1} are $0(n^{-4})$ it follows from the variational principles of [2] that $\lambda_{kn}=(n^{-4})$ for $n>0$. Now for k fixed, the operator $n=1, 2, \dots$. For each t, the eigenvalues approach the eigenvalues $L_k+tM_k(t\geq 0)$ has a sequence of eigenvalues which we write as $k^2\lambda_{kn}^{-1}(t)$, $\mu_{kn}(n>0)$ of L_k; that is, $|k^2\lambda_{kn}(t)-\mu_{kn}|\to 0$ as $n\to\infty$ (cf. [10]). The eigenvalues $\lambda_{kn}(n>0)$ of (3.12) are precisely the values $\lambda_{kn}(t)$ for which $\lambda_{kn}(t)=t$ and as such, can be put in one to one correspondence with μ_{kn} in such a way that $|\lambda_{kn}^{-1}-\mu_{kn}|\to 0$ as $n\to\infty$. Using the contour integral perturbation method one can show as in [11] that

$$\|w_{kn}-e_{kn}\| \leq d_k\cdot\lambda_{kn}(n>0) \tag{3.17}$$

where e_{kn} are the normalized eigenfunctions of L_k and d_k is a constant. Now, if there were an $f\in L^2$ satisfying $(f, w_{kn})=0$ for each $n>0$ (k still fixed), then $(L_k^{-1/2}f, L_k^{1/2}w_{kn})=0$ would follow and since $L_k^{1/2}w_{kn}$ spans L^2 we would obtain $L_k^{-1/2}f=0$, implying $f=0$. Since $\sum_{n>0}\|w_{kn}-e_{kn}\|^2\leq$ $\leq d_k^2\ \Sigma\,\lambda_{kn}^2<\infty$ we may appeal to Lemma 2.9 to conclude that $w_{kn}(n>0)$ is an unconditional basis of $L^2[0, 1]$.

Now a vector $\Phi=(w, \rho)$ perpendicular to $\mathfrak{L}$ certainly satisfies (3.15) for $n>0$ yielding

$$|(\rho,\ w_{kn})| = \frac{\lambda_{kn}}{k^2}\,|(M_k^{1/2}w,\ M_k^{1/2}L_k^{-1/2}L_k^{1/2}w_{kn})|$$

and since

$$\| M_k^{1/2} L_k^{-1/2} \| \leq \sqrt{2} k^{-1},$$

$$|(\rho, w_{kn})| \leq \sqrt{2} k^{-3} \lambda_{kn} \| M_k^{1/2} w \| \cdot \| L_k^{1/2} w_{kn} \|.$$

Since

$$\begin{aligned} \| L_k^{1/2} w_{kn} \|^2 &= (L_k w_{kn}, w_{kn}) = \\ &= -(\lambda_{kn} M_k w_{kn}, w_{kn}) + k^2 \lambda_{kn} \| w_{kn} \|^2 \\ &\leq k^2 \lambda_{kn}^{-1} \| w_{kn} \|^2, \end{aligned}$$

we obtain

$$|(\rho, w_{kn})| \leq \sqrt{2} k^{-1} \lambda_{kn}^{1/2} \| M_k^{1/2} w \|.$$

As w_{kn} is an unconditional basis, and hence is « similar » to an orthonormal basis (cf. [12])

$$\begin{aligned} \| \rho \|^2 &\leq \text{const} \sum_{n>0} |(\rho, w_{kn})|^2 \\ &\leq \text{const} \| M_k^{1/2} w \|^2 \cdot \sum_{n>0} \lambda_{kn} \\ &= c_k^2 \| M_k^{1/2} w \|^2 \end{aligned}$$

yielding (3.16).

Returning to (3.14) we note that if $\mathfrak{L} \neq \mathfrak{D}(\tilde{P}_k^{1/2})$, then $\sigma > 0$, since for $\Phi \neq 0$

$$\begin{aligned} \frac{(\tilde{P}_k^{1/2}\Phi, \tilde{P}_k^{1/2}\Phi)}{(Q_k\Phi, \Phi)} &\geq \frac{1}{3} \frac{((P_k')^{1/2}\Phi, (P_k')^{1/2}\Phi)}{(Q_k\Phi, \Phi)} \\ &\geq \frac{1}{6k^2} \frac{(Q_k\Phi, \Phi)}{(Q_k\Phi, \Phi)} \end{aligned}$$

using the proof of Lemma 2.5. Suppose that $\Phi_j = (w_j, \rho_j)$ is a minimizing sequence for (3.14), with $(Q_k\Phi_j, \Phi_j) = 1$. Inasmuch as $(Q_k\Phi_j, \Phi_j) = (M_k w_j, w_j) + k^2(\rho_j, \rho_j)$, $(M_k w_j, w_j)$ is bounded by 1 for all j and as M_k^{-1} is compact, a subsequence of the w_j, still called w_j, will converge strongly to w^0 in L^2. At the same time we know $\| L_k^{1/2} w_j \|^2$ must be bounded and since $M_k^{1/2} w_j = M_k^{1/2} L_k^{-1/2} L_k^{1/2} w_j$ with $M_k^{1/2} L_k^{-1/2}$ compact as map of L^2 (cf. [2], p. 153), $M_k^{1/2} w_j$ must have a convergent subsequence. Since $M_k^{1/2}$ is closed we can choose the subsequence so that it converges to $M_k^{1/2}w^0$. From (3.16) we see that ρ_j must converge to some $\rho^0 \in L^2$ giving $\Phi^0=(w^0, \rho^0)$ with $(Q_k\Phi^0, \Phi^0)=1$. We may also assume $L_k^{1/2}w_j$ converges weakly to $L_k^{1/2}w^0$ or that $\tilde{P}_k^{1/2}\Phi_j$ converges weakly to $\tilde{P}_k^{1/2}\Phi^0$. Since $\| \tilde{P}^{1/2}\Phi^0 \|^2 \leq \lim_j \| \tilde{P}_k^{1/2}\Phi_j \|^2$, we see that $\| \tilde{P}_k^{1/2}\Phi^0 \|^2=\sigma$. Then letting $\Psi^0=P_k^{1/2}\Phi^0$ we have

$$\frac{(\tilde{P}_k^{-1/2}Q_k\tilde{P}_k^{-1/2}\Psi^0, \Psi^0)}{(\Psi^0, \Psi^0)}=\sigma^{-1}$$

and

$$\frac{(\tilde{P}_k^{-1/2}Q_k\tilde{P}_k^{-1/2}\Psi, \Psi)}{(\Psi, \Psi)}\leq\sigma^{-1}$$

if $(\Psi, \tilde{P}_k^{1/2}\Phi_{kn})=0$ for all n. It then follows from standard variational arguments that

$$\tilde{P}_k^{-1/2}Q_k\tilde{P}_k^{-1/2}\Psi^0-\sigma^{-1}\Psi^0=0.$$

Since $Q_k\tilde{P}_k^{-1/2}$ is bounded, Ψ^0 is in $\mathfrak{D}(\tilde{P}_k^{1/2})$. Letting $\Phi^0=\tilde{P}_k^{-1/2}\Psi^0$ we see that $\Phi^0 \in \mathfrak{D}(\tilde{P}_k)$ and $\tilde{P}\Phi^0=\sigma Q_k\Phi^0$. But Φ^0 must then be one of the eigenvectors obtained from (3.13), requiring $(Q_k\Phi^0, \Phi^0)=0$, a contradiction. We can conclude, then, that the vectors Φ_{kn} (properly normalized) provide an orthonormal basis in $\mathfrak{D}(\tilde{P}_k^{1/2})$.

Suppose now that $k=0$. As we have already observed, we must have $w_0(z)\equiv 0$. Suppose $\tilde{u}_0(z)$ and $\tilde{\rho}_0(z)$ are given. Then since $\partial\widehat{\rho}_0/\partial t=0$, $\widehat{\rho}_0(z, t)=\widehat{\rho}_0(z)$ and $\widehat{P}_0(z, t)=-\int^z\widehat{\rho}_0(z)dz+c_0$ where c_0 is an arbitrary constant. The first equation of (3.3) becomes the diffusion equation

$$(1+z)\frac{\partial\widehat{u}_0(z, t)}{\partial t}=A_0\widehat{u}_0(z, t) \tag{3.18}$$

where A_0 denotes the selfadjoint operator with domain $\mathfrak{D}(A_0)= =H_1^2[0, 1]$ and defined by $A_0f=D^2f$. Since A_0^{-1} is known to be compact in L^2, it follows from Lemma 2.8 and well-known techniques (to be used below) that (3.18) has a solution in the L^2 sense, given by an eigenfunction expansion, and that the initial data is assumed in the sense of $\mathfrak{D}(A_0^{1/2})=H_1^1[0, 1]$ (cf. [2], p. 153).

As far as the existence is concerned, we will henceforth consider the case $k=0$ finished and consider only $k\neq 0$.

Having $(w_{kn}, \rho_{kn})=(w_{kn}, \lambda_{kn}^{-1}w_{kn})\in\mathfrak{D}(L_k)\times L^2$, we let

$$(3.19)\qquad \begin{cases} w_{kn}(x, z, t)=w_{kn}(z)e^{ikx+\lambda_{kn}t} \\ \rho_{kn}(x, z, t)=\rho_{kn}(z)e^{ikx+\lambda_{kn}t} \\ u_{kn}(x, z, t)=-(ik)^{-1}Dw_{kn}(x, z, t) \\ P_{kn}(x, z, t)=(ik)^{-1}((D^2-k^2)u_{kn}(x, z, t)-\lambda_{kn}(1+ \\ \qquad +z)u_{kn}(x, z, t). \end{cases}$$

The quadruple defined in (3.19) is a solution of (3.1) in the sense of $L^2(\Omega, 2\pi)$. Thus, provided it converges in the appropriate derivative norm, a sum

$$(3.20)\qquad \begin{bmatrix} u(x, z, t) \\ w(x, z, t) \\ \rho(x, z, t) \\ p(x, z, t) \end{bmatrix} = \sum_{k,\, n\neq 0} a_{kn} \begin{Bmatrix} u_{kn}(x, z, t) \\ w_{kn}(x, z, t) \\ \rho_{kn}(x, z, t) \\ p_{kn}(x, z, t) \end{Bmatrix}$$

will also be a solution. Consider now the pair

$$(3.21)\qquad \begin{bmatrix} \rho(x, z, t) \\ w(x, z, t) \end{bmatrix} = \sum_{k,\, n\neq 0} a_{kn}\Phi_{kn}(z)e^{ikx+\lambda_{kn}t}.$$

Since the initial vector $\tilde{V}=(\tilde{u}, \tilde{w})$ is in $S(\Omega, 2\pi)$, we can conclude from the solenoidal property and Lemma 2.2 that $\tilde{w}\in H_2^2(\Omega, 2\pi)$. We can expand $\tilde{\Phi}=(\tilde{w}, \tilde{\rho})$ as

$$\tilde{\Phi}=\sum_{k\neq 0}\Phi_k e^{ikx}=\sum_{k\neq 0}\begin{bmatrix} \tilde{w}_k(z) \\ \tilde{\rho}_k(z) \end{bmatrix} e^{ikx}$$

convergent in $H_2^2(\Omega, 2\pi)\times L^2(\Omega, 2\pi)$ with each $\tilde{w}_k(z)$ in $H_2^2[0, 1]$. Since for each k, $\Phi_{kn}(n\neq 0)$ is an orthonormal basis in $\mathfrak{D}(\tilde{P}_k^{1/2})= =H_2^2[0, 1]\times L^2$, if we let $a_{kn}=(\tilde{\Phi}_k, \tilde{P}_k\Phi_{kn})$, then using lemma 2.4 and inequality 3.8 we have

$$
\begin{aligned}
&\| \sum_{|k|<k_0} \sum_{n|<n_0} a_{kn}\Phi_{kn}(z)e^{ikx} \|^2_{H^2\times L^2}\\
&\leq \eta_2 \sum_{|k|<k_0} \| \sum_{|n|<n_0} (P'_k)^{1/2} a_{kn}\Phi_{kn} \|^2\\
&\leq 3\eta_2 \sum_{|k|<k_0} \| \sum_{|n|<n_0} \tilde{P}_k^{1/2} a_{kn}\Phi_{kn} \|^2\\
&= \eta_2 \sum_{|k|<k_0} \sum_{|n|<n_0} |a_{kn}|^2\\
&\leq 18\eta_2^2(\| \tilde{w} \|^2 + \| \tilde{\rho} \|_0^2)
\end{aligned}
$$

and hence

$$
\tilde{\Phi}(x, z) = \sum_{k\neq 0} \sum_{n\neq 0} a_{kn}\Phi_{kn}e^{ikx} \tag{3.22}
$$

convergent in $H^2(\Omega, 2\pi)\times L^2(\Omega, 2\pi)$. It follows immediately that with a_{kn} so defined, $w(x, z, t)$, $\rho(x, z, t)$ given by (3.20) approaches $(\tilde{w}, \tilde{\rho})$ in $H^2(\Omega, 2\pi)\times L^2(\Omega, 2\pi)$ as $t\to 0$. Further, from the definition of $u_{kn}(x, z, t)$ we see that $\Sigma\, a_{kn}u_{kn}(x, z, t)$ converges in $H^1(\Omega, 2\pi)$ to the function $\tilde{u}$. Thus the functions $(V(x, z, t), \rho(x, z, t))=(u(x, z, t), w(x, z, t), \rho(x, z, t))$ defined by (3.20) satisfy requirement 4) of the theorem. As regards item 1) of the theorem we see from (3.19) that $\Sigma\, a_{kn}u_{kn}(x, z, t)$ will converge in $H^2(\Omega, 2\pi)$ if $\Sigma\, a_{kn}k^{-1}\Phi_{kn}(x, z, t)$ converges in $H^3(\Omega, 2\pi)\times L^2(\Omega, 2\pi)$ and that $\Sigma\, a_{kn}p_{kn}(x, z, t)$ will converge in $H^4(\Omega, 2\pi)$ if $\Sigma\, a_{kn}k^{-2}\Phi_{kn}(x, z, t)$ converges in $H^4(\Omega, 2\pi)$ taking into account that the term $\lambda_{kn}(1+z)u_{kn}(x, z, t)$ in the expression for p_{kn} really has a multiplier $\lambda_{kn}e^{\lambda_{kn}t}$ which, for each $t<0$, is uniformly bounded in k and n and, thus does not effect the $H^1(\Omega, 2\pi)$ convergence of the expansion $\Sigma\, a_{kn}p_{kn}(x, z, t)$.

For conclusion 2) of the theorem it suffices to observe that a time derivative merely introduces a factor λ_{kn} which, as we have noted, is compensated for by $e^{\lambda_{kn}t}$, at least in the case of time derivatives of w and ρ. For the time derivative of u to exist in $H^2(\Omega, 2\pi)$ the convergence of $\Sigma\, a_{kn}k^{-1}\lambda_{kn}\Phi_{kn}(z)e^{ikx+\lambda_{kn}t}$ in $H^3(\Omega, 2\pi)\times L^2(\Omega, 2\pi)$

will suffice. Conclusion 3) of the theorem follows form 1) and 2) since the elementary summands are solutions.

We will show only the convergence of $\Sigma\, a_{kn}k^{-2}\Phi_{kn}(z)e^{ikx+\lambda_{kn}t}$ in $H^4(\Omega,\ 2\pi)\times L^2(\Omega,\ 2\pi)$. The convergence of $\Sigma\, a_{kn}k^{-1}\lambda_{kn}\Phi_{kn}e^{ikx+\lambda_{kn}t}$ in $H^4\times L^2$, and hence in $H^3\times L^2$, can be shown in a similar fashion. Recalling that $(P_k+\tilde{\lambda}_{kn}Q_k)\Phi_{kn}=0$ with $\tilde{\lambda}_{kn}=\lambda_{kn}-2k^{-2}$, we have

$$
\begin{aligned}
&\Big\|\sum_{|k|<k_0}\sum_{|n|<n_0} a_{kn}k^{-2}\Phi_{kn}(z)e^{ikx+\lambda_{kn}t}\Big\|_{H^4\times L^2}\\
&\le \eta_1\sum_{|k|<k_0}\Big\| L_k\sum_{|n|<n_0} a_{kn}e^{\lambda_{kn}t}k^{-2}w_{kn}(z)\Big\|^2\\
&+\eta_1\sum_{|k|<k_0}\Big\|\sum_{|n|<n_0} a_{kn}e^{\lambda_{kn}t}k^{-2}\rho_{kn}(z)\Big\|^2\\
&=\eta_1\sum_{|k|<k_0}\Big\| P'_k\sum_{|n|<n_0} a_{kn}e^{\lambda_{kn}t}k^{-2}\Phi_{kn}(z)\Big\|^2\\
&=\eta_1\sum_{|k|<k_0}\Big\| P'_k\tilde{P}_k^{-1}\sum_{|n|<n_0} a_{kn}e^{\lambda_{kn}t}k^{-2}\tilde{P}_k\Phi_{kn}(z)\Big\|^2\\
&\le\eta_1\eta_3^{-1}\sum_{|k|<k_0}\Big\|\sum_{|n|<n_0} a_{kn}e^{\lambda_{kn}t}\tilde{\lambda}_{kn}Q_k\Phi_{kn}(z)\Big\|^2\\
&=\eta_1\eta_3^{-1}\sum_{|k|<k_0}\Big\| Q_k\tilde{P}_k^{-1/2}\sum_{|n|<n_0} a_{kn}e^{\lambda_{kn}t}\tilde{\lambda}_{kn}\tilde{P}_k^{1/2}\Phi_{kn}\Big\|^2\\
&\le\eta_1\eta_3^{-1}\sup_{k,n}\big|\sqrt{2}k^2e^{\lambda_{kn}t}\tilde{\lambda}_{kn}\big|\sum_{|k|<k_0}\sum_{|n|<n_0}|a_{kn}|^2
\end{aligned}
$$

using Lemma 2.3, and the inequalities (3.9) and (3.11). Recalling that λ_{kn} lies in $(-\infty,\ -k^2/2)$ or in $(0,\ k^{-2})$ one sees that the last expression in (3.23) is bounded for each $t>0$ and hence that the series converges as desired. One can, of course, avoid using the term $e^{\lambda_{kn}t}$ to produce convergence and hence obtain convergence for $0\le t\le T$, uniformly in t, by placing further conditions on initial data resulting in a more rapid decreasse of a_{kn} to zero. This concludes the proof of the existence of a solution satisfying items 1)-4) of the theorem.

The proof of uniqueness is standard. If there were another solution $(u',\ w',\ \rho',\ p')$ then it would have a Fourier expansion satysfying (3.4) with primes on all the functions. Again we assume $k\neq 0$, the case $k=0$ being similar. Letting $\Phi'_k=(w'_k(z,\ t),\ \rho'_k(z,\ t)$ be the coefficients we would find

$$\left(P_k+\frac{\partial}{\partial t}Q_k\right)\Phi'_k$$

as before. Expanding $\Phi'_k(z, t)$ as $\Sigma\, a'_{kn}(t)\Phi_k e^{(z)}$ we would have to have

$$a'_{kn}(t)+\frac{\partial}{\partial t}\lambda_{kn}^{-1}a'_{kn}(t)=0 \qquad (t>0)$$

with $a'_{kn}(0)=a_{kn}$. Since the ordinary differential equation has a unique solution, we are finished.

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