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L. FUCHS

F. LOONSTRA

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ON DIRECT DECOMPOSITIONS OF TORSION-FREE
ABELIAN GROUPS OF FINITE RANK

L. FUCHS and F. LOONSTRA *)

1. In his book [5], I. Kaplansky asked: *if $G \oplus G$ and $H \oplus H$ are isomorphic, are G and H isomorphic?* B. Jónsson [4] has given an example of two torsion-free indecomposable abelian groups¹⁾ G and H of rank 2 which are not isomorphic, but their direct squares are isomorphic. Puzzled by this example, we have investigated the direct powers of torsion-free indecomposable groups, and have found that to every integer $m \geq 2$, there exist two indecomposable groups, G and H , of rank 2, such that the direct sum of n copies of G is isomorphic to the direct sum of n copies of H exactly if m divides n (Theorem 1). Moreover, we can choose G such that the direct sum of m copies of G decomposes into the direct sum of m pairwise non-isomorphic groups (Theorem 2)²⁾.

Any two direct decompositions of a torsion-free group of rank ≤ 2 are isomorphic: in fact, the group is either indecomposable or completely decomposable³⁾. However, it is an open problem whether or not a torsion-free group of rank $m \geq 3$ can have infinitely many, pairwise non-isomorphic direct decompositions. We make a step towards the solution in our Theorem 3 which asserts that to every $m \geq 3$ and every

*) Indirizzo degli AA.: L. Fuchs: Tulane University, Dept. of Mathematics, New Orleans, Louisiana 70118, U.S.A.

F. Loonstra: Technical University, Delft, The Netherlands.

¹⁾ All the groups in this note are additively written abelian groups. For the definition of rank and related concepts see e.g. Fuchs [2].

²⁾ For another type of pathological direct decomposition of torsion-free groups of finite rank, see Corner [1].

³⁾ See e.g. Fuchs [2].

large t there is a torsion-free group of rank m which has at least t non-isomorphic direct decompositions. This yields a negative solution to Problem 15 in Fuchs [3] ⁴).

2. We start off with the construction of groups needed in the proofs.

Let $A_i (i=1, \dots, m)$ be isomorphic groups of rank 1, and $B_i (i=1, \dots, m)$ isomorphic groups of rank 1 such that

$$(1) \quad \text{Hom}(A_i, B_i) = 0 = \text{Hom}(B_i, A_i)$$

and for their endomorphism rings ⁵)

$$(2) \quad \text{End } A_i \cong \mathbb{Z} \cong \text{End } B_i$$

holds. For instance, we can choose two disjoint, infinite sets of primes, say P_1 and P_2 , and define A_i as a subgroup of the rational vector space with basis a_i , generated by $p_1^{-1}a_i$ for all $p_1 \in P_1$:

$$A_i = \langle p_1^{-1}a_i \mid p_1 \in P_1 \rangle,$$

and similarly,

$$B_i = \langle p_2^{-1}b_i \mid p_2 \in P_2 \rangle$$

for $i=1, \dots, m$. In view of (2), we can choose elements $a_i \in A_i$ and $b_i \in B_i$ such that a_1, \dots, a_m and b_1, \dots, b_m correspond to each other under isomorphisms of the A_i and the B_i , respectively, and $p \nmid a_i$, $p \nmid b_i$ for some prime p (which will be suitably chosen later on). Then the groups

$$(3) \quad X_i = \langle A_i \oplus B_i, p^{-1}(a_i + b_i) \rangle \quad \text{for } i=1, \dots, m$$

are isomorphic among themselves, and as standard arguments show, they are indecomposable.

⁴) This problem asks for the existence of an integer $k(n)$ such that every torsion-free group of rank n has at most $k(n)$ non-isomorphic direct decompositions into indecomposable summands.

⁵) \mathbb{Z} denotes the ring of integers.

Let $C_i \cong A_i (i=1, \dots, m)$ and $D_i \cong B_i (i=1, \dots, m)$ be other sets of groups, and let $c_i \in C_i, d_i \in D_i$ correspond to a_i, b_i under some isomorphisms. Then for any choice of $k_i=1, \dots, p-1$, we can form the groups

$$(4) \quad Y_i = \langle C_i \oplus D_i, p^{-1}(c_i + k_i d_i) \rangle \quad \text{for } i=1, \dots, m.$$

These are again indecomposable. All X_i, Y_i are of rank 2.

Assume there is an isomorphism

$$\varphi : X^* = X_1 \oplus \dots \oplus X_m \rightarrow Y^* = Y_1 \oplus \dots \oplus Y_m.$$

From (1) we conclude that $A = A_1 \oplus \dots \oplus A_m$ and $B = B_1 \oplus \dots \oplus B_m$ are fully invariant subgroups of X^* , thus φ must map A onto $C = C_1 \oplus \dots \oplus C_m$ and B onto $D = D_1 \oplus \dots \oplus D_m$:

$$(5) \quad \varphi : a_i \mapsto \sum_{j=1}^m \alpha_{ij} c_j \quad \text{and} \quad b_i \mapsto \sum_{j=1}^m \beta_{ij} d_j$$

where owing to (2) the α_{ij}, β_{ij} are integers such that the matrices $\|\alpha_{ij}\|, \|\beta_{ij}\|$ are invertible, i.e. the determinants

$$(6) \quad \det(\alpha_{ij}) = \pm 1, \quad \det(\beta_{ij}) = \pm 1.$$

Notice that, for every i ,

$$(7) \quad \varphi : a_i + b_i \mapsto \sum_{j=1}^m \alpha_{ij}(c_j + k_j d_j) + \sum_{j=1}^m (\beta_{ij} - k_j \alpha_{ij}) d_j.$$

An isomorphism must map an element divisible by p upon an element of the same sort, thus from the independence of the d_j we obtain

$$(8) \quad \beta_{ij} - k_j \alpha_{ij} \equiv 0 \pmod{p}$$

for all i and j . From (6) and (8) it follows that

$$(9) \quad k_1 \dots k_m \equiv \pm 1 \pmod{p}.$$

Let us observe that (6) and (8) together guarantee that (5) and (7) yield an isomorphism between X^* and Y^* , since (6) implies that (5) induces an isomorphism between $A \oplus B$ and $C \oplus D$, while (8) implies, because of (7) and (6), that all $a_i + b_i$ are divisible by p if and only

if all the $c_j + k_j d_j$ are divisible by p .

3. Now we are ready to prove:

THEOREM 1. *Given any integer $m \geq 2$, there exist two torsion-free indecomposable groups X and Y , of rank 2 such that*

$$X \oplus \dots \oplus X \cong Y \dots \oplus \dots \oplus Y \quad (n \text{ summands})$$

if and only if

$$n \equiv 0 \pmod{m}.$$

We choose X as in (3) and Y as in (4) with $k_1 = \dots = k_m = k$ in order to have isomorphic groups Y_i . Our goal is to assure that the direct sums of m copies of X and Y , respectively, are isomorphic, but the same fails to hold for a smaller number of copies. In view of (9), this means

$$(10) \quad k^m \equiv -1 \pmod{p},$$

but

$$(11) \quad k^n \not\equiv \pm 1 \pmod{p} \text{ for } n = 1, \dots, m-1.$$

(We dropped the possibility $k^m \equiv 1$, because this can occur in the presence of (11) only for odd m). In order to get a solution for (10) and (11), let us choose the prime p so as to satisfy

$$p \equiv 1 \pmod{2m};$$

by Dirichlet's theorem on primes in arithmetic progressions, such a p does exist. If l is a primitive root mod p , then set $k = l^{\frac{p-1}{2m}}$; it is easily seen that both (10) and (11) hold.

Next we show α_{ij} and β_{ij} can be chosen so as to satisfy (6) and (8) with $k_j = k$ as chosen above. Let x satisfy $kx \equiv 1 \pmod{p}$ and choose, for instance, the lower triangular $m \times m$ -matrix

better result for odd integres m . Namely, to any odd m and any integer t , there exist t groups of rank 2 such that any two of them behave like X and Y in Theorem 1 (in this case we replace p by a product of primes). For even integers m we could construct t non-isomorphic groups such that the direct sums of m copies are isomorphic (but m is not necessarily the smallest number with this property).

4. Let us point out that for the direct sum of m copies of X in the last theorem we might have a completely different sort of decomposition:

THEOREM 2. *Given an integer $m \geq 2$, there exist pairwise non-isomorphic torsion-free indecomposable groups X, Y_1, \dots, Y_m of rank 2 such that*

$$X \oplus \dots \oplus X \cong Y_1 \oplus \dots \oplus Y_m \text{ (} m \text{ summands),}$$

but $X \oplus \dots \oplus X$ (n summands) is not isomorphic to the direct sum of n groups

$$Y_{i_1}, \dots, Y_{i_n} \text{ with } 1 \leq i_1 < \dots < i_n \leq m \text{ and } n < m.$$

Choose $X \cong X_i$ in (3) and the Y_i as in (4). In this case we choose incongruent k_i so as to satisfy

$$(12) \quad k_1 \dots k_m \equiv -1 \pmod{p},$$

but

$$(13) \quad k_{i_1} \dots k_{i_n} \not\equiv \pm 1 \pmod{p} \quad \text{for } 1 \leq n < m.$$

For instance, if we let $k_1 = k, k_2 = k^3, \dots, k_m = k^{3^{m-1}}$ for a k satisfying

$$k^{\frac{1}{2}(3^m - 1)} \equiv -1 \pmod{p}$$

such that no power of k with a smaller exponent is $\equiv \pm 1 \pmod{p}$, then (12) and (13) will be fulfilled. As in the proof of Theorem 1, we can find a prime p and a k with this property. We choose $\|\alpha_{ij}\|$ as earlier, but the x 's in the j th column will be replaced by $x^{3^j - 1}$, while

in the earlier form of $\|\beta_{ij}\|$, the k 's in the j th column ought to be replaced by $k^{3^{j-1}}$, and $y = (-1)^{m+1}(k^{\frac{1}{2}(3^m-1)} + 1)$. This completes the proof of Theorem 2.

REMARK 1. It is straightforward to show that our choice of k_i implies that

$$Y_{i_1} \oplus \dots \oplus Y_{i_n} \cong Y_{j_1} \oplus \dots \oplus Y_{j_n}$$

with

$$1 \leq i_1 < \dots < i_n \leq m, \quad 1 \leq j_1 < \dots < j_n \leq m$$

can hold only if

$$i_1 = j_1, \dots, i_n = j_n.$$

REMARK 2. The fact that the summands in Theorem 1 and 2 are of rank 2 is not essential. In fact, X_i in (3) can be replaced by

$$\langle A_i \oplus B_i \oplus \dots \oplus E_i, p^{-1}(a_i + b_i), \dots, p^{-1}(a_i + e_i) \rangle,$$

and Y_i in (4) by

$$\langle C_i \oplus D_i \oplus \dots \oplus F_i, p^{-1}(c_i + k_i d_i), \dots, p^{-1}(c_i + k_i f_i) \rangle$$

where every pair among A_i, B_i, \dots, E_i satisfies (1) and (2).

5. We proceed to the problem of constructing a group of rank $m \geq 3$ which has many direct decompositions. It is obvious that it suffices to consider the rank 3 case only.

THEOREM 3. *To any given integer t , there exists a torsion-free group of rank 3 which has at least t non-isomorphic direct decompositions.*

Let $A_1 \cong B_1$ and C be rank 1 groups whose endomorphism rings are isomorphic to Z and $\text{Hom}(A_1, C) = 0 = \text{Hom}(C, A_1)$. We choose elements $a_1 \in A, b_1 \in B$ and $c \in C$ which are not divisible by a given prime p (to be specified later) such that $a_1 \leftrightarrow b_1$ under some isomorphism.

Put

$$G = A_1 \oplus H_1 \text{ with } H_1 = \langle B_1 \oplus C, p^{-1}(b_1 + c) \rangle$$

where H_1 is indecomposable of rank 2.

If $\alpha_i, \beta_i, \gamma_i, \delta_i$ are integers such that $\alpha_i \delta_i - \beta_i \gamma_i = 1$, and if we put

$$a_i = \alpha_i a_1 + \beta_i b_1, \quad b_i = \gamma_i a_1 + \delta_i b_1 \quad (i=2, \dots, t),$$

then $A_1 \oplus B_1 = A_i \oplus B_i$ where A_i, B_i are the pure subgroups generated by a_i, b_i , and $a_1 \leftrightarrow a_i, b_1 \leftrightarrow b_i$ under suitable isomorphisms $A_1 \cong A_i, B_1 \cong B_i$. We wish to choose the A_i, B_i so that, for some integer $1 < k_i < p$, we have

$$(14) \quad G = A_i \oplus H_i \text{ with } H_i = \langle B_i \oplus C, p^{-1}(k_i b_i + c) \rangle$$

with indecomposable H_i . Then

$$k_i b_i + c = k_i \gamma_i a_1 + (k_i \delta_i - 1) b_1 + (b_1 + c)$$

must be divisible by p , thus

$$(15) \quad k_i \gamma_i \equiv 0 \text{ and } k_i \delta_i \equiv 1 \pmod{p}.$$

It is easily seen that conversely, (15) implies that (14) holds. If to any $1 < k_i < p$ we choose $\gamma_i = p, \alpha_i = k_i$ and β_i, δ_i so as to satisfy $k_i \delta_i - p \beta_i = 1$, then (15) will be satisfied. Notice that an isomorphism $\varphi_{ij} : H_i \rightarrow H_j$ must map B_i upon B_j and C upon itself such that $b_i \mapsto \pm b_j$ and $c \mapsto \pm c$. Consequently, φ_{ij} maps $k_i b_i + c$ upon $\pm(k_i b_j \pm c)$. Since divisibility by p is preserved under isomorphism and $p \nmid b_j$, we conclude that φ_{ij} can exist only if $k_i \equiv \pm k_j \pmod{p}$.

Therefore, if we choose

$$(k_1 = 1), k_2 = 2, \dots, k_t = t \text{ and } p > 2t + 1,$$

then $k_i \not\equiv \pm k_j \pmod{p}$ for $i \neq j$, and thus the groups H_1, \dots, H_t are pairwise non-isomorphic and indecomposable, establishing the existence of at least t non-isomorphic decompositions for G .

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