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L. CARLITZ

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# SOME APPLICATIONS OF SAALSCHÜTZ'S THEOREM

L. CARLITZ \*)

1. Put

$$(1) \quad {}_rF_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_r)_k}{k! (b_1)_k \dots (b_s)_k},$$

where

$$(a)_k = a(a+1) \dots (a+k-1), \quad (a)_0 = 1.$$

The formula

$$(2) \quad {}_3F_2 \left[ \begin{matrix} c, d \\ -n, a, b \end{matrix} \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}$$

where  $n$  is a nonnegative integer and

$$(3) \quad c+d = -n+a+b+1,$$

is Saalschütz's theorem [1, p. 9].

Changing the notation slightly, (2) becomes

$$(4) \quad \sum_{j=0}^k \frac{(-k)_j (a+k)_j (-a+b+c+1)_j}{j! (b+1)_j (c+1)_j} = \frac{(a-b)_k (a-c)_k}{(b+1)_k (c+1)_k}.$$

Then

$$\sum_{k=0}^{\infty} \frac{(a)_k (a-b)_k (a-c)_k}{k! (b+1)_k (c+1)_k} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^k \frac{(-k)_j (a)_{j+k} (-a+b+c+1)_j}{j! (b+1)_j (c+1)_j}$$

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\*) Indirizzo dell'A.: Depart. of Math., Duke Univ., Durham, North Carolina, U.S.A.

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$$= \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j} x^j \sum_{k=0}^{\infty} \frac{(a+2j)_k}{k!} x^k.$$

Since

$$\sum_{k=0}^{\infty} \frac{(a+2j)_k}{k!} x^k = (1-x)^{-a-2j},$$

we get the identity

$$(5) \quad \sum_{k=0}^{\infty} \frac{(a)_k(a-b)_k(a-c)_k}{k!(b+1)_k(c+1)_k} x^k = \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j} x^j (1-x)^{-a-2j}.$$

In particular if we take  $a = -2n$ ,  $x = 1$ , (5) becomes

$$(6) \quad \sum_{k=0}^{2n} \frac{(-2n)_k(-2n-b)_k(-2n-c)_k}{k!(b+1)_k(c+1)_k} = (-1)^n \frac{(2n)!(b+c+2n+1)_n}{n!(b+1)_n(c+1)_n}$$

In the hypergeometric notation (1), we have

$$(7) \quad {}_3F_2 \left[ \begin{matrix} -2n, & -2n-b, & -2n-c \\ & b+1, & c+1 \end{matrix} \right] = (-1)^n \frac{n!(b+1)_n(c+1)_n}{(2n)!(b+c+2n+1)_n},$$

which is the finite version of Dixon's theorem [1, p. 13].

We remark that, for  $x = -1$ , (5) reduces to

$$(8) \quad \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(a-b)_k(a-c)_k}{j!(b+1)_j(c+1)_j} = \sum_{j=0}^{\infty} \frac{(a)_{2j}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j} 2^{-a-2j}.$$

Also, for  $a = -n$ ,  $b = c = 0$ , (5) becomes

$$(9) \quad \sum_{k=0}^{\infty} \binom{n}{k}^3 x^k = \sum_{2j \leq n} \frac{(j!)^3(n-2j)!}{(n+j)!} x^j (1+x)^{n-2j}.$$

This formula, attributed to Foata, is proved in an entirely different way in [3, p. 41]. The special case

$$(10) \quad \sum_{k=0}^{\infty} \binom{n}{k}^3 = \sum_{2j \leq n} \frac{(n+j)!}{(j!)^3(n-2j)!} 2^{n-2j}$$

is due to MacMahon.

2. In the next place we consider the sum

$$(11) \quad S = \sum_{k=0}^{\infty} \frac{k!(b+1)_k(c+1)_k(d+1)_k(e+1)_k}{(a)_k(a-b)_k(a-c)_k(a-d)_k(a-e)_k} = \\ = {}_5F_4 \left[ \begin{matrix} a, a-b, a-c, a-d, a-e \\ b+1, c+1, d+1, e+1 \end{matrix} \right].$$

By (4) we have

$$S = \sum_{k=0}^{\infty} \frac{(a)_k(a-d)_k(a-e)_k}{k!(d+1)_k(e+1)_k} \sum_{j=0}^k \frac{(-k)_j(a+k)_j(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j} = \\ S = \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j(a-d)_j(a-e)_j}{j!(b+1)_j(c+1)_j(d+1)_j(e+1)_j} \cdot \\ \cdot \sum_{k=0}^{\infty} \frac{(a+2j)_k(a-d+j)_k(a-e+j)_k}{k!(d+j+1)_k(e+j+1)_k}.$$

If we take  $a = -2n$  the inner sum is

$${}_3F_2 \left[ \begin{matrix} -2n+2j, -2n-d+j, -2n-e+j \\ d+j+1, e+j+1 \end{matrix} \right]$$

which, by (7), is equal to

$$(-1)^{n-j} \frac{(2n-2j)!(d+e+2n+1)}{(n-j)!(d+j+1)_{n-j}(e+j+1)_{n-j}}.$$

Thus

$$S = (-1)^n \sum_{j=0}^n \frac{(-2n)_{2j}(b+c+2n+1)_j(-2n-d)_j(-2n-e)_j}{j!(b+1)_j(c+1)_j(d+1)_j(e+1)_j} \cdot \\ \cdot \frac{(2n-2j)!(d+e+2n+1)_{n-j}}{(nj)!(d+j+1)_{n-j}(e+j+1)_{n-j}} = \\ = (-1)^n \sum_{j=0}^n \frac{(2n)!(b+c+2n+1)_j(-2n-d)_j(-2n-e)_j(d+e+2n+1)_{n-j}}{j!(n-j)!(b+1)_j(c+1)_j(d+1)_j(e+1)_n} = \\ = (-1)^n \frac{(2n)!(d+e+2n+1)_n}{n!(d+1)_n(e+1)_n}.$$

$$\cdot \sum_{0=j}^n \frac{(-n)_j (b+c+2n+1)_j (-2n-d)_j (-2n-e)_j}{j! (b+1)_j (c+1)_j (-d-e-3-n)_j}.$$

If we assume that

$$(12) \quad b+c+d+e+5n+1=0,$$

the last sum reduces to

$$(13) \quad \sum_{j=0}^n \frac{(-n)_j (-2n-d)_j (-2n-e)_j}{j! (b+1)_j (c+1)_j}.$$

In view of (12), we can sum (13) by means of (2) to get

$$\frac{(b+d+2n+1)_n (c+d+2n+1)_n}{(b+1)_n (c+1)_n}.$$

We have therefore

$$(14) \quad S = (-1)^n \frac{(2n)! (b+d+2n+1)_n (c+d+2n+1)_n (d+e+2n+1)_n}{n! (b+1)_n (c+1)_n (d+1)_n (e+1)_n},$$

provided (12) is satisfied.

We may restate this result in the following form.

$$(15) \quad {}_5F_4 \left[ \begin{matrix} a, a-b, a-c, a-d, a-e \\ b+1, c+1, d+1, e+1 \end{matrix} \right] = \\ = (-1)^n \left[ \frac{(2n)! (b+d+2n+1)_n (c+d+2n+1)_n (d+e+2n+1)_n}{n! (b+1)_n (c+1)_n (d+1)_n (e+1)_n} \right] = \\ = (-1)^n \frac{(2n)! (b+e+2n+1)_n (c+e+2n+1)_n (d+e+2n+1)_n}{n! (b+1)_n (c+1)_n (d+1)_n (e+1)_n},$$

where  $a = -2n$  and

$$b+c+d+e+5n+1=0.$$

In particular, for  $b=c=d=0$ ,  $e = -5n-1$ , (15) reduces to

$$(16) \quad {}_5F_4 \left[ \begin{matrix} -2n, -2n, -2n, -2n, 3n+1 \\ 1, 1, 1, -5n \end{matrix} \right] = (-1)^n \frac{((3n)!)^3(4n)!}{(n!)^4((2n)!)^2(5n)!}.$$

If  $p=5n+1$  is prime, then (16) implies the following curious result.

$$(17) \quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^5 \equiv - \frac{1}{[n!(2n)!]^5} \pmod{p}.$$

For other congruences of this kind see [2].

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