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STATISTICAL STUDY OF NAVIER-STOKES EQUATIONS, I

C. FOIAŞ *)

ABSTRACT. This article is devoted to the study of the evolution of the statistical distribution in the velocity functional phase space associated with the initial value problem for the Navier-Stokes equations. This approach yields a rigorous mathematical treatment of turbulence in the framework of which existence and uniqueness theorems are proved for the statistical distribution.

The article has its origin and its core in the joint research done by G. Prodi and the author during 1968-1970 on these problems. Actually it constitutes the first complete exposition of this joint research.

The paper contains 9 paragraphs, among which the first one discusses in some detail the meaning of the present research while the second lists the main prerequisits concerning the stationary and nonstationary solutions of the Navier-Stokes equations (other facts concerning such individual solutions are scattered, where they are necessary, through the paper).

In § 3 we first establish the basic equation (see Eq. (3.13_{I})) for the evolution of the statistical distribution; then we construct, by an adequate Faedo-Galerkin method, a solution of the initial value problem for our basic equation (see Th. 1 in Sec. 3.2). Various improvements of this construction are given in Sec. 3.3 (Th. 2), Sec. 4.1 (Prop. 1), Sec. 4.4 (Th. 2), Sec. 5.3 (Th. 4) and Sec. 5.4 (Th. 5). In Sec. 3.4, we prove that the usual individual (weak) solution of the Navier-Stokes equations are precisely those solutions of our basic Eq. (3.13_{I}) which are Dirac measure valued. The next § 4 is devoted to the study of the energy relations for statistical distributions (see for instance Cor. 2 in Sec. 4.1) and some compactness consequences of the energy inequality (Th. 1 in Sec. 4.2). The § 5 constitutes one of the core of

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our research. It contains the uniqueness theorems for the evolution of the statistical distribution. These theorems are of two kinds (in a complete analogy with the usual nonstatistic behaviour of the solutions of Navier-Stokes equations), namely: For plane flows if, for the initial statistical distribution, flows with large kinetical energy have enough small probability to occur, then its evolution is uniquely determined in the future (Th. 1 in Sec. 5.1.*b*) and Th. 3 in Sec. 5.2); for three dimensional flows, this uniqueness holds only for the near future provided that moreover large enough energy vorticities have probability 0 with respect to the initial statistical distribution (Th. 2 in Sec. 5.1). Let us also mention Th. 6 in Sec. 5.4.*b*) which essentially shows that the evolution of the statistical distribution is uniquely determined under some mild conditions.

In § 6 we study the stationary (i.e. time independent) statistical distributions showing that such distributions automatically occur in the study of the time averages of individual solutions (see Th. 1 in Sec. 6.1.*b*) and Th. 2 in Sec. 6.2.*c*) as well as Prop. 7 in Sec. 6.3.6)). For plane fluids, the stationary statistical distributions are precisely the probabilities on the functional phase space, invariant to the functional flow which corresponds in this space to the initial value problem for the Navier-Stokes equations (see Sec. 6.2). Therefore in the next § 7 we study in more detail such invariant probabilities. Our main results concern the position in the functional phase space of the supports of the invariant probabilities (see Prop. 2 and Th. 2 in Sec. 7.2.6)) and an upper estimation of the functional dimension of these supports (see Th. 3 in Sec. 7.3). One should observe that these results are steps towards a rigorous mathematical proof that no fully developed turbulence exists for plane flows. Finally we show that statistically, plane flows behave as a random point in an *m*-dimensional space (see Prop. 3 in Sec. 7.4.*b*) and Prop. 4 in Sec. 7.4.*c*)) where the dimension *m* behaves similarly to the Reynolds number. In § 8 we give a consistent mathematical foundation for the Reynolds equations based on our basic Eq. (3.13_{I}) (see Sec. 8.1). Moreover for the time independent Reynolds equations we show that there exists a unique mean velocity majorizing the amount of energy transfered from the mean velocity to the turbulent fluctuations (see Sec. 2.*a*) and especially Th. 2). Finally in § 9, we establish Hopf's functional equation for turbulence as a consequence of our basic Eq. (3.13_{I}) (see Th. 1 in Sec. 9.1) and thus solve the initial value problem for Hopf's equation (see Th. 2 in Sec. 9.2.*a*) and Th. 3 in Sec. 9.2.*b*)).

§ 1. Introduction, comments and motivation.

1. In recent years much progress was made in some non linear problems of the theory of partial differential equations, as illustrated by Lions [1]. However many old problems in mathematical physics still have an incomplete rigorous solution in contemporary mathematics. A striking example of this kind is given by the *Navier-Stokes equations*

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u, \operatorname{grad})u = f - \operatorname{grad} p, \quad \operatorname{div} u = 0. \tag{1.1}$$

(Recall that u denotes the Eulerian fluid velocity, f the external body force, p the pressure and ν the viscosity while the density $=1$, u and f are R^n-valued functions in (t, x) belonging to an enough regular cylinder $[0, T] \times \Omega$ of R^{n+1}; here $\Omega \subset R^n$ and n is the so called dimension of the fluid or that of the spatial variable x in (1.1).) Indeed, concerning the initial value problem

$$u/(t=0) = u_0 \text{ (given)}, \quad u/\partial\Omega = 0 \tag{1.2}$$

for (1.1), the main question faced by Leray [2], [3] almost 40 years ago (namely the existence of a *unique* solution u for t in an interval of time independent of the initial data u_0) is, in case the spatial dimension n is ≥ 3, still without definite answer in spite of the researches of many mathematicians.

On the other hand the experimental study of turbulence leads rather to a statistical approach than to a deterministic one. Among the first attempts of connecting turbulence with the Navier-Stokes equations was made by Reynolds [1] in the following way: Consider that the velocity u is the sum of the velocity $\bar{u}$ of the mean flow and of the fluctuation δu of the velocity due to the turbulence. Then taking formally the mean values in (1.1) one obtains the *Reynolds equations*

$$\frac{\partial \bar{u}}{\partial t} - \nu\Delta\bar{u} + (\bar{u}, \operatorname{grad})\bar{u} = f - \overline{(\delta u, \operatorname{grad})\delta u} - \operatorname{grad} \bar{p}, \quad \operatorname{div} \bar{u} = 0, \tag{1.3}$$

where the additional term in the right member of the first equation (1.3) represents the average of the forces corresponding to the turbulent stresses due to the turbulent fluctuations of the velocity. Though these equations are not causal and rather mathematically inconsistent, they have been constantly used in the description of turbulent phenomena; hence much work was done by mathematicians in order to build a framework in which (1.3) shall have a consistent meaning (see for instance the survey by Kampé de Feriet [1]). Another more recent approach to turbulence was proposed by Hopf [3], consisting essentially in a statistical mechanics, based on the Navier-Stokes equations instead of the Hamiltonian canonical system, in which the basic equation concerns the evolution of the characteristic function of the statistical distribution of velocities. However, since no existence theorems have been proved, this new statistical mechanics was developed more as a formalism in theoretical physics than as a coherent mathematical theory.

Finally let us mention that in 1960, G. Prodi proposed a measure theoretical approach to the study of the Navier-Stokes equations (1.1)-(1.2), with the purpose of establishing the uniqueness of the solution u for all initial datae u_0 in a set of total measure with respect to certain measures suitably connected with the Navier-Stokes equations (see Prodi [3], [4]). Though this program is not yet achieved, the remark that the measures involved in these researches give a resonable coherent mathematical model for stationary turbulence was at the basis of the present paper. We shall try now to give a brief account of the contents and purposes of this article together with their possible significance in the theory of turbulence.

2. Let us begin by remarking that the phase space for (1.1)-(1.2) (i.e. the space of the initial data) is a real Hilbert space $\boldsymbol{N}$ of infinite dimension [1]), while the initial value problem (1.1)-(1.2) becomes the initial value problem for an equation of evolution in $\boldsymbol{N}$ of the following type

$$u'+Au+B(u,\ u)=g, \quad u(0)=u_0, \tag{1.4}$$

where u is a $\boldsymbol{N}$-valued function on $[0,\ T]$, u' is its derivative in t, g is a given $\boldsymbol{N}$-valued function while A is a positive selfadjoint operator and B is a bilinear mapping of the domain $\mathfrak{D}_A$ of A into $\boldsymbol{N}$. The equation (1.4) corresponding to (1.1)-(1.2) always has a solution (in a certain

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1) For the meaning of the objects involved in this section, which are not completely defined, see §§ 2-3.

weak sense) satisfying

$$u(\cdot)\in L^\infty(0,\ T;\ \boldsymbol{N})\cap L^2(0,\ T;\ \boldsymbol{N}^1) \tag{1.5}$$

where $\boldsymbol{N}^1$ stands for the domain of the square root $A^{1/2}$ of A. If in the equations (1.1) the space dimension n is $=2$, then this solution u is uniquely determined by its initial data u_0 and if $S(t)$ is the map $u_0 \mapsto u(t)$, then $\{S(t)\}_{t\ge 0}$ possesses nice smoothness properties; in particular

$$\int \Phi(S(t)u)d\mu(u) \tag{1.6}$$

is a smooth function in t for any (Borel) probability μ on $\boldsymbol{N}$ and for a suitably large class of real functionals $\Phi(\cdot)$ on $\boldsymbol{N}$. Now statistical mechanics postulates (with a very persuasive heuristic argument based on the identification of probabilities to frequencies) that if μ denotes the probability on phase space $\boldsymbol{N}$ giving the statistical distribution of the initial data, then the statistical distribution at the moment $t>0$ must be given by the probability μ_t defined by

$$\mu_t(\omega)=\mu(S(t)^{-1}\omega) \qquad (\omega \text{ Borel set } \subset \boldsymbol{N}). \tag{1.7}$$

Therefore (1.6) has to be equal to

$$\int \Phi(u)d\mu_t(u). \tag{1.6'}$$

Using this identification, we establish that, for rather a large class of real test functionals $\Phi(\cdot,\ \cdot)$ defined on $[0,\ T]\times \boldsymbol{N}$, and vanishing for t near T, the following equation holds

$$\begin{aligned} &-\int_0^T \left(\int_{\boldsymbol{N}} \frac{\partial\Phi}{\partial t} d\mu_t\right)dt+ \\ &+\int_0^T \left\{\int_{\boldsymbol{N}} [(Au,\ \Phi'_u(t,\ u))+(B(u,\ u),\ \Phi'_u(t,\ u))]\cdot d\mu_t(u)\right\}dt= \\ &=\int_{\boldsymbol{N}} \Phi(0,\ u)d\mu(u)+\int_0^T \left[\int_{\boldsymbol{N}} (g(t),\ \Phi'_u(t,\ u))d\mu_t(u)\right]dt, \end{aligned} \tag{1.8}$$

where Φ'_u stands for the Frechet derivative of Φ with respect to u. Equation (1.8) (or rather some equivalent form of it which is valid for a larger class of functionals Φ) makes sense even if the functional flow $\{S(t)\}$ is missing. Therefore, (1.8) *can be considered as the equation of evolution for the statistical distribution* μ_t *which for* $t=0$ *coincides with* μ. This equation deserves to be studied for its own sake, independently of the dimension n of the underlying domain Ω in (1.1)-(1.2).

Our first main result will be (see Sect. 3.3) that if $n=2$, 3 or 4, and, if the initial probability verifies

$$(1.9) \qquad \int_{\boldsymbol{N}} | u |^2 \, d\mu(u) \infty$$

($| u |$ denotes the $\boldsymbol{N}$-norm of u), then the equation (1.8) always has a solution $\{\mu_t\}$ satisfying also the properties

$$\int_{\boldsymbol{N}} | u |^2 \, d\mu_.(u) \in L^\infty(0, T) \text{ and } \int_{\boldsymbol{N}} | A^{1/2} u | \, d\mu_.(u) \in L^2(0, T)\,^{1)}.$$

Moreover one can choose the solution $\{\mu_t\}$ to verify an energy inequality (see § 4) which is stronger than that obtained by a formal integration with respect to μ of Leray's energy inequality for individual solutions. It is this strenghened version of the energy inequality which is useful in the study of the solutions of (1.8). For instance it implies that if the support of μ is bounded in $\boldsymbol{N}$ then the support of μ_t is uniformly (in t) bounded in $\boldsymbol{N}$; moreover it plays an essential role in the proof of the fact that the solutions of (1.8) (for a given μ) form a convex compact set in a convenient locally convex vector space; this suggests that perhaps a well chosen strictly convex functional of $\{\mu_t\}$ may select a unique natural solution. We shall illustrate this idea later on the Reynolds equations.

The study of (1.8) presents, as one should expect, similar aspects to that of (1.4) (i.e. (1.1)-(1.2)), especially in what concerns the pecu-

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1) It should be noted that $| u |$ (the norm in $\boldsymbol{N}$) is proportional to the kinetic energy corresponding to the velocity u; therefore (1.9) means that the initial mean kinetic energy is finite.

liarity arising in the passage from plane to spatial flows (i.e. from space dimension $n=2$ to $n=3$).

Let us give a sample of this peculiarity. In case $n=2$, at least if the support of the initial probability μ is bounded in $\boldsymbol{N}$, the only solution of (1.8) turns out to be that given by (1.7) (see § 5). Therefore if we start with initial data u_0, the statistical solution μ_t with initial value the Dirac measure δ_{u_0} concentrated in u_0 will be the Dirac measure $\delta_{S(t)u_0}$ concentrated in $S(t)u_0$. In the more interesting case $n=3$, any individual solution $u(t)$ with initial value u_0 yields a statistical solution $\delta_{u(t)}$ with initial value δ_{u_0} (see Sec. 3.4). Therefore if the uniqueness of the initial value problem for individual solution fails in the case $n=3$, the same will be true for the uniqueness of the initial value problem for statistical solutions, and even more, there would exist non-Dirac-measure-valued statistical solutions corresponding to initial Dirac measures. We dont't know if, conversely, this kind of behaviour of the statistical solution implies the non-uniqueness of the individual solutions.

Let us discuss more extensively the peculiarity of the three dimensional case, relating it to the theory of turbulence. For us, unlike Leray [2], [3], a turbulent non-stationary solution of the Navier-Stokes equations means a solution of (1.8) even if the initial date u_0 is determined, i.e. if $u=\delta_{u_0}$. This point of view agrees with that of Hopf [3] and Batchelor [1]. Now turbulence may be produced at least in two ways. The first one, depending on a high instability of the individual solutions, can occur even with the uniqueness of the problem (1.1)-(1.2) (whose proof in dimension $n=3$ is missing, perhaps, of the weakness of the presently available mathematical methods). This is certainly the case for the plane flows, as pointed out before; it is moreover consistent with the fact that in this case

$$(1.10) \qquad \int_{\boldsymbol{N}} |u-\overline{u(t)}|^2 \, d\mu_t(u) \leqq e^{c_2 \int_{\boldsymbol{N}} |u|^2 d\mu(u) \cdot \int_{\boldsymbol{N}} |u-\bar{u}|^2 d\mu(u)}$$

where c_1, c_2 are constants and

$$(1.11) \qquad \overline{u(t)}=\int_{\boldsymbol{N}} u \, d\mu_t(u), \quad \bar{u}=\int_{\boldsymbol{N}} u \, d\mu(u).$$

Indeed, (1.10) permits a very great increase of the dispersion of μ_t in a very small time even if the dispersion of μ is very small, once large initial mean energy $\int_N |u|^2 d\mu(u)$ is admitted. This kind of turbulence is obviously possible in the three dimensional case too. However, in the latter case we have in mind a second way by which turbulence may occur and which would be without analogue in the present day physical theories. More precisely, it is possible as pointed out before, that for the solution $\{\mu_t\}$ of (1.8) with initial data $\mu=\delta_{u_0}$ the measures μ_t shall no more be of Dirac type for all $u_0 \in \boldsymbol{N}$. This would be an *intrinsic turbulence* due, not to the impossibility of our precise determination of the initial data but to the absence of our smooth description of motion, on which the Navier-Stokes equations are based. In this description turbulence may be also the reflection of molecular phenomena. Unhappily we have been unable as yet to prove that this kind of intrinsic turbulence exists.

In any case our definition of a turbulent solution for Navier-Stokes equations permits a simple understanding of the Reynolds equations. Indeed if $\overline{u(t)}$ is the mean velocity defined by (1.11) then an easy computation gives

$$\overline{u(t)}'+A\overline{u(t)}+B(\overline{u(t)}, \overline{u(t)})=g-\int B(u-\overline{u(t)}, u-\overline{u(t)})d\mu_t(u) \tag{1.12}$$

which obviously is the functional form for the Reynolds equations (1.3), exactly as (1.4) is for (1.1)-(1.2). In this way we can consider that solving our equation (1.8) one obtains explicit solutions for the Reynolds equations. The compactness properties of the solutions $\{\mu_t\}$ for a given μ yield easily (see § 8) a unique mean velocity function $\overline{u(\cdot)}$ on $[0, t_0]$ (for any $t_0 \in [0, T)$) such that

$$\int_0^{t_0} | A^{1/2}\overline{u(t)} |^2 \, dt$$

shall be a minimum. In this way in the Reynolds equations the mean velocity is uniquely determined (even if the turbulence $\{\mu_t\}$ is not) by a certain supplementary extremum principle.

Our approach yields also solutions for the *Hopf's equation for turbulence* (see Hopf [3]). Indeed if $\{\mu_t\}$ is a solution of (1.8) then it is easy to establish (see § 9) that the functional

$$\Phi(t, v)=\int_N e^{i(u, v)}d\mu_t(u),$$

where $(t, v)\in[0, T)\times \boldsymbol{N}$ and $(\cdot, \cdot)$ denotes the scalar product in $\boldsymbol{N}$, is a weak solution of Hopf's equation. Thus our existence theorem, for (1.8), gives an existence theorem for the Hopf equation, provided the equation

$$\Phi(v)=\int_N e^{i(u, v)}d\mu(u), \qquad \text{for all } v\in \boldsymbol{N},$$

determines a μ satisfying (1.9).

3. Special attention is paid in the paper to the stationary turbulence, that is to the stationary (i.e. time-independent) solutions of (1.8) [1]. In dimension $n=3$, the results concerning stationary solutions of (1.8) essentially coincide with those in Prodi [4]; we have not succeeded in going beyond them. In dimension $n=2$, however we prove that the stationary solutions are exactly the invariant measures, i.e. measures μ satisfying

$$\mu(\omega)=\mu(S(t)^{-1}\omega) \qquad (\omega \text{ Borel set} \subset \boldsymbol{N}), \tag{1.7'}$$

and that they are carried by rather thin sets in $\boldsymbol{N}$ (see §§ 6-7). Moreover we shall prove that the motion given in $\boldsymbol{N}$ by $\{S(t)\}$ is from a probabilistic point of view isomorphic to that of a random particle in R^k, where the number k behaves in the same manner as the Reynolds number. A further justification of our way of regarding Reynolds equations is that if $u(t)=S(t)u$ is any solution of (1.4) and u^* is any cluster point of the time averages

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[1]) Of course in this case g will be taken independent of t.

$$\frac{1}{t}\int_0^t u(\tau)d\tau \text{ for } t \to \infty$$

then for a suitable invariant measure we have

$$u^* = \int u d\mu(u),$$

and obviously

$$Au^* + B(u^*, u^*) = g - \int_N B(u-u^*, u-u^*) d\mu(u)$$

which is the functional form of the time independent Reynolds equations. In this particular case there is a unique mean velocity u^* minimizing $|A^{1/2}u^*|$.

4. Some of the results of this paper were already announced by Foiaş [1] but this is the first time that they are published together with their complete proofs. Of course many questions remained open including unfortunately some of the most important ones such as that of the existence of intrinsic turbulence.

Finishing this introduction we wish to mention another such question: Is there any physically meaningfully principle enabling us in case $n=3$ to select a unique solution of (1.8)? Perhaps before answering this question one has to understand what happens with the uniqueness theorem for $\{\mu_t\}$ even in the case $n=2$ once no boundedness condition is imposed on the support of μ.

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As already stated, §§ 3-5 are the result of the joint research of G. Prodi with the author. The lattere expresses his deep gratitude to prof. Prodi for all he did concerning this research.

Also the § 1-3 of the present English version of the paper were written during the stay at Stanford University (October-December 1970) of the author whose work there was supported by the NSF (i.e. National Science Foundation). Expressing his gratitude to these Institutions (i.e. CNR and NSF) the author also thanks professors D. Gilbarg and J. L. Lions for their kind interest in the subject ».

§ 2. Preliminaries on individual solutions.

1. We begin by recalling the basic notions and definitions as well as the basic results on individual solutions of Navier-Stokes equations.

Let $\Omega \subset R^n$ be a *bounded* domain and let $\boldsymbol{L}^p$ denote the space of measurable vector-valued functions $u=(u_1, u_2, \ldots, u_n) \in R^n$ defined on Ω such that

$$| u |_p = \left| \int_\Omega \left(\sum_{i=1}^n u_i(x)^2 \right)^{p/2} dx \right|^{1/p} < \infty .$$

Let $\boldsymbol{H}^l$ $(l \geq 1)$ be the space of those $u \in \boldsymbol{L}^2$ which satisfy $D^\alpha u = (D^\alpha u_1, \ldots, D^\alpha u_n) \in \boldsymbol{L}^2$ for all $|\alpha| \leq l$. (Here $D^\alpha = \partial^{\alpha_1 + \cdots + \alpha_n} / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and the derivatives are taken in the sense of the theory of distributions). The norm in $\boldsymbol{H}^l$ will be the usual one:

$$\| u \|_l = \left(\sum_{|\alpha| \leq l} | D^\alpha u |_2^2 \right)^{1/2} .$$

Moreover we shall use the following notations:

$$(u, v) = \int_\Omega \sum_{i=1}^n u_i v_i dx \qquad (u, v \in \boldsymbol{L}^2)$$

$$((u, v)) = \int_\Omega \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx \qquad (u, v \in \boldsymbol{H}^1)$$

and

$$| u | = (u, u)^{1/2} (= | u |_2), \quad \| v \| = ((v, v))^{1/2} \qquad (u \in \boldsymbol{L}^2, v \in \boldsymbol{H}^1).$$

Let $\mathfrak{N}$ be the space of those vector-valued functions $u=(u_1, u_2, \ldots, u_n)$ defined on Ω such that $u_j \in C_0^\infty(\Omega)$ $(j=1, 2, \ldots, n)$ (i.e. u_j is a C^∞-function with compact support in Ω), and that

$$\operatorname{div} u = \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j} = 0 \text{ in } \Omega.$$

We shall denote by $\boldsymbol{N}$, resp. $\boldsymbol{N}^1$, the closure of $\mathfrak{N}$ in $\boldsymbol{L}^2$, resp. $\boldsymbol{H}^1$; moreover for $l>1$ we shall put $\boldsymbol{N}^l=\boldsymbol{N}^1 \cap \boldsymbol{H}^l$. Let D denote Friedrichs' selfadjoint extension (in $\boldsymbol{N}$) of the operator $-\Delta \mid \mathfrak{N}$ where Δ denotes the Laplace operator; note that D is the only selfadjoint operator ≥ 0 in $\boldsymbol{N}$ such that its domain $\mathfrak{D}_D$ is contained in $\boldsymbol{N}^1$ and satisfies $D \mid \mathfrak{N} = -\Delta \mid \mathfrak{N}$. It is plain that actually

$$\mathfrak{D}_{D^{1/2}} = \boldsymbol{N}^1 \text{ and } |D^{1/2} u| = \| u \| \text{ for } u \in \boldsymbol{N}^1.$$

Moreover since Ω is bounded, D^{-1} is compact in $\boldsymbol{N}$. Consequently there exists an orthonormal basis $\{w_n\}$ of $\boldsymbol{N}$ such that $Dw_n=\lambda_n w_n$ where $\lambda_n \geq 0$. By a suitable choice of the indices we can suppose that

$$0<\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \to \infty.$$

Obviously λ_1 can be also defined by

$$\lambda_1^{-1/2} = \sup \{ |u| / \|u\| : u \in \boldsymbol{N}^1 \}.$$

We shall denote by P_m the orthogonal projection of $\boldsymbol{N}$ onto the subspace spanned by $w_1, w_2, \ldots, w_m$ (for $m=0$, put $P_m=0$). These projections $P_m (m=0, 1, 2, \ldots)$ will play an essential role in the sequel.

For $l \geq 1$ let us denote by $\boldsymbol{N}^{-l}$ the conjugate space $(\boldsymbol{N}^l)^*$ where the duality extends that given by (u, v) (with $v \in \boldsymbol{N}^l$, $u \in \boldsymbol{L}^2$). This means that $\boldsymbol{L}^2$ is identified with a subspace of $\boldsymbol{N}^{-l}$ in such a way that the value of $u \in \boldsymbol{L}^2$, regarded in $\boldsymbol{N}^{-l}$, in an element $v \in \boldsymbol{N}^l$ is (u, v). For $v \in \boldsymbol{N}^l$ and $u \in \boldsymbol{N}^{-l}$ we shall keep the notation (u, v) for the value of u in the point v. With this notations we can write

$$((u, v)) = (D_e u, v) \text{ for all } u, v \in \boldsymbol{N}^1,$$

where D_e is a linear continuous operator from $\boldsymbol{N}^1$ into $\boldsymbol{N}^{-1}$ extending D.

2. We shall also consider the trilinear functional

$$b(u, v, w)=\sum_{i,j=1}^{n}\int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

defined whenever the integrals make sense; in particular, this is the case if $u, w \in \boldsymbol{L}^4$ and $v \in \boldsymbol{N}^1$, in which case we have also

$$|b(u, v, w)| \leq |u|_4 \,\|v\|\, |w|_4 . \tag{2.1}$$

In the sequel we shall always suppose that the dimension $n \leq 4$. In virtue of the following particular case of Sobolev's imbedding theorem

$$\boldsymbol{H}^1 \subset \begin{cases} \boldsymbol{L}^{\frac{2n}{n-2}} \text{ if } n \geq 3 \\ \boldsymbol{L}^q \text{ for all } 1 \leq q < \infty \text{ if } n=2 \end{cases} \tag{2.2}$$

(where the imbedding is continuous) we deduce from (1) the inequality

$$|b(u, v, w)| \leq c_1 \|u\| \cdot \|v\| \cdot \|w\| \qquad (u, v, w \in \boldsymbol{N}^1)\,^{1)} \tag{2.3}$$

i.e. b is continuous on $\boldsymbol{N}^1$. For $n=2$ or 3 we shall also use the following obvious inequality

$$\left.\begin{matrix} |b(u, v, w)| \\ |b(w, v, u)| \end{matrix}\right\} \leq |u|_p \,|\operatorname{grad} v|_{\frac{2p}{p-2}} \,|w| \tag{2.3'}$$

where

$$|\operatorname{grad} v|_q = \left\{ \int_{\Omega} \left[\sum_{i,j=1}^{n} \left(\frac{\partial v_i}{\partial x_j} \right)^2 \right]^{q/2} dx \right\}^{1/q} .$$

In case $n=2$, the inequality (Ladyzenskaya [1]; see Lions [1], Ch. I, § 6)

$$|u|_4 \leq 2^{1/4} |u|^{1/2} \cdot \|u\|^{1/2} \qquad (u \in \boldsymbol{N}^1) \tag{2.2'}$$

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1) Here c_1 denotes a constant not depending on u, v, w; in the sequel c_2, c_3, ... will denote similar constants.

gives readily, by (2.1.),

$$|b(u, v, w)| \leq 2^{1/2} |u|^{1/2} \cdot \|u\|^{1/2} \cdot |w|^{1/2} \cdot \|w\|^{1/2} \cdot \|v\| \tag{2.3''}$$

for all $u, v, w \in N^1$. Finally let us remark that

$$b(u, v, w) = -b(u, w, v) \qquad (u, v, w \in \boldsymbol{N}^1) \tag{2.4}$$

since this relation is immediate if $u, v, w \in \mathfrak{N}$. In virtue of (2.3) there exists e continuous bilinear map B of $\boldsymbol{N}^1 \times \boldsymbol{N}^1$ into $\boldsymbol{N}^{-1}$ such that

$$b(u, v, w) = (B(u, v), w) \qquad (u, v, w \in \boldsymbol{N}^1). \tag{2.4'}$$

With these notations, let us now recall the definition of a (weak) *stationary solution of the Navier-Stokes equations with right term* $f \in \boldsymbol{L}^2$ independent of t. By definition, such a solution is an element $u \in \boldsymbol{N}^1$ satisfying

$$D_e u + B(u, u) = Pf \qquad \text{in } \boldsymbol{N}^{-1}; \tag{2.5}$$

here P denotes the orthogonal projection of $\boldsymbol{L}^2$ onto $\boldsymbol{N}$. Obviously any such solution satisfies the following energy equation

$$\nu \|u\|^2 = (f, u). \tag{2.6}$$

A stationary solution always exists (Leray [1], Ladyzenskaya [2]; see also Lions [1], Ch. I, § 7). If $\|u\|$ is sufficiently small (for instance this is the case if $\nu^{-1}\lambda_1^{-1/2}|f|$ is small enough), then the solution u is unique. In the case $n=2$ and if the boundary $\partial\Omega$ of Ω is of class C^2 then for m large enough, P_m is on the set $\boldsymbol{S}$ of all possible stationary solutions (with a fixed f) a homeomorphic map from $\boldsymbol{S}$ into $P_m\boldsymbol{N}$; thus $\boldsymbol{S}$ is a compact of $\boldsymbol{N}$ of finite dimension (see Foiaş-Prodi [1]).

3. To recall the fact concerning non-stationary solutions of the Navier-Stokes equations let us introduce the following notations. For a given Banach space $\boldsymbol{B}$, we shall denote by $L^p(a, b; \boldsymbol{B})$ $(1 \leq p \leq \infty)$ the usual Lebesgue space L^p of the (classes of) $\boldsymbol{B}$-valued functions defined on (a, b), endowed with the usual norm. That is, $f \in L^p(a, b; \boldsymbol{B})$

means that f is strongly measurable and that

$$\| f \|_{L^p(a, b; \boldsymbol{B})} = \begin{cases} \left[\int_a^b \| f(t) \|_{\boldsymbol{B}}^p dt \right]^{1/p} & \text{if } p < \infty \\ \underset{a<t<b}{l \cdot u \cdot b} \| f(t) \|_{\boldsymbol{B}} & \text{if } p = \infty \end{cases}$$

is finite. $L^p_{\text{loc}}(a, b; \boldsymbol{B})$ will denote the space of those functions g defined on (a, b) such that $g \mid (a_1, b_1) \in L^p(a_1, b_1; \boldsymbol{B})$ for all intervals $[a_1, b_1] \subset \subset (a, b)$. If $\boldsymbol{B} = R$, we shall write simply $L^p(a, b)$ and $L^p_{\text{loc}}(a, b)$.

Concerning the Navier-Stokes equations, *we shall suppose moreover that the right term* $f(t, \cdot) = (f_1(t, \cdot), \ldots, f_n(t, \cdot))$ *belongs to* $L^2(0, T; \boldsymbol{L}^2)$. By definition, *a* (*weak non-stationary*) *solution of Navier Stokes equations* (in $(0, T) \times \Omega$) with initial value $u_0 \in \boldsymbol{N}$ (see (1.1)-(1.2)) is a function

$$u(\cdot) \in L^2(0, T_1; \boldsymbol{N}^1) \cap L^\infty(0, T_1; \boldsymbol{N}) \tag{2.7}$$

for every $T_1 \in (0, T)$, satisfying the equation

$$\int_0^T [-(u(t), v'(t)) + \nu((u(t), v(t))) + b(u(t), u(t), v(t))] dt = \tag{2.8}$$
$$= (u_0, v(0)) + \int_0^T (f(t), v(t)) dt$$

for all functions $v(\cdot)$ verifying the following conditions:

(i) $v(\cdot) \in C([0, T); \boldsymbol{N}^1)$, i.e. it is continuous from $[0, T)$ to $\boldsymbol{N}^1$,

(ii) $v(\cdot)$ is differentiable (in $\boldsymbol{N}$) and its derivative $v'(\cdot)$ belongs to $L^2(0, T; \boldsymbol{N})$,

(iii) $v(\cdot)$ has compact support in $[0, T)$.

Obviously, the integrals in (2.8) make sense. Note also that it is sufficient to verify (2.8) for functions $v(\cdot)$ of the particular form $\varphi(\cdot)v$ with $v \in \boldsymbol{N}^1$ and $\varphi(\cdot)$ a real continuously differentiable function with compact support in $[0, T)$. (Indeed if $v(\cdot)$ satisfies (i)-(iii), then $v(\cdot)$, resp. $v'(\cdot)$, can be approached in $C([0, T); \boldsymbol{N}^1)$, resp. $L^2(0, T; \boldsymbol{N})$, by the functions

$$w_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} v(\tau)\,d\tau \quad \text{resp.}\ w'_\varepsilon(t) \qquad (t \in [0, T))$$

which for $\varepsilon > 0$ small enough belong to $C_0^1([0, T); \boldsymbol{N}^1)$, i.e. are $\boldsymbol{N}^1$-valued continuously differentiable functions with compact support in $[0, T)$. Such functions can be approached in $C_0^1([0, T); \boldsymbol{N}^1)$ by sums of functions of the form $\varphi(\cdot)v$ indicated above.)

One has to point out the relation between (2.8) and (1.4). Put $\nu D_e = A_e$ and $\nu D = A$. Then it is not hard to prove (in virtue of the preceding remark) that $u(\cdot)$ is a solution in the sense of the above given definition if and only if $u(\cdot)$ verifies (2.7) and as $\boldsymbol{N}^{-1}$-valued function is an absolutely continuous function whose derivatives $u'(\cdot)$ verifies a.e. (i.e. almost everywhere with respect to the Lebesgue measure) on $(0, T)$ the differential equation

$$u'(t) + A_e u(t) + B(u(t), u(t)) = Pf(t) \tag{2.9}$$

and the initial condition

$$u(0) = u_0 . \tag{2.10}$$

The relations (2.9)-(2.10) are to be considered in $\boldsymbol{N}^{-1}$. Obviously in case $f(t)$ does not depend on t, a stationary solutions is precisely a solution $u(t)$ which does not depend on t.

The basic result due to Leray [2], [3], Hopf [2] (see also Lions [1], Ch. I, § 6) is that for any initial data $u_0 \in \boldsymbol{N}$, there exists a solution of the Navier-Stokes equations (i.e. satisfying (2.7) and (2.8) for any $v(\cdot)$ with properties (i)-(iii), or equivalenty satisfying (2.7) and (2.9)-(2.10)). Moreover this solution can be chosen in such way that the following *energy inequality* holds

$$\frac{1}{2}\,|\,u(t)\,|^2 + \nu \int_0^t ||\,u(\tau)\,||^2\,d\tau \leq \frac{1}{2}\,|\,u_0\,|^2 + \tag{2.12}$$

$$+\int_0^t (f(\tau),\ u(\tau))d\tau \qquad \text{a.e. in } (0,\ T).$$

In case $n=2$, any solution can be considered as belonging to $C([0,\ T);\ \boldsymbol{N})$ and satisfying the *energy equation*

$$\frac{1}{2}\,|\,u(t)\,|^2+\nu\int_0^t \|\,u(\tau)\,\|^2\,d\tau=\frac{1}{2}\,|\,u_0\,|^2+ \tag{2.13}$$

$$+\int_0^t (f(\tau),\ u(\tau))d\tau \quad \text{everywhere in } (0,\ T)$$

(Prodi [2]). Moreover this solution is uniquely determined (Lions- Prodi [1]; see Lions [1], Ch. I, § 6.2).

4. In the sequel *we shall always suppose that the boundary* $\partial\Omega$ *of* Ω *is of class* C^2. This assumption implies, in virtue of a deep result concerning the linearized Navier-Stokes equations (see Cattabriga [1] [1]), Vorovich-Yudovich [1]), that the domain $\mathfrak{D}_D$ of D coincides with $\boldsymbol{N}^2$ and thus

$$c_2\,|\,u\,|_{\boldsymbol{N}^2}\leq|\,Du\,|\leq c_2^{-1}\,|\,u\,|_{\boldsymbol{N}^2} \text{ for } u\in\boldsymbol{N}^2. \tag{2.14}$$

In particular this inequality allows us to connect $|\,\mathrm{grad}\,u\,|_q$ (which occurs in (2.3′)) with $|D^\alpha u|$ for suitable $\alpha>0$. Indeed, the map D_j $(j=1,2,\ldots,n)$ being continuous

$$\text{from } \boldsymbol{N}^1 \text{ into } \boldsymbol{N}\subset\boldsymbol{L}^2,$$

$$\text{from } \boldsymbol{N}^2=\mathfrak{D}_D \text{ into } \boldsymbol{N}^1 \text{ thus in } \boldsymbol{L}^{\frac{2n}{n-2}}$$

for $n=3$, 4 by Sobolev's inequality (2.2), we can deduce, using the

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1) The particular case which interests us is the following: Let $\partial\Omega$ be of class C^l, $l\geq 2$ and let $u\in\boldsymbol{N}^{l-2}$ (where $\boldsymbol{N}^0=\boldsymbol{N}$). Then there exists $v\in\mathfrak{D}_D\cap\boldsymbol{N}^l$ such that $Dv=u$.

Interpolation Theory (see Lions-Peetre [1]), that D_j will map continuously the domain of the power $\frac{1}{2}\theta+1-\theta$ of D into $\boldsymbol{L}^q$ where

$$\frac{1}{q}=\frac{1}{2}\theta+\left(\frac{2n}{n-2}\right)^{-1}(1-\theta).$$

That gives

$$(2.15)\qquad |\operatorname{grad} u|_q \leq c_3 \left| D^{\frac{1}{2}+n\frac{q-2}{4q}} u \right|$$

for any u belonging to the domain of $D^{\frac{1}{2}+n\frac{q-2}{4q}}$ and any $q\in\left[2, \frac{2n}{n-2}\right]$ ($n=3, 4$). Let us note two particular significant cases of (2.15), namely

$$(2.15')\qquad |\operatorname{grad} u|_q \leq c_3 |D^{3/4}u| \leq c_3 |Du|^{1/2} \|u\|^{1/2}$$

where $u\in\mathfrak{D}_D$ is arbitrary and

$$(2.15'')\qquad q=3 \text{ if } n=3\,^{1)},\ q=\frac{8}{3} \text{ if } n=4.$$

We can supplement (2.15')-(2.15'') in case $n=2$ with

$$(2.15''')\qquad |\operatorname{grad} u|_4 \leq c_4 |Du|^{1/2} \|u\|^{1/2} \text{ for } u\in\mathfrak{D}_D$$

(see Foiaş-Prodi [1], p. 11). Using (2.15''') together with (2.3') one obtains for any solution $u(\cdot)$ in dimension $n=2$ the following estimate

$$(2.16)\qquad \|u(t)\| \leq c_5\left(\frac{|u_0|}{t^{1/2}}+1\right) \exp\{c_6(|u_0|^2 e^{-\gamma_1 \nu t}+1)^2\}$$

for all $t\in(0, T)$, whenever $f(\cdot)\in L^\infty(0, T; \boldsymbol{L}^2)$. In case $T=\infty$ this obviously gives

$$(2.16')\qquad \lim_{t\to\infty} \|u(t)\| \leq c_7$$

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[1]) See Prodi [4], [6].

(where c_7 is a constant depending on Ω, ν, $\| f \|_{L^\infty(0,\infty;\, \boldsymbol{L}^2)}$ but independent of $u(\cdot)$). For (2.16)-(2-16′) and other consequences of (2.15‴) consult Foiaş-Prodi [1]. In case $n=3$, the inequality (2.15′), leads to the existence of a unique solution $u(\cdot)$ with initial value $u_0 \in \mathbf{N}^1$ defined on an interval $[0, t(u_0))$ of (2.9)-(2.10), where the end $t(u_0)$ satisfies an inequality

$$t(u_0) \geq c'_7(\| u_0 \|^2 + 1)^{-2},$$

c'_7 being a constant depending only on Ω, ν and $\| f \|_{L^\infty(0,T;\, \boldsymbol{L}^2)}$, where $f(\cdot)$ is supposed in $L^\infty(0, T;\, \boldsymbol{L}^2)$ (see Prodi [4], [6]).

5. We shall finish this paragraph by pointing out that the initial value problem for the Navier-Stokes equations can be considered as a particular case of the following: Let A be positive selfadjoint operator in a real Hilbert space $\boldsymbol{N}$, let $\boldsymbol{N}^\alpha$ ($\alpha=0, 1, 2$) denote the domain of $A^{\alpha/2}$ normed with (an equivalent norm to) the natural norm of $\mathfrak{D}_{A^{\alpha/2}}$ (i.e. $u \to | A^{\alpha/2}u |$) and let $\boldsymbol{N}^{-\alpha}$ be the dual space of $\boldsymbol{N}^\alpha$ the duality extending that of $\boldsymbol{N}^0$ to $\boldsymbol{N}^0$. Let moreover $B(\cdot, \cdot)$ a bilinear map of $\boldsymbol{N}^1 \times \boldsymbol{N}^1$ into $\boldsymbol{N}^{-1}$ which extends by continuity from $\boldsymbol{N} \times \boldsymbol{N}^1$ and $\boldsymbol{N}^1 \times \boldsymbol{N}$ to $\boldsymbol{N}^{-2}$ and which verifies

$$(B(u, v), v)=0 \tag{2.17}$$

(whenever the left term makes sense). Let moreover $f(\cdot) \in L^2(0, T;\, \boldsymbol{N})$. Then for any $u_0 \in \boldsymbol{N}$ there exists a function

$$u(\cdot) \in L^\infty(0, T;\, \boldsymbol{N}) \cap L^2(0, T;\, \boldsymbol{N}^1), \tag{2.18}$$

which, if considered with values in $\boldsymbol{N}^{-2}$, is strongly absolutely continuous and such that

$$u'(t) + A_e u(t) + B(u(t), u(t)) = f(t), \quad \text{a.e. in } (0, T) \tag{2.19}$$

and

$$u(0) = u_0 . \tag{2.19'}$$

Here A_e is the extension of A as continuous operator from $\boldsymbol{N}^1$ to $\boldsymbol{N}^{-1}$. The case $n=2$, enters in this abstract framework with the additional property (see (2.3″)).

$$(2.20) \qquad |(B(u, v), w)| \leq c_8 |u|^{1/2} \ \|u\|^{1/2} \|v\| \ |w|^{1/2} \ \|w\|^{1/2}$$

for all $u, v, w \in \boldsymbol{N}^1$. For the study of this abstract evolution equation we refer to Foiaş [2]. In particular let us mention that if (2.20) is valid then there exists a family of mappings $\{S(t)\}_{0 \leq t \leq T}$ of $\boldsymbol{N}$ into $\boldsymbol{N}$ such that the unique solution of (2.19)-(2.19′) is given by $S(\cdot)u_0$, and

$$(2.21) \qquad |S(t)u_0 - S(t)v_0| \leq c_9 |u_0 - v_0| \exp(c_{10} |v_0|^2)$$

for all $u_0, v_0 \in \boldsymbol{N}$; moreover $S(t)u$ is continuous in $(t, u) \in [0, T] \times \boldsymbol{N}$ In case $f(t)$ does not depend on t we have obviously $S(t_1)S(t_2) = S(t_1 + t_2)$ for all $t_1, t_2 \geq 0$.

In the sequel any solution of (2.19) will be called an *individual solution* of (2.19). Part of the results which we obtained on the statistical solutions of Navier-Stokes equations will be valid also for the abstract equation (2.19). Therefore the next paragraph will be devoted entirely to the study of statistical solutions for an abstract equation (2.19) with the properties which were given above and which obviously are fulfilled by the Navier-Stokes equations, as plainly follows from the facts presented in the other preceding sections of this paragraph.

§ 3. Statistical solutions and their existence.

1. *Definition of a statistical solution. a)* By a (Borel) measure μ on a real Hilbert space $\boldsymbol{N}$ we shall understand a countably additive function defined on all Borel sets $\subset \boldsymbol{N}$. Let us recall that the σ-algebra of all Borel sets in $\boldsymbol{N}$ is the smallest containing the open (or the closed) subsets of $\boldsymbol{N}$. Let S be a continuous map from $\boldsymbol{N}$ into $\boldsymbol{N}$. Define

$$(3.1) \qquad \nu(\omega) = \mu(S^{-1}\omega) \text{ for all } \omega \text{ Borel } \subset \boldsymbol{N}.$$

Plainly ν is also a (Borel) measure on $\boldsymbol{N}$. If Φ is any real continuous functional on $\boldsymbol{N}$ and if $\Phi \circ S$ is μ-integrable then obviously Φ will be

ν-integrable and

$$\int \Phi d\nu = \int \Phi \circ S d\mu. \tag{3.1'}$$

Let us now suppose that we are given the equation (1.19) with all the properties listed in the last section of § 2, together with the supplementary property (2.20). This will allow us to consider the family $\{S(t)\}_{0 \leq t \leq T}$. Suppose now that we are given also a probability (Borel) measure μ on $\boldsymbol{N}$ (i.e. and $\mu(\boldsymbol{N})=1$) and that this probability μ representes the statistical distributions of the initial data of (1.19). Which will be the probability μ_t representing the statistical distribution of the individual solutions of (1.19) at the time t if at the initial time $t=0$ it was μ? The natural definition is in this case

$$\mu_t(\omega) = \mu(S(t)^{-1}\omega) \quad \text{for all } \omega \text{ Borel } \subset \boldsymbol{N}. \tag{3.2}$$

The usual heuristic justification of (3.2) given in any text book of statistical mechanics is the following: Let us consider N initial possible data with N sufficiently large, so that we might suppose that $\mu(\omega)$ is « approximatively » equal to the quotient $N(\omega)/N$, where $N(\omega)$ denotes the number of those initial data which belong to ω. Denote by $N_t(\omega)$ the number of those data which in their evolution will belong to ω at the time t. Obviously

$$\frac{N_t(\omega)}{N} = \frac{N(S(t)^{-1}\omega)}{N}$$

so that again supposing that $\mu_t(\omega)$ is approximatively equal to $N_t(\omega)/N$ we « obtain » (3.2).

The unpleasant feature of the definition (3.2) is that it is connected with the equation (1.19), by the intermediate of $\{S(t)\}$ which can not be defined in case no supplementary condition on B (as for instance is (1.20)) occurs. Therefore we shall establish an equation which is satisfied by $\{\mu_t\}$ defined as in (3.2) whenever $\{S(t)\}$ exists but which shall make sense even if this does not happen.

b) To this purpose let us remark that the energy inequality (2.12) for the individual solutions $u(\cdot)$ leads to

$$(3.3)\qquad \begin{cases} | u(t) |^2 \leq c_{11} | u_0 |^2 \text{ for al } 0 \leq t \leq T, \\ \int\limits_0^T \| u(t) \|^2 \, dt \leq c_{12} | u_0 |^2, \end{cases}$$

where for simplicity we have taken $T < \infty$ and $f \in L^2(0, T; \boldsymbol{N}^{-1})$. Now in the particular case when $u(t) = S(t)u$ the functions

$$(t, u_0) \to | S(t)u_0 |^2$$

and

$$(t, u_0) \to \| S(t)u_0 \|^2$$

are obviously Borel functions on $[0, T] \times \boldsymbol{N}$. Therefore (3.3) gives that

$$(3.4)\qquad \int\limits_{\boldsymbol{N}} | S(t)u |^2 \, d\mu(u) \leq c_{11} \int\limits_{\boldsymbol{N}} | u |^2 \, d\mu(u) \text{ for all } 0 \leq t \leq T,$$

the left term being a Borel function in t, and

$$(3.4')\qquad \int\limits_0^T \left(\int\limits_{\boldsymbol{N}} \| S(t)u \|^2 \, d\mu(u) \right) dt = \int\limits_{\boldsymbol{N}} \left(\int\limits_0^T \| S(t)u \|^2 \, dt \right) d\mu(u) \leq$$
$$\leq c_{12} \int\limits_{\boldsymbol{N}} | u |^2 \, d\mu(u).$$

In this way if

$$(3.5)\qquad \int\limits_{\boldsymbol{N}} | u |^2 \, d\mu(u) < \infty$$

condition which will be always imposed tu μ *from now on,* then (3.4), (3.4′) together with (3.1-1′) and (3.2), yield

$$(3.6)\qquad \int\limits_{\boldsymbol{N}} | u |^2 \, d\mu_.(u) \in L^\infty(0, T),$$

$$(3.6')\qquad \int\limits_{\boldsymbol{N}} \| u \|^2 \, d\mu_.(u) \in L^1(0, T).$$

Let us now consider a real functional $\Phi(t, u)$ defined for $(t, u) \in \in [0, T] \times \boldsymbol{N}$ such that

$$\Phi(t, P_m u) = \Phi(t, u) \text{ for all } t, u, \tag{3.7}$$

and for a certain $m = 1, 2, \ldots$ (For the definition of P_m see Sec. 2.1). Moreover let us suppose that $\Phi(\cdot, \cdot)$ is Fréchet differentiable from $[0, T] \times \boldsymbol{N}$ into $\mathbb{R}$ with continuous differential [1]) and that

$$|\Phi'_u(t, u)| \leq c_{13} \text{ and } |\Phi'_t(t, u)| \leq c_{14} + c_{15} |u| \text{ for all } t, u, \tag{3.8}$$

c_{13}, c_{14}, c_{15} being constants depending on Φ.

Let moreover $u(\cdot)$ be a solution of (2.19). Let us recall that $u(\cdot)$, as function with values in $\boldsymbol{N}^{-2}$, is absolutely continuous (hence the derivative $u'(t)$ exists in $\boldsymbol{N}^{-2}$, a.e. on $(0, T)$) and verifies (2.19); moreover let us note that since $w_1, w_2, \ldots, w_m \in \boldsymbol{N}^2$ and

$$P_m u = (u, w_1) w_1 + \ldots + (u, w_m) w_m,$$

P_m can be extended, by continuity, into a continuous mapping $(P_m)_e$ of $\boldsymbol{N}^{-2}$ into $\boldsymbol{N}$ (even $\boldsymbol{N}^2$). Therefore in virtue of (3.7), $\Phi(t, u(t))$ is

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[1]) Let us recall that a real function $\Phi(\cdot)$ defined in a neighbourhood of a point u_0 of a Banach space $\boldsymbol{B}$ is called Fréchet differentiable in u_0, if there exists an element $\Phi^*_0 \in \boldsymbol{B}^*$ such that

$$|\Phi(u) - \Phi(u_0) - (\Phi^*_0, u - u_0)| \leq 0(|u - u_0|)$$

where (v^*, v) denotes the duality between $\boldsymbol{B}^*$ and $\boldsymbol{B}$, i.e. $(v^*, v) = v^*(v)$ for $v \in \boldsymbol{B}$, $v^* \in \boldsymbol{B}^*$ and

$$\frac{0(|u - u_0|)}{|u - u_0|} \to 0 \text{ for } u \to u_0.$$

If this happens for all $u - u_0$ belonging to a subspace $\boldsymbol{B}' \subset \boldsymbol{B}$ we shall call Φ *Fréchet $\boldsymbol{B}$-differentiable in the direction of $\boldsymbol{B}'$*. In both cases Φ^*_0 is called the Frechet differential of Φ in u_0 and is denoted by $\Phi'_{u_0}(u_0)$.

Finally let us note that if $\boldsymbol{B} = \boldsymbol{X} \times \boldsymbol{Y}$ then $\boldsymbol{B}^* = \boldsymbol{X}^* \times \boldsymbol{Y}^*$ and

$$\Phi'_{x_0, y_0}(x_0, y_0) = (\Phi'_{x_0}(x_0, y_0), \Phi'_{y_0}(x_0, y_0))$$

where the components constitute the partial Fréchet differentials.

absolutely continuous in t and a.e. on $(0, T)$ we have

$$\frac{d\Phi(t, u(t))}{dt} = \Phi'_t(t, u(t)) + (u'(t), \Phi'_v(t, v))/v = u(t); \tag{3.9}$$

where the brackets represent the duality between $\boldsymbol{N}^{-2}$ and $\boldsymbol{N}^2$ since, in virtue of (3.7), we have

$$\Phi'_u(t, u) = P_m \Phi'_{P_m u}(t, P_m u) \in P_m \boldsymbol{N} \subset \boldsymbol{N}^2. \tag{3.10}$$

Using (3.9-10), as well as (2.19), (3.3) and (3.8), we obtain readily

$$\begin{aligned}\left|\frac{d}{dt}\Phi(t, u(t))\right| &\leq |u'(t)|_{-2}\ |P_m\Phi'_v(t, v)/v = u(t)|_2 + \\ &\quad + c_{14} + c_{15}|u(t)| + c_{16}|u'(t)|_{-2} \leq \\ &\leq c_{14} + c_{15}|u(t)| + c_{16}|f(t)| + c_{17}|u(t)| + c_{18}|u(t)| \cdot \|u(t)\| \leq \\ &\leq c_{19} + c_{20}|u| + c_{21}(1 + |u|)\,\|u(t)\| \ (\text{where } u = u(0)),\end{aligned}$$

whence (by (3.3))

$$\begin{aligned}\int_0^T \left|\frac{d}{dt}\Phi(t, u(t))\right| dt &\leq c_{22}(1 + |u_0|)\cdot \\ &\cdot \left(1 + \int_0^T \|u(t)\|^2\, dt\right)^{1/2} \leq c_{23}(1 + |u_0|^2).\end{aligned} \tag{3.11}$$

Using (3.5) we deduce

$$\int_{\boldsymbol{N}}\int_0^T \left|\frac{d}{dt}\Phi(t, S(t)u)\right| dt\, d\mu(u) < \infty.$$

By Fubini theorem, the function

$$\Psi : t \to \int_{\boldsymbol{N}} \frac{d}{dt}\Phi(t, S(t)u)\, d\mu(u)$$

is Lebesgue-integrable on $(0, T)$. Let $r(\cdot)$ be any function of class C^1 with compact support in $(0, T)$. Then, by (3.11),

$$-\int_0^T r'(t)\left[\int_N \Phi(t, S(t)u)d\mu(u)\right]dt=$$
$$=\int_N\left[-\int_0^T r'(t)\Phi(t, u(t))dt\right]d\mu(u)=$$
$$=\int_N\left[\int_0^T r(t)\frac{d}{dt}\Phi(t, u(t))dt\right]d\mu(u)=\int_0^T r(t)\Psi(t)dt,$$

so that

$$\int_N \Phi(t, S(t)u)d\mu(u)$$

is absolutely continuous and a.e. on $(0, T)$

$$\frac{d}{dt}\int_N \Phi(t, S(t)u)d\mu(u)=\int_N \frac{d}{dt}\Phi(t, S(t)u)d\mu(u). \tag{3.12}$$

On the other hand in virtue of (2.19) and (3.9) we have

$$\frac{d}{dt}\Phi(t, S(t)u)=\Phi'_{\square}(\boxed{t}, S(t)u)^{1)}+ \tag{3.12'}$$
$$+(f(t)-A_eS(t)u-B(S(t)u, S(t)u), \Phi'_{\square}(t, \boxed{S(t)u}))=$$
$$=\Phi'_{\square}(\boxed{t}, S(t)u)-\nu((S(t)u, \Phi'_{\square}(t, \boxed{S(t)u})))-$$
$$-b(S(t)u, S(t)u, \Phi'_{\square}(t, \boxed{S(t)u}))+(f(t), \Phi'_{\square}(t, \boxed{S(t)u})),$$

so that (3.12) can be written under the form

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[1]) We shall use the symbol $\square$ to indicate with respect to which variable is the differentiation taken.

$$\frac{d}{dt}\int_N \Phi(t, \cdot)\circ S(t)ud\mu(u)+\int_N [-\Phi'_\square(\boxed{t}, \cdot)\circ S(t)u+$$
$$+\nu((\cdot, \Phi'_\square(t, \boxed{\cdot}))\circ S(t)u+b(\cdot, \cdot, \Phi'_\square(t, \boxed{\cdot}))\circ S(t)u)]d\mu(u)=$$
$$=\int_N (f(t), \Phi'_\square(t, \boxed{\cdot}))\circ S(t)ud\mu(u),$$

and using (3.1′) we can rewrite this relation under the form

$$\frac{d}{dt}\int_N \Phi(t, u)d\mu_t(u)+\int_N [-\Phi'_t(t, v)+((u, \Phi'_u(t, u)))+ \\ +b(u, u, \Phi'_u(t, u)))]d\mu_t(u)=\int_N (f(t), \Phi'_u(t, u))d\mu_t(u). \tag{3.13}$$

The noteworthy fact is that in (3.13) the system of maps $\{S(t)\}$ does no more occur but instead the equation (3.13) is written only in terms of the measures μ_t and of the given abstract differential equation (2.19), that is, in the case in which we are concerned, only in terms of the Navier-Stokes equations and the functional spaces and operators associated with them. To avoid technical difficulties we subject $\Phi(\cdot, \cdot)$ to the supplementary condition

$$\Phi(t, \cdot)=0 \text{ for } t \text{ near } T, \tag{3.8′}$$

and integrate (3.13) with respect to $t\in(0, T)$, thus obtaining the equation

$$\boxed{\begin{aligned}&\int_0^T\left\{\int_N -\Phi'_t(t, u)+\nu((u, \Phi'_u(t, u)))+\right.\\ &\left.+b(u, u, \Phi'_u(t, u))]d\mu_t(u)\right\}dt=\int_N \Phi(0, u)d\mu(u)+\\ &+\int_0^T \int_N\left[(f(t), \Phi'_u(t, u))d\mu_t(u)\right]dt.\end{aligned}} \tag{3.13$_\mathrm{I}$}$$

The equation (3.13_{I}) *will be the basic equation which our study concerns.* Therefore we begin with a simple discussion of properties connected with it.

c) Let us define some new functional spaces, namely: $\mathcal{C}_\alpha$, for $\alpha \geq 0$, will be the space of all real continuous functionals $\Phi(\cdot)$ on $\boldsymbol{N}$ such that

$$\| \Phi(\cdot) \|_{\mathcal{C}_\alpha} = \sup_{u \in N} \frac{|\Phi(u)|}{1+|u|^\alpha} < \infty . \tag{3.14}$$

$\mathcal{C}_{1,1}$ will be the space of all real continuous functional $\Phi(\cdot)$ on $\boldsymbol{N}^1$ such that

$$\| \Phi(\cdot) \|_{\mathcal{C}_{1,1}} = \sup_{u \in N^1} \frac{|\Phi(u)|}{1+|u| \cdot \| u \|} < \infty . \tag{3.14'}$$

Moreover we shall put

$$\begin{cases} \mathcal{L}_\alpha = L^1(0,\ T;\ \mathcal{C}_\alpha), \\ \mathcal{L}_{1,1} = L^2(0,\ T;\ \mathcal{C}_{1,1}). \end{cases} \tag{3.15}$$

A family $\{\mu_t\}_{0 \leq t \leq T}$ of positive Borel measures μ_t on $\boldsymbol{N}$, will be called *basic* [1]) if it satisfies the following conditions

$$\int_N (1+|u|^2) d\mu_\cdot(u) \in L^\infty(0,\ T), \tag{3.16}$$

$$\int_N \| u \|^2 d\mu_\cdot(u) \in L^1(0,\ T), \text{[2]}) \tag{3.16'}$$

$$\int_N \Phi(u) d\mu_\cdot(u) \text{ is measurable} \tag{3.16''}$$

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[1]) The reason of this terminology is such family of measures will be at the base of our research in the sequel.

[2]) Since $\| u \|^2 = \lim\limits_{u \to \infty} \| P_m u \|^2$ and $\| P_m \cdot \|^2$ is continuous on $\boldsymbol{N}$ we deduce that $\| \cdot \|^2$ is a Borel function from $\boldsymbol{N}$ to $[0,\ \infty]$, hence the integral $\int_N \| u \|^2 d\mu_t(u)$ makes sense for all $t \in [0,\ T]$ being perhaps ∞ for some t.

for any non-negative functional Φ defined on $\boldsymbol{N}$ and weakly continuous (i.e. continuous from $\boldsymbol{N}_{\text{weak}}$ to $\mathbb{R}$, i.e. from $\boldsymbol{N}$ endowed with the weak topology, to $\mathbb{R}$). Plainly *the family of measures defined by* (3.2), *once* (3.5) *is valid, is basic* (see (3.6-6′)).

A first remark to be made is that

$$\mathcal{C}_2 \subset \mathcal{C}_{1,1} \text{ (with continuous embedding)} \tag{3.17}$$

and

$$\mathcal{L}_{1,1} \cap \mathcal{L}_2 \textit{ is dense in } \mathcal{L}_2 . \tag{3.17′}$$

The first inclusion is obvious, while for the second we have only to note that, by (3.17),

$$L^2(0, T; \mathcal{C}_2) \subset \mathcal{L}_{1,1} \tag{3.17″}$$

(with continuous embedding), the first space being obviously dense in $\mathcal{L}_2$.

In virtue of the property (3.17′), any functional $F \in \mathcal{L}^*_{1,1}$ [1]), whose restriction to $\mathcal{L}_2 \cap \mathcal{L}_{1,1}$ is $\mathcal{L}_2$-continuous, uniquely extends to a functional $\in \mathcal{L}^*_2$. The set of these functionals of $\mathcal{L}^*_{1,1}$ will be denoted by $\mathcal{L}^*_{1,1;2}$. Obviously the extension map is a surjection of $\mathcal{L}^*_{1,1;2}$ onto the set of those functionals $\in \mathcal{L}^*_2$ whose restriction to $\mathcal{L}_{1,1} \cap \mathcal{L}_2$ is $\mathcal{L}_{1,1}$-continuous. For $F \in \mathcal{L}^*_{1,1;2}$ we can consider $F(\Phi)$ either for $\Phi \in \mathcal{L}_{1,1}$ or for $\Phi \in \mathcal{L}_2$.

LEMMA 1. *Let* $\{\mu_t\}_{0<t<T}$ *be a basic family of measures on* $\boldsymbol{N}$. *Then*

$$F(\Phi) = \int_0^T \left[\int_N \Phi(t, u) d\mu_t(u) \right] dt \tag{3.18}$$

makes sense for any $\Phi(\cdot, \cdot)$ *belonging to* $\mathcal{L}_2$ *or* $\mathcal{L}_{1,1}$, *and is a linear satisfying* [2])

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[1]) $\mathcal{L}^*_2$ and $\mathcal{L}^*_{1,1}$ denote the dual space of $\mathcal{L}_2$, resp. $\mathcal{L}_{1,1}$.

[2]) Thus $F \in \mathcal{L}^*_{1,1;2}$.

(3.18₂) $$\| F \|_{\mathfrak{L}_2^*} \leq \left\| \int_N (1+|u|^2) d\mu_\cdot(u) \right\|_{L^\infty(0,T)}$$

(3.18₁,₁) $$\| F \|_{\mathfrak{L}_{1,1}^*} \leq \left\| \int_N (1+|u|^2) d\mu_\cdot(u) \right\|_{L^\infty(0,T)}^{1/2} \cdot \left\| \int_N (1+\| u \|^2 d\mu_\cdot(u) \right\|_{L^1(0,T)}^{1/2}.$$

PROOF. Let $\Phi(\cdot)$ be any non-negative functional defined on $\boldsymbol{N}$, continuous from $\boldsymbol{N}$ to $\mathbb{R}$. The function Φ_m, defined by

$$\Phi_m(u) = \Phi(P_m u) \qquad \text{for } u \in \boldsymbol{N},$$

is weakly continuous on $\boldsymbol{N}$, thus

$$\varphi_m(\cdot) = \int_N \Phi_m(u) d\mu_\cdot(u) \qquad (m=1, 2, \ldots)$$

are measurable functions in t on $(0, T)$. Letting $m \to \infty$, we deduce that their limit

(3.19) $$\int_N \Phi(u) d\mu_\cdot(u)$$

is measurable on $(0, T)$. Now in virtue of (3.16′) we have

(3.16‴) $$\mu_t(\boldsymbol{N} \setminus \boldsymbol{N}^1) = 0 \qquad \text{a.e. on } (0, T)$$

so that the preceding argument applies also to any nonnegative functional Φ defined on $\boldsymbol{N}$ and continuous from $\boldsymbol{N}^1$ to $\mathbb{R}$. Plainly (3.16), (3.16′) will now imply that (3.19) is defined a.e. on $(0, T)$ and is measurable, for any $\Phi \in \mathcal{C}_2$ or $\Phi \in \mathcal{C}_{1,1}$. Thus it remains only to prove (3.18₂) and (3.18₁,₁).

For (3.18₂) we have only to note that

$$| F(\Phi) | \leq \int_0 \left[\int_N | \Phi(t, u) | \, d\mu_t(u) \right] dt \leq$$

$$\leq \int_0^T \left[\int_N \| \Phi(t, \cdot) \|_{\mathcal{C}_2} (1+|u|^2) d\mu_t(u) \right] dt =$$

$$= \int_0^T \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_2} \left[\int_N (1+|u|^2) d\mu_t(u) \right] dt \leq$$

$$\leq \left[\int_0^T \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_2} dt \right] \left\| \int_N (1+|u|^2 d\mu_.(u) \right\|_{L^\infty(0,T)} =$$

$$= \| \Phi(\cdot,\ \cdot) \|_{\mathfrak{L}_2} \cdot \left\| \int_N (1+|u|^2) d\mu_.(u) \right\|_{L^\infty(0,T)},$$

while for $(3.18_{1,1})$ that

$$| F(\Phi) | \leq \int_0^T \left| \int_N | \Phi(t,\ u) | \, d\mu_t(u) \right] dt \leq$$

$$\leq \int_0^T \left[\int_N \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_{1,1}} (1+|u| \cdot \|u\|) d\mu_t(u) \right] dt \leq$$

$$\leq \int_0^T \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_{1,1}} (1+|u|^2)^{1/2} (1+\|u\|^2)^{1/2} d\mu_t(u) \Big] dt \leq$$

$$\leq \int_0^T \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_{1,1}} \left| \int_N (1+|u|^2 d\mu_t(u) \right]^{1/2} \left| \int_N (1+\|u\|^2) d\mu_t(u) \right]^{1/2} dt \leq$$

$$\leq \left\| \int_N (1+|u|^2) d\mu_.(u) \right\|_{L^\infty(0,T)}^{1/2} \cdot$$

$$\cdot \int_0^T \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_{1,1}} \left[\int_N (1+\|u\|^2) d\mu_t(u) \right|^{1/2} dt \leq$$

$$\leq \left\| \int_N (1+|u|^2) d\mu_.(u) \right\|_{L^\infty(0,T)}^{1/2} \cdot \left\| \int_N (1+\|u\|^2) d\mu_.(u) \right\|_{L^1(0,T)}^{1/2} \cdot$$

$$\cdot \left[\int_0^T \| \Phi(t,\ \cdot) \|_{\mathfrak{C}_{1,1}}^2 dt \right]^{1/2} = \left\| \int_N (1+|u|^2) d\mu_.(u) \right\|_{L^\infty(0,T)}^{1/2} \cdot$$

$$\cdot \left\| \int_N (1+\|u\|^2) d\mu_.(u) \right\|_{L^1(0,T)}^{1/2} \cdot \| \Phi(\cdot,\ \cdot) \|_{\mathfrak{L}_{1,1}} .$$

This finishes the proof of Lemma 1.

d) Let now $\mathfrak{T}$ denote the space of all real functions $\Phi(\cdot, \cdot)$ defined on $[0, T] \times \boldsymbol{N}^1$ and satisying the following conditions:

(i) *$\Phi(t, u)$ is continuous in $(t, u) \in [0, T] \times \boldsymbol{N}^1$ and verifies the second condition* (3.8),

(ii) *$\Phi(t, u)$ is Fréchet $\boldsymbol{N}$-differentiable in the direction of $\boldsymbol{N}^1$,* i.e. there exists $\Phi'_u(t, u) \in \boldsymbol{N}$ such that

$$\frac{1}{|v|} \,|\, \Phi(u+v) - \Phi - (\Phi'_u(t, u), v) \,| \to 0 \text{ for } v \in \boldsymbol{N}^1, \ |v| \to 0,$$

(iii) *$\Phi'(\cdot, |\cdot|)$ is continuous from $[0, T] \times \boldsymbol{N}^1$ to $\boldsymbol{N}^1$ and is bonuded.*

Let denote by $\mathfrak{T}_0$ the space of those funtions which were defined after formulae (3.6-6′). A fuction of $\mathfrak{T}$ will be called a *test functional.* In virtue of (3.10). we have $\mathfrak{T}_0 \subset \mathfrak{T}$. We shall call the functions of $\mathfrak{T}_0$, *elementary test functionals.*

LEMMA 2. *Let $\{\mu_t\}_{0 \le t < T}$ be a basic family of measures on $\boldsymbol{N}$, let $f(\cdot) \in L^1(0, T; \boldsymbol{N}^{-1})$ and let $\Phi(\cdot, \cdot) \in \mathfrak{T}$. Then*

$$\varphi(\Phi) = \int_0^T \left\{ \int_{\boldsymbol{N}} [-\Phi'_t(t, u) + \nu((u, \Phi'_u(t, u))) + \right. \tag{3.20}$$
$$\left. + b(u, u, \Phi'_u(t, u)) - (f(t), \Phi'_u(t, u))] d\mu_t(u) \right\} dt$$

makes sense. Moreover if

$$\Phi_m(t, u) \text{ def } = \Phi(t, P_m u) \text{ for all } (t, u) \in [0, T] \times \boldsymbol{N}, \tag{3.20′}$$

then $\Phi_m \in \mathfrak{T}_0$ (for all $m = 1, 2, \ldots$) and

$$\varphi(\Phi_m) \to \varphi(\Phi) \text{ for } m \to \infty. \tag{3.20″}$$

PROOF. That $\Phi_m \in \mathfrak{T}_0$ is obvious; also it is obvious that $\Phi_m \to \Phi$ pointwise. Moreover let us remark that for all $m = 1, 2, \ldots$

$$\| (\Phi_m)'_u(t, u) \| = \| P_m \Phi'_\square(t, \boxed{P_m u}) \| \le \| \Phi'_\square(t, \boxed{P_m u}) \| \le c_{24}, \tag{3.21}$$

where c_{24} is a constant depending on Φ. Moreover $(\Phi_m)'_u$ is continuous from $(t, u) \in [0, T] \times \boldsymbol{N}$ to $\boldsymbol{N}^1$ so that if

$$\Psi_{\Phi_m}(t, u) = -(\Phi_m)'_t(t, u) + ((u, (\Phi_m)'_u(t, u))) +$$
$$+ b(u, u, (\Phi_m)'_u(t, u)) - (f(t), (\Phi_m)'_u(t, u)),$$

the function $\Psi_{\Phi_m}(\cdot, \cdot)$ is continuous from $[0, T] \times \boldsymbol{N}^1$ to $\mathbb{R}$ and

$$(3.21') \qquad |\Psi_{\Phi_m}(t, u)| \leq c_{25} + c_{24} \| u \| + c_{26} \| u \|^2 + c_{24} | f(t) |_{N^{-1}}$$

where $c_{24} - c_{26}$ are constants (depending on Φ but independent of t, u and m). Using (3.16′) we can easily verify that $\varphi(\Phi_m)$ makes sense for all $m = 1, 2, \ldots$. Let us define also

$$(3.22) \qquad \Psi_{\Phi}(t, u) = -\Phi'_t(t, u) + \nu((u, \Phi'_u(t, u))) +$$
$$+ b(u, u, \Phi'_u(t, u)) - (f(t), \Phi'_u(t, u))$$

for all $(t, u) \in [0, T] \times \boldsymbol{N}^1$. Since

$$\| \Phi'_u(t, u) - (\Phi_m)'_u(t, u) \| \leq \| \Phi'(t, u) - P_m \Phi'_u(t, u) \| +$$
$$+ \| \Theta(t, u) - \Theta(t, P_m u) \| \quad \text{where } \Theta(t, u) = \Phi'_u(t, u),$$

we deduce by property (iii) of a test function that

$$(3.22') \qquad \Psi_{\Phi_m}(t, u) \to \Psi_{\Phi}(t, u) \text{ for all } (t, u) \in [0, T] \times \boldsymbol{N}^1.$$

Moreover (3.21′) shows that

$$(3.22'') \qquad |\Psi_{\Phi_m}(t, u)| \leq c_{27}(1 + \| u \|^2) + c_{24} | f(t) |_{N^{-1}},$$

so that applying Lebesgue's dominated convergence theorem we obtain that (3.20) makes sense and (3.20) is valid. Indeed for a value $t \in (0, T)$ such that

$$\int_{\boldsymbol{N}} (1 + \| u \|^2) d\mu_t(u) < \infty \text{ and } | f(t) |_{N^{-1}} < \infty$$

by Lebesgue's theorem we obtain that $\Psi_\Phi(t, u)$ is μ_t-integrable and

$$(3.23)\qquad \int_N \Psi_{\Phi_m}(t, u)d\mu_t(u) \to \int \Psi_\Phi(t, u)d\mu_t(u) \text{ for } m \to \infty.$$

Denote by $\psi_m(t)$ and $\psi(t)$ the integrals occurring in (3.23). We have a.e. on $(0, T)$

$$\psi_m(t) \to \psi(t).$$

Moreover on $(0, t)$ we have also

$$|\psi_m(t)| \leq c_{27} \int_N (1 + \|u\|^2) d\mu_t(u) + c_{24} |f(t)|_{N^{-1}}$$

where the right term is an integrable function (in t). Applying once again Lebesgue's theorem we infer that $\psi(\cdot)$ is integrable on $(0, T)$ and that

$$\int_0^T \psi_m(t)dt \to \int_0^T \psi(t)dt.$$

This conclusion coincides with our claim.

e) We are now in the state to give our basic definition. *A statistical solution of the Navier-Stokes equations* (or more generally of an abstract evolution equation of the type described in Sec. 2.5) *is, by definition, a basic family of probabilities* $\{\mu_t\}_{0<t<T}$ *on* $\boldsymbol{N}$ *satisfying* (3.13_I) *for all test functions* $\Phi(\cdot, \cdot)$ *which satisfy* (3.8′). *The probability* μ *is called the initial data and is subjected to condition* (3.5), *while the function* $f(\cdot)$ *is called the right member*[1]) *and is subjected to the condition* $f(\cdot) \in L^1(0, T; \boldsymbol{N}^{-1})$.

REMARKS. 1. In virtue of Lemma 2 above in order that $\{\mu_t\}$ be a statistical solution it is sufficient (and obviously necessary) that

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[1]) Recall that it represents the external body force.

(3.13_{I}) shall be satisfied only by all elementary test functions $\Phi(\cdot,\ \cdot)$ satisfying (3.8′).

2. *The above definition was chosen in such a way that in case the space dimension n of the fluid is* $=2$, *i.e. in case the system* $\{S(t)\}_{0\le t<T}$ *exists (in this case* $f(\cdot)\in L^2(0,\ T;\ \boldsymbol{N}^{-1})$), *the family* $\{\mu_t\}_{0<t<T}$, *defined by* (3.2), *yields a statistical solution of the Navier-Stokes equations.* The important feature of our definition is that it does not involve the existence of the system $\{S(t)\}_{0<t<T}$ thus it is valid also for the case where the space dimension of the fluid n is $=3$ or 4 (remember that our abstract scheme concerns the Navier- Stokes equations only in dimension $n=2$, 3 or 4).

3. For $\Phi\in\mathcal{T}_0$, it is plain that $\Psi_\Phi(\cdot,\ \cdot)$ as defined by (3.22) belongs to $\mathfrak{L}_{1,1}$. The condition that the basic family $\{\mu_t\}$ be a statistical solution of the Navier-Stokes equations can be obviously given the following equivalent form: *Let* $F\in\mathfrak{L}_{1,1;2}$ *be the functional attached to* $\{\mu_t\}_{0<t<T}$ *by formula* (3.18). *Then* $\{\mu_t\}_{0<t<T}$ *is a statistical of the Navier-Stokes equations if and only if*

$$F(\Psi_\Phi)=\int_{\boldsymbol{N}}\Phi(0,\ u)d\mu(u) \tag{3.13_{II}}$$

for all elementary test functionals $\Phi\in\mathcal{T}_0$ *satisfying* (3.8′).

This equivalent definition will be very used in the sequel.

4. The study of the test functionals is of great interest in the theory of statistical solutions of the Navier-Stokes equations. A very significant progress can be achieved by enlarging the class of test functionals in the basic equation (3.13_{I}). Suppose for instance that for a statistical solution the equation (3.13_{I}) would remain valid whenever it makes sense. But this is then the case with $\Phi(t,\ u)=\frac{1}{2}r(t)\ |\ u\ |^2$ where $r(\cdot)\in C_0^\infty\ ([0,\ T))$ is arbitrary. But such choices in (3.13_{I}) lead readily to the energy equation

$$\frac{1}{2}\int_{\boldsymbol{N}}|\ u\ |^2 d\mu_t(u)+\int_0^t\Big[\int_{\boldsymbol{N}}\|\ u(\tau)\ \|^2 d\mu_\tau(u)\Big]d\tau=$$

$$=\frac{1}{2}\int_{N}|u|^2 d\mu(u)+\int_0^t\left[\int_{N}(f(\tau),\ u)d\mu_\tau(u)\right]d\tau$$

a.e. on $(0, T)$. This would already be a very interesting fact concerning statistical solutions. The main reason for which in our approach this substitution for $\Phi(\cdot,\ \cdot)$ is not allowed is that the Frechet derivative of $\frac{1}{2}|u|^2$ is u which is not bounded in the $\boldsymbol{N}^1$-norm.

5. In connection to the preceding remark let us point out some restritive properties of our test functionals:

Any test functional $\Phi(\cdot,\ \cdot)$ can be extended to $[0,\ T]\times\boldsymbol{N}^{-1}$ such that

$$|\Phi(t,\ u)-\Phi(t,\ v)|\leq c_{28}\|u-v\|_{N^{-1}} \tag{3.24}$$

for all $t\in[0,\ T]$, $u,\ v\in\boldsymbol{N}^{-1}$, where

$$c_{28}=\sup_{u\in N^1}\|\Phi'_u(t,\ u)\|\,. \tag{3.24'}$$

Moreover this extension is continuous from $[0,\ T]\times\boldsymbol{N}_{\text{weak}}$ to $\mathbb{R}$.

PROOF. We have for any $s,\ t\in[0,\ T]$ and $u,\ v\in\boldsymbol{N}^1$ (using the notations of (3.8) and (3.24′))

$$|\Phi(t,\ v)-\Phi(s,\ u)|=\left|\int_0^1(\Phi'_{\square}(t,\ \underline{\overline{|\theta v+(1-\theta)u|}}),\ v-u)d\theta+\right. \tag{3.24''}$$

$$\left.+\int_s^t\Phi'_{\square}(\underline{|\tau|},\ u)d\tau\right|\leq c_{28}\|u-v\|_{N^{-1}}+$$

$$+|t-s|(c_{14}+c_{15}|u|).$$

Taking $s=t$ in (3.24″), we readily deduce that $\Phi(t,\ \cdot)$ can be extended by continuity to whole $\boldsymbol{N}^{-1}$ (since $\boldsymbol{N}^1$ is dense in $\boldsymbol{N}^{-1}$) and that this extension will satisfy (3.24). For this extension, (3.24″) remains valid whenever $s,\ t\in[0,\ T]$, $v\in\boldsymbol{N}^{-1}$ and $u\in\boldsymbol{N}$. Since the imbedding $\boldsymbol{N}\subset\boldsymbol{N}^{-1}$ is compact, this extended version of (3.24″) shows that

$\Phi(t, v) \to \Phi(s, u)$ if $t \to s$, $v, u \in \boldsymbol{N}$, and $v \to u$ weakly in $\boldsymbol{N}$.

6. One can easily deduce from the preceding remark that for any test functional $\Phi(\cdot, \cdot)$ we have (for its extension on $[0, T] \times \boldsymbol{N}$

$$(3.25) \qquad |\Phi(t, u)| \leq c_{29} + c_{30} |u| \text{ for all } (t, u),$$

with some suitable constants $c_{29} - c_{30}$ depending only on $\Phi(\cdot, \cdot)$. Another remark is that if a test functional $\Phi(\cdot, \cdot)$ satisfies $\Phi(t, u) = 0$ for a certain $t \in [0, T]$ and for all u with $\|u\|$ enough large, then actually $\Phi(t, \cdot) = 0$ (i.e. $\Phi(t, u) = 0$ for all u). Indeed if $u \in \boldsymbol{N} \backslash \boldsymbol{N}^1$ there exists $u_h \in \boldsymbol{N}^1$ $(h = 1, 2, \ldots)$ such that $u_h \to u$ in $\boldsymbol{N}$ and $\|u_h\| \to \infty$. Therefore $\Phi(t, u_h) = 0$ for enough large h. By Remark 5., $\Phi(t, u_h) \to \to \Phi(t, u)$, thus $\Phi(t, u) = 0$ for all $u \in \boldsymbol{N} \backslash \boldsymbol{N}^1$ which is dense in $\boldsymbol{N}$, so that this implies $\Phi(t, u) = 0$ for all $u \in \boldsymbol{N}$.

2. *Existence of statistical solutions. a*) The aim of this section is to prove the following basic result.

THEOREM 1. *For any initial data* μ *satisfying* (3.5) *and any right term* $f(\cdot) \in L^2(0, T; \boldsymbol{N}^{-1})$,*there exists a statistical solution* $\{\mu_t\}_{0<t<T}$ *of the Navier-Stokes equations* (i.e. satisfying (3.13$_{\text{I}}$) for all test functional $\Phi(\cdot, \cdot)$ subjected to the condition (3.8′)).

The proof of this theorem will be rather big and parts of it will be used in the sequel to improve the above existence theorem.

The main part of the proof lies on the following

LEMMA 3. *Let* $\{\{\mu_t^{(m)}\}_{0<t<T}\}$ *be a sequence of basic families of (Borel) measures on* $\boldsymbol{N}$ *such that*

$$(3.26) \qquad \sup_m \left\| \int_{\boldsymbol{N}} (1 + |u|^2) d\mu_{\cdot}(u) \right\|_{L^\infty(0,T)} < \infty,$$

$$(3.26') \qquad \sup_m \left\| \int_{\boldsymbol{N}} (1 + \|u\|^2) d\mu_{\cdot}(u) \right\|_{L^1(0,T)} < \infty.$$

Let moreover $F^{(m)}$, $m = 1, 2, \ldots$ *be the functional* $\in \mathfrak{L}^*_{1,1;2}$ *defined as in Lemma* 1 *for* $\{\mu_t\} = \{\mu_t^{(m)}\}$. *Let moreover* $F \in \mathfrak{L}^*_{1,1}$ *be a* w^**-cluster point* *of* $\{F^{(m)}\}_{m=1}^{\infty}$ [1]). *Then* $F\in\mathfrak{L}^*_{1,1;2}$, *F is also a* w^*-*cluster point of* $\{F^{(m)}\}_{m=1}^{\infty}$ *in* $\mathfrak{L}^*_2$ *and there exists a basic family* $\{\mu_t\}_{0<t<\infty}$ *of (Borel) measures on* $\boldsymbol{N}$ *such that* (3.18) *be valid for all* $\Phi(\cdot,\ \cdot)\in L^1(0,\ T;\ \mathfrak{C}_0)$ [2]).

PROOF. Let c_{31} and c'_{31} be the suprema occurring in (3.26) and (3.26′). By (3.18_2) we have

$$(3.27)\qquad \|F^{(m)}\|_{\mathfrak{L}^*_2}\leq c_{31}\ \text{for all}\ m=1,\ 2,\ \dots$$

Therefore for $\Phi\in\mathfrak{L}_2\cap\mathfrak{L}_{1,1}$ we have

$$|F(\Phi)|\leq c_{31}\|\Phi\|_{\mathfrak{L}_2}$$

and this shows that $F\in\mathfrak{L}^*_{1,1;2}$. We shall denote the extension, by continuity, to whole $\mathfrak{L}_2$ of our functional, also by F. On account of (3.27) and the fact that $\mathfrak{L}_{1,1}\cap\mathfrak{L}_2$ is dense in $\mathfrak{L}_2$, and since F is a w^*-cluster point in $\mathfrak{L}^*_{1,1}$ of $\{F^{(m)}\}_{m=1}^{\infty}$, it is easy to verify that F is also a w^*-cluster point of $\{F^{(m)}\}_{m=1}^{\infty}$ in $\mathfrak{L}^*_2$ too.

Since F can be considered in $\mathfrak{L}^*_2=(L^1(0,\ T;\ \mathfrak{C}_2))^*$ it is natural to use the integral representation of the continuous linear functionals on L^1-spaces. However since $\mathfrak{C}_2$ is not separable, one has to refer to a more involved representation theorem (particular form a general theorem of A. and C. Ionescu Tulcea [1]; see also Dinculeanu [1], §§ 10-13):

Let $\mathfrak{X}$ be a Banach space (which is not supposed neither separable nor reflexive), let $F\in(L^1(0,\ T;\ \mathfrak{X}))^*$ and let λ be a strong lifting of $L^\infty(0,\ T)$ [3]). Then there exists a family $\{F_t\}_{0<t<T}\subset\mathfrak{X}^*$ such that

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[1]) Recall that a w^*-cluster point of a sequence $\{b^*_m\}\subset\boldsymbol{B}^*$, where $\boldsymbol{B}$ is a Banach space, is a functional $b^*\in\boldsymbol{B}^*$, which is in the w^*-closure of the set $\{b_m\}_{m=k}^{\infty}$ for all $k=1,\ 2,\ \dots$ This means that for any finite set of elements $b_1,\ b_2,\ \dots,\ b_l$ of $\boldsymbol{B}$, $\varepsilon>0$ and any k, there exists an $m\geq k$ such that $|b^*(b_j)-b^*_m(b_j)|<\varepsilon$ for all $j=1,\ 2,\ \dots,\ l$. The existence of a w^*-cluster point F is a consequence of the fact that in virtue of (3.18_2-$18_{1,1}$) and (3.26-26′), the sequence $\{F^{(m)}\}_{m=1}^{\infty}$ is bounded in $\mathfrak{L}^*_{1,1}$ (and also in $\mathfrak{L}^*_2$); indeed in $\boldsymbol{B}^*$ any ball $\{b^*;\ \|b^*\|_{\boldsymbol{B}^*}\leq r\}$ (with $r<\infty$) is w^*-compact (see DUNFORD-SCHWARTZ [1], Ch. V, 4.2).

[2]) It is plain that any $\Phi(\cdot,\ \cdot)\in L^1(0,\ T;\ \mathfrak{C}_0)$ belongs to $\mathfrak{L}_2$ so that the assertion makes sense since F is defined on $L^1(0,\ T;\ \mathfrak{C}_0)$.

[3]) Let us recall that a lifting λ of $L^\infty(0,\ T)$ is an algebraic application of $L^\infty(0,\ T)$ into the algebra of all measurable bounded functions (defined everywhere

$$F(x(\cdot))=\int_0^T F_t(x(t))dt \text{ for every } x(\cdot)\in L^1(0,\,T;\,\mathcal{X}), \tag{3.28}$$

$$\sup_{0<t<T} \| F_t \| = \| F \|, \tag{3.28'}$$

$$\lambda(F_{\cdot}(x))(t)=F_t(x) \text{ for all } t\in(0,\,T) \text{ and } x\in\mathcal{X}. \tag{3.28''}$$

In our case $\mathcal{X}=\mathcal{C}_2$ and the functional F is just the w^*-cluster point in $\mathfrak{L}^*_2(=(L^1(0,T;\,\mathcal{C}_2))^*)$ of $\{F^{(m)}\}_{m=1}^{\infty}$ with which we are concerned.

Our first aim is now to find for almost all $t\in(0,\,T)$ a Borel measure $\mu_t\geq 0$ on $\boldsymbol{N}$ such that

$$F_t(\Phi)=\int_N \Phi(u)d\mu_t(u) \text{ for all } \Phi\in\mathcal{C}_0\,^{1)}. \tag{3.29}$$

We start by remarking that $F^{(m)}\geq 0$, i.e. $F^{(m)}(\Phi)\geq 0$ whenever $\Phi\in\mathfrak{L}_2$, $\Phi\geq 0$. Since F is a w^*-cluster point in $\mathfrak{L}^*_2$ of $\{F^{(m)}\}_{m=1}^{\infty}$ the functional F will also be ≥ 0. Thus

$$\int_0^T r(t)F_t(\Phi)dt=F(r\otimes\Phi)\geq 0 \tag{3.30}$$

(where $r\otimes\Phi$ denotes the function defined by

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on $(0,\,T)$) such that $\lambda\varphi\in\varphi$ for all $\varphi\in L^\infty(0,\,T)$ and

(*) $\varphi\geq 0$ in $L^\infty(0,\,T)$ implies $(\lambda\varphi)(t)\geq 0$ for all $t\in(0,\,T)$,

(**) $\varphi(1)\equiv 1$ on $(0,\,T)$.

The lifting λ is called strong if for any $\varphi\in C([0,\,T])$ we have $(\lambda\varphi)(t)=\varphi(t)$ for all $0\leq t\leq T$, where in the left side, φ is considered in $L^\infty(0,\,T)$. There exists always a strong lifting of $L^\infty(0,\,T)$ (see DINCULEANU [1], § 20, Sec. 2) and in the sequel such a lifting λ will be fixed. The property of λ which will be constantly used and which follows easily from ($*$) is that if for an everywhere defined bounded measurable function we have $(\lambda\psi)(t)=\psi(t)$ everywhere and if $\psi(t)=0$ a.e. on $(0,\,T)$, then $\psi(t)=0$ everywhere.

1) Recall that $\mathcal{C}_0\subset\mathcal{C}_2$, thus $F_t(\Phi)$ makes sense for $\Phi\in\mathcal{C}_0$.

$$(r \otimes \Phi)(t, u) = r(t)\Phi(u) \qquad (t \in (0, T),\ u \in \boldsymbol{N}))$$

whenever $r(\cdot) \in L^1(0, T)$, $\Phi(\cdot) \in \mathcal{C}_2$ and $r \geq 0$, $\Phi \geqslant 0$. By the arbitrariness of $r(\cdot)$, (3.30) implies $F_t(\Phi) \geq 0$ a.e. on $(0, T)$. Using (3.28″) and the properties of the lifting we deduce $F_t(\Phi) \geq 0$ for all $t \in (0, T)$. Thus for any $t \in (0, T)$ we have $F_t(\Phi) \geq 0$ whenever $\Phi(\cdot) \in \mathcal{C}_2$, $\Phi \geq 0$.

Denote, for $p = 1, 2, \ldots,$

(3.31) $$\boldsymbol{b}_p = \{u;\ u \in \boldsymbol{N}^1,\ \| u \| \leq p\}.$$

Then for any $\Phi(\cdot) \in \mathcal{C}_0$ (i.e. real continuous bounded functional on N) we have

(3.32) $$| F^{(m)}(\mathbf{1} \otimes \Phi) | \leq \int_0^T \left| \int_{\boldsymbol{N}} | \Phi(u) | \, d\mu_t^{(m)}(u) \right| dt \leq$$

$$\leq \int_0^T \left| \int_{\boldsymbol{b}_p} | \Phi(u) | \, d\mu_t^{(m)}(u) + \int_{\boldsymbol{N} \setminus \boldsymbol{b}_p} | \Phi(u) | \, d\mu_t^{(m)}(u) \right| dt \leq$$

$$\leq (\sup_{\boldsymbol{b}_p} | (\Phi(u) |) T c_{31} + \| \Phi \|_{\mathcal{C}_0} \int_0^T \mu_t^{(m)}(\boldsymbol{N} \setminus \boldsymbol{b}_p) dt.$$

But, using (3.26′),

(3.33) $$p^2 \int_0^T \mu_{(m)}^t(\boldsymbol{N} \setminus \boldsymbol{b}_p) dt \leq \int_0^T \left| \int_{\boldsymbol{N} \setminus \boldsymbol{b}_p} \| u \|^2 \, d\mu_t(u) \right| dt \leq c'_{31}$$

so that (3.32) gives

(3.32′) $$| F^{(m)}(\mathbf{1} \otimes \Phi) | \leq c_{31} \cdot T \cdot \sup_{\boldsymbol{b}_p} | \Phi(u) | + \frac{c'_{31}}{p^2} \| \Phi \|_{\mathcal{C}_0}$$

for all $m = 1, 2, \ldots$ Since $\mathbf{1} \otimes \Phi$ (recall $(\mathbf{1} \otimes \Phi)(t, u) = 1 \cdot \Phi(u) = \Phi(u)$ for all $(t, u) \in (0, T) \times \boldsymbol{N}$) belongs to $\mathfrak{L}_2$ we can deduce, from (3.32′), the relation

(3.32″) $$| F(\mathbf{1} \otimes \Phi) | \leq c_{31} \cdot T \cdot \sup_{\boldsymbol{b}_p} | \Phi(u) | + \frac{c'_{31}}{p^2} \| \Phi \|_{\mathcal{C}_0}$$

valid for all $p = 1, 2, \ldots$ and all $\Phi(\cdot) \in \mathcal{C}_0$.

Let now $\sigma=\{\Phi_r\}_{r=1}^{\infty}$ be a sequence of functionals $\Phi_r(\cdot)\in\mathcal{C}_0$ such that

$$\Phi_1(u)\geq\Phi_2(u)\geq \dots \rightarrow 0 \text{ for all } u\in\boldsymbol{N}. \tag{3.34}$$

Taking into account that $\boldsymbol{b}_p$ is compact in N and applying the Dini theorem we have that

$$\sup_{\boldsymbol{b}_p} |\varphi_r| \rightarrow 0 \text{ for } r\rightarrow\infty .$$

Since $F_t\geq 0$ we have

$$F_t(\Phi_1)\geq F_t(\Phi_2)\geq \dots \geq F_t(\Phi_r)\geq \dots \geq 0 \tag{3.34'}$$

for all $t\in(0, T)$, therefore by Fatou's lemma can infer from (3.32″) that

$$\int_0^T \lim_r F_t(\Phi_r)dt=\lim_r \int_0^T F_t(\Phi_r)dt=\lim_r F(\mathbf{1}\otimes\Phi_r)\leq\|\Phi_1\|_{\mathcal{C}_0}\frac{c'_{31}}{p^2};$$

since this is valid for all $p=1, 2, \dots$, it results

$$\lim_{r\rightarrow\infty} F_t(\Phi_r)=0 \quad \text{a.e. on } (0, T). \tag{3.35}$$

In this conslusion the exceptional set depends on the sequence σ.

We shall use this partial result to show that actually the exceptional set can be chosen independent of σ. To this purpose let us introduce the function defined on $\boldsymbol{N}$ by

$$d_{p,q}(u)=\begin{cases} q \text{ (distance in } \boldsymbol{N} \text{ of } u \text{ to } \boldsymbol{b}_p) \text{ if this distance is } \leq \dfrac{1}{q} \\ 1 \text{ if the distance is } > \dfrac{1}{q}. \end{cases}$$

Obviously $d_{p,q}\in\mathcal{C}_0$ for all p, $q=1, 2, 3, \dots$ For p, q fixed, let

$$\varphi_r(u)=\min\left\{d_{p,q}(u), \frac{1}{p^2}\|P_r u\|^2\right\}$$

for all $u \in \boldsymbol{N}$ and $r=1, 2, \ldots$ Then $\varphi_r(\cdot)$ and $\psi_r(\cdot)=d_{p,q}(\cdot)-\varphi_r(\cdot)$ belong to $\mathcal{C}_0$. Moreover since

$$\| P_r u \| \leq \| P_{r+1} u \| \leq \| u \| \tag{3.36}$$

for all $u \in \boldsymbol{N}$ (where $\| u \|$ is taken $=\infty$ if $u \notin \boldsymbol{N}^1$), the sequence $\{\varphi_r\}_{r=1}^{\infty}$ is increasing and

$$\varphi_r(u) \rightarrow \min \left\{ d_{p,q}(u), \frac{1}{p^2} \| u \|^2 \right\} = d_{p,q}(u)$$

for all $u \in \boldsymbol{N}$. Thus the sequence $\{\psi_r\}_{r=1}^{\infty}$ satisfies (3.34). In virtue of (3.35), there exists an exceptional set $E_{p,q} \subset (0, T)$ of Lebesgue measure 0 such that

$$\lim_{r \rightarrow \infty} F_t(\psi_r)=0 \text{ for all } t \in (0, T) \backslash E_{p,q}$$

that is

$$\lim_{r \rightarrow \infty} F_t(\varphi_r)=F_t(d_{p,q}) \text{ for all } t \in (0, T) \backslash E_{p,q} . \tag{3.37}$$

But

$$\varphi_r(u) \leq \frac{1}{p^2} \| P_r u \|^2 \text{ for all } u \in \boldsymbol{N}$$

so that

$$F_t(\varphi_r) \leq \frac{1}{p^2} F_t(\| P_r \cdot \|^2)$$

for all $t \in (0, T)$ and $r=1, 2, \ldots$, whence (by (3.37))

$$F_t(d_{p,q}) \leq \frac{1}{p^2} \lim_{r \rightarrow \infty} F_t(\| P_r \cdot \|^2) \tag{3.37'}$$

for all $t \in (0, T) \backslash E_{p,q}$.

Let

$$\theta(t)= \lim_{r \rightarrow \infty} F_t(\| P_r \cdot \|^2), \tag{3.37''}$$

where the limit exists (being perhaps also $=\infty$) because F_t is ≥ 0 and thus, by (3.36), $\{F_t(\| P_r\cdot \|^2)\}_{r=1}^{\infty}$ is non-decreasing in r, for all $t\in(0, T)$. Note now that

$$F^{(m)}(\mathbf{1}\otimes \| P_r\cdot \|^2)=\int_0^T \left| \int_N \| P_r u \|^2 \, d\mu_t^{(m)}(u) \right| dt \leq$$

$$\leq \int_0^T \left[\int_N \| u \|^2 \, d\mu_t^{(m)}(n) \right] dt \leq c'_{31}$$

for all m, $r=1, 2, \ldots$; since for a fixed r, $\mathbf{1}\otimes \| P_r\cdot \|^2 \in \mathfrak{L}_2$ and since F is a w^*-cluster point in $\mathfrak{L}^*_2$ too, we infer that

$$\int_0^T F_t(\| P_r\cdot \|^2)dt=F(\mathbf{1}\otimes \| P_r\cdot \|^2)\leq c'_{31} \qquad \text{for all } r=1, 2, \ldots$$

Applying the Beppo-Levi theorem we deduce that

$$\text{(3.37''')} \qquad \int_0^T \theta(t)dt=\lim_r \int_0^T F_t(\| P_r\cdot \|^2)dt\leq c'_{31}.$$

Thus if

$$E=\{t: t\in(0, T),\ \theta(t)=\infty\}\cup\Big(\bigcup_{p,q=1}^{\infty} E_{p,q}\Big),$$

the Lebesgue measure of E will be 0.

We shall show now that

$$\text{(3.35')} \qquad \lim_{r\to\infty} F_t(\Phi_r)=0 \text{ for all } t\in(0, T)\setminus E$$

and for all sequences $\{\Phi_r\}_{r=1}^{\infty}\subset \mathfrak{C}_0$ satisfying (3.34).

Indeed, if

$$\varepsilon_{p,r}=\max_{\boldsymbol{b}_p} \Phi_r$$

and if for every $u_0\in \boldsymbol{b}_p$, $\mathbf{O}_0=\{u: |u-u_0|<\varepsilon_0\}$ is a sufficiently small ball such that

$$|\Phi_r(u)-\Phi_r(u_0)|<\varepsilon_{p,r} \text{ for all } u\in \mathbf{O}_0,$$

then by the compactness of $\boldsymbol{b}_p$, there exists a finite number $\mathbf{O}'_0$, $\mathbf{O}''_0$, ..., $\mathbf{O}_0^{(l)}$ of such balls such that

$$\boldsymbol{b}_p \subset \mathbf{O}'_0 \cup \mathbf{O}''_0 \cup \dots \cup \mathbf{O}_0^{(l)} = \mathbf{O}.$$

Obviously $\mathbf{O}$ is an open neighbourhood of $\boldsymbol{b}_p$ so that for q sufficiently large we have

$$\left\{ u : \text{distance in } \boldsymbol{N} \text{ of } u \text{ to } \boldsymbol{b}_p \leq \frac{1}{q} \right\} \subset \mathbf{O}.$$

Therefore it is clear that we have

$$|\Phi_r(u)| \leq 2\varepsilon_{p,r} + \|\Phi_r\|_{\mathcal{C}_0} \cdot d_{p,q}(u) \text{ for all } u \in \boldsymbol{N},$$

and for all r, $p = 1, 2, \dots$ It results

$$0 \leq F_t(\Phi_r) \leq \varepsilon_{p,r} F_t(\mathbf{1}) + \|\Phi_r\|_{\mathcal{C}_0} \cdot F_t(d_{p,q})$$

for all p, $r = 1, 2, \dots$ and all $t \in (0, T)$. For fixed p, by Dini's theorem, $\varepsilon_{p,r} \to 0$ for $r \to \infty$, so that for $t \in (0, T) \backslash E_{p,q}$ we have

$$0 \leq \lim_r F_t(\varphi_r) \leq \|\Phi_1\|_{\mathcal{C}_0} \frac{1}{p^2} \theta(t). \tag{3.35''}$$

For $t \in (0, T) \backslash E$, the relation (3.35″) will be valid for all $p = 1, 2, \dots$, and $\theta(t)$ is $< \infty$, so that (3.35′) results from (3.35″) making $p \to \infty$.

We proved in this manner that for any $t \in (0, T) \backslash E$, the functional $F_t \in \mathcal{C}_0^*$ satisfies Daniell's condition; therefore by the well known theory of the Daniell integral (see for instance Loomis [1], Ch. III, §§ 12-13) it results that there exists a Borel positive measure μ_t on $\boldsymbol{N}$ such that (3.29) is true; recall that this assertion is valid for all $t \in (0, T) \backslash E$.

Let for $k = 1, 2, \dots$,

$$\varphi_k(r) = r \text{ for } 0 \leq r \leq k \text{ and } = k \text{ for } r \geq k.$$

Then $\Phi(\cdot) = \varphi_k(1 + |\cdot|^2) \in \mathcal{C}_0$ and is $\leq 1 + |\cdot|^2 \in \mathcal{C}_2$. Therefore for $t \in (0, T) \backslash E$ and all $k = 1, 2, \dots$

$$\int_N \varphi_k(1+|u|^2)d\mu_t(u)=F_t(\Phi)\leq F_t(1+|\cdot|^2)\leq \| F_t \|_{\mathcal{C}_2^*} \| 1+|\cdot|^2 \|_{\mathcal{C}_2} =$$
$$= \| F_t \|_{\mathcal{C}_2^*} \leq \| F \| = c_{31}\,.$$

Letting $k\to\infty$ and applying the theorem of Beppo Levi we obtain

$$\int_N (1+|u|^2)d\mu_t(u)\leq c_{31} \text{ for all } t\in(0,\, T)\backslash E. \tag{3.38}$$

The same argument can be applied to $1+\| P_r\cdot \|^2$ instead of $1+|\cdot|^2$. It leads to the conclusion

$$\int_N (1+\| P_r u \|^2)d\mu_t(u)\leq \| F \| + \theta(t) \text{ for all } t\in(0,\, T)\backslash E$$

and all $r=1, 2, \ldots$ Making $r\to\infty$ we deduce finally

$$\int_N (1+\| u \|^2)d\mu_t(u)\leq \| F \| + \theta(t) \text{ for all } t\in(0,\, T)\backslash E. \tag{3.38$'$}$$

Let us define $\mu_t\equiv 0$ for $t\in E$. Then (3.38) and (3.38′) (because of (3.37‴)) show that the family $\{\mu_t\}_{0<t<T}$ satisfies the conditions (3.16-16′). Moreover if Φ is any weakly continuous functional ≥ 0 on $\boldsymbol{N}$, then $\Phi_k(\cdot)$ defined by

$$\Phi_k(u)=\min\{\Phi(u),\, k\} \qquad (u\in\boldsymbol{N})$$

for all $k=1, 2, \ldots$, belongs to $\mathcal{C}_0$, thus, since

$$\int_N \Phi_k(u)d\mu_t(u)=F_t(\Phi_k) \text{ for all } t\in(0,\, T)\backslash E,$$

the function in the first term is measurable in t. Letting $k\to\infty$, $\Phi_k(u)$ tends increasing to $\Phi(u)$ (for all $u\in N$). Thus applying the Beppo Levi theorem we deduce that

$$\int_{N} \Phi(u)d\mu_t(u) = \lim \int_{N} \Phi_k(u)d\mu_t(u) \text{ for all } t\in(0, T)$$

is a measurable function in t. Thus $\{\mu_t\}_{0<t<T}$ is a basic family of measures on $\boldsymbol{N}$. Finally if $\Phi(\cdot, \cdot)\in L^1(0, t; \mathcal{C}_0)$ then $\Phi(t, \cdot)\in\mathcal{C}_0$ a.e. on $(0, T)$, thus, by (3.29), a.e. on $(0, T)$

$$F_t(\Phi(t, \cdot)) = \int_{N} \Phi(t, u)d\mu_t(u).$$

Therefore (3.18) is valid for this $\Phi(\cdot, \cdot)$ in virtue of the representation (3.28).

This finishes the proof of Lemma 3.

REMARK. *If all measures $\mu_t^{(m)}$ $(0<t<T, m=1, 2, ...)$ are probabilities* (i.e. if $\mu_t^{(m)}(\boldsymbol{N})=1$ for all m and t) *then in the basic family $\{\mu_t\}_{0<t<T}$ yielded by Lemma 3, for almost all $t\in(0, T)$, the measures μ_t are probabilities.*

Indeed if all measures $\mu_t^{(m)}$ are probabilities, then for any $r(\cdot)\in$ $\in L^1(0, T)$ we have

$$F^{(m)}(r\otimes\mathbf{1}) = \int_0^T \left| \int_{N} (r\otimes\mathbf{1})(t, u)d\mu_t^{(m)}(u) \right| dt =$$

$$= \int_0^T r(t)\mu_t^{(m)}(\boldsymbol{N})dt = \int_0^T r(t)dt,$$

so that using the fact that $r\otimes\mathbf{1}\in\mathfrak{L}_2$ we can infer

$$F(r\otimes\mathbf{1}) = \int_0^T r(t)dt.$$

But $r\otimes\mathbf{1}$ belongs even to $L^1(0, T; \mathcal{C}_0)$, thus

$$\int_0^T r(t)\mu_t(\boldsymbol{N})dt = \int_0^T F_t((r\otimes\mathbf{1})(t, \cdot))dt = F(r\otimes\mathbf{1}) = \int_0^T r(t)dt$$

for all $r(\cdot)\in L^1(0, T)$ so that

$$\mu_t(\boldsymbol{N})=1 \text{ a.e. on } (0, T).$$

b) Lemma 3 can be completed with a less immediate remark, given by the following useful

LEMMA 4. *If in Lemma 3, the function* $|\cdot|^2$ *is uniformly integrable with respect to almost all* $\mu_t^{(m)}$ $(0<t<T,\ m=1, 2, \ldots)$ [1]) *then* (3.18) *is valid also for every* $\Phi(\cdot, \cdot)$ *belonging to* $\mathfrak{L}_2$.

PROOF. The hypothesis of uniform integrability yields us a function $\varepsilon(r)$ of $r\in(1, \infty)$ such that

$$\varepsilon(r)\to 0 \quad \text{for } r\to\infty \tag{3.39}$$

$$\int\limits_{\{u:\,|u|\geq r\}} (1+|u|^2)d\mu_t^{(m)}(u)\leq\varepsilon(r) \tag{3.39'}$$

for all $r\geq 1$, almost all $t\in(0, T)$, and $m=1, 2, \ldots$

Let now $\Phi(\cdot)\in\mathfrak{C}_2$ satisfy

$$\Phi(u)=0 \quad \text{for } |u|\leq r \tag{3.40}$$

for a certain $r\geq 1$, and let $\rho(\cdot)\in L^1(0, T)$.

Then

$$|F^{(m)}(\rho\otimes\Phi)|=\left|\int\limits_0^T\left[\int\limits_N \rho(t)\Phi(u)d\mu_t^{(m)}(u)\right]dt\right|=$$

$$=\left|\int\limits_0^T r(t)\left[\int\limits_{\{u:\,|u|\geq r\}}\Phi(u)d\mu_t^{(m)}(u)\right]dt\right|\leq\int\limits_0^T|\rho(t)|\left|\int\limits_{\{u:\,|u|\geq r\}}|\Phi(u)|\,d\mu_t^{(m)}(u)\right|dt\leq$$

$$\leq\int\limits_0^T|\rho(t)|\cdot\|\Phi\|_{\mathfrak{C}_2}\left[\int\limits_{\{u:\,|u|\geq r\}}(1+|u|^2)d\mu_t^{(m)}(u)\right]dt\leq$$

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[1]) This means that for any $\varepsilon>0$ there exists an $0<r_\varepsilon<\infty$ such that $\int\limits_{\{u:\,u\in N,\,|u|\geq r_\varepsilon\}}|u|^2\,d\mu_t(u)\leq\varepsilon$ for all m and almost all t.

$$\leq \varepsilon(r) \| \Phi \|_{\mathcal{C}_2} \int_0^T | \rho(t) | \, dt,$$

so that (since $\rho \otimes \varphi \in \mathfrak{L}_2$) we deduce that for the w^*-cluster point F in $\mathfrak{L}^*_2$ we have

$$| F(\rho \otimes \Phi) | \leq \varepsilon(r) \| \Phi \|_{\mathcal{C}_2} \int_0^T | \rho(t) | \, dt$$

and this is valid for all $\rho(\cdot) \in L^1(0, T)$. This conclusion together with (3.28) leads to

$$| F_t(\Phi) | \leq \varepsilon(r) \| \Phi \|_{\mathcal{C}_2} \tag{3.40'}$$

a.e. on $(0, T)$. Using (3.28″) and the properties of the lifting λ we deduce that actually (3.40′) is valid for all $t \in (0, T)$.

Let now Φ be an arbitrary functional $\in \mathcal{C}_2$, let $t \in (0, T) \backslash E$ (see the proof of Lemma 3) and let for $r \geq 2$

$$\varphi_r(u) = \begin{cases} 1 & \text{for } | u | \leq r-1 \\ 0 & \text{for } | u | \geq r \\ \text{linear in } | u | & \text{for } r-1 \leq | u | \leq r. \end{cases} \tag{3.41}$$

Then $(1-\varphi_r)\Phi \in \mathcal{C}_2$, satisfies (3.40) and $\| (1-\varphi_r)\Phi \|_{\mathcal{C}_2} \leq \| \Phi \|_{\mathcal{C}_2}$. Therefore in virtue of (3.40′) we have

$$F_t(\varphi_r \Phi) \to F_t(\Phi) \text{ for } r \to \infty . \tag{3.40''}$$

But since $\varphi_r \Phi \in \mathcal{C}_0$ and $t \in (0, T) \backslash E$, we have also

$$F_t(\varphi_r \Phi) = \int_N \varphi_r(u) \Phi(u) d\mu_t(u)$$

where the integral converges to $\int_N \Phi(u) d\mu_t(u)$. This can be shown in the same way as (3.40″), using (3.38) instead of the uniform integrability. Thus (3.29) is valid for all $\Phi \in \mathcal{C}_2$ and consequently (3.28), with $x(\cdot)$ of the form $x(t) \equiv \Phi(t, \cdot)$, $\Phi(\cdot, \cdot) \in \mathfrak{L}_2 = L^1(0, T; \mathcal{C}_2)$ coincides with (3.18).

This finishes the proof of Lemma 4.

The above result does not assure that in the conditions of Lemmas 3 and 4, the relation (3.18) is valid also for all $\Phi(\cdot, \cdot)$ of $\mathfrak{L}_{1,1}$. There is however an important functional of $\mathfrak{L}_{1,1}$ for which this is still true, namely the functional $\Psi_\Phi(\cdot, \cdot)$ defined by (3.22), in case $\Phi(\cdot, \cdot)$ is an elementary test functional. As we already remarked, the functional Ψ_Φ belongs in this case to $\mathfrak{L}_{1,1}$, so that if F is the functional yielded by Lemma 3, then $F(\Psi_\Phi)$ makes sense. With this remark we can pass to the following supplement to Lemmas 3-4:

COROLLARY. *In the conditions of Lemmas* 3 *and* 4, *the formula* (3.18) (for the functional F yielded by Lemma 3) *is valid for all functionals* $\Psi_\Phi(\cdot, \cdot)$ *with* $\Phi \in \mathcal{T}_0$.

PROOF. Firstly note that $\Phi \in \mathcal{T}_0$ implies that $\Phi'_t(t, \cdot)$ is continuous in t from (0, T) to $\mathcal{C}_2$, because for all $t \in (0, T)$ and $u \in \boldsymbol{N}$:

$$\frac{|\Phi'_t(t, u)|}{1+|u|^2} \leq \frac{c_{32}(1+|u|^2)^{1/2}}{1+|u|^2} = \frac{c_{32}}{(1+|u|^2)^{1/2}}$$

and

$$\Phi'_t(t, u) = \Phi'_t(t, P_m u),$$

with a suitable constant c_{32} and integer $m(=1, 2, \ldots)$ depending only on Φ. Sencondly note that $\Phi'_u(t, u)$ is (see (3.10)) continuous and bounded from $(0, T) \times \boldsymbol{N}$ to $\boldsymbol{N}^2$. This implies that the functions in t

$$((\cdot, \Phi'_\square(t, \boxed{\cdot}))) \text{ and } (f(t), \Phi'_\square(t, \boxed{\cdot}))$$

belong to $L^\infty(0, T; \mathcal{C}_2)$. In this way

$$\Psi_\Phi(\cdot, \cdot) = \Theta(\cdot, \cdot) + b(\cdot, \cdot, \Phi'_\square(\cdot, \boxed{\cdot}))$$

where

$$\Theta(\cdot, \cdot) \in L^\infty(0, T; \mathcal{C}_2) \subset L^1(0, T; \mathcal{C}_2) = \mathfrak{L}_2,$$

so that by Lemma 4, to prove our corollary it is sufficient to show that

$$(3.42)\qquad F(b(\cdot,\ \cdot,\ \Phi'_{\square}(\cdot,\ \overline{\underline{|\cdot|}})) = \int_0^T \left[\int_N b(u,\ u,\ \Phi'_u(t,\ u))d\mu_t(u) \right| dt.$$

Let denote by $\Psi(\cdot,\ \cdot)$ the functional occurring in (3.42) and let for a $k=1,\ 2,\ \ldots$ denote by $\Psi_k(\cdot,\ \cdot)$ the functional defined by

$$(3.42')\qquad \Psi_k(t,\ u) = b(P_k u,\ u,\ \Phi'_u(t,\ u)) = (B(P_k u,\ u),\ \Phi'_u(t,\ u)).$$

Since $(B(u,\ v),\ w)$ is continuous in $(u,\ v,\ w) \in \boldsymbol{N}^1 \times \boldsymbol{N} \times \boldsymbol{N}^2$ (see Sec. 2.5) and P_k is continuous from $\boldsymbol{N}$ into $\boldsymbol{N}^1$, it is clear that $\Psi_k(t,\ \cdot)$ is continuous in t from $(0,\ T)$ into $\mathfrak{C}_2$; therefore

$$(3.42_k)\qquad F(\Psi_k) = \int_0^T \left| \int_N \Psi_k(t,\ u)d\mu_t(u) \right| dt$$

for all $k=12,\ \ldots$ Using also the fact that $(B(u,\ v),\ w)$ is continuous in $(u,\ v,\ w) \in \boldsymbol{N} \times \boldsymbol{N}^1 \times \boldsymbol{N}^2$ we deduce that

$$(3.42'_k)\qquad |\Psi(t,\ u) - \Psi_k(t,\ u)| \leq c_{33} |(I-P_k)u| \cdot \| u \| \qquad \text{for all } (t,\ u) \in (0,\ T) \times \boldsymbol{N}^1,$$

where c_{33} is a constant depending on Φ. In virtue of $(3.42'_k)$ it results

$$\left| \int_0^T \left| \int_N \Psi(t,\ u)d\mu_t(u) \right| dt - \int_0^T \left| \int_N \Psi_k(t,\ u)d\mu_t(u) \right| dt \right| \leq$$

$$\leq c_{33} \int_0^T \left[\int_N |(I-P_k)u| \cdot \| u \| \, d\mu_t(u) \right] dt \leq$$

$$\leq c_{33} \left\{ \int_0^T \left[\int_N |(I-P_k)u|^2 \, d\mu_t(u) \right| dt \right\}^{1/2} \left\{ \int_0^T \| u \|^2 \, d\mu_t(u) \right| dt \right\}^{1/2}$$

where in virtue of Lebesgue's dominated convergence theorem the first integral tends to 0 for $k \to \infty$. In this manner to infer (3.42) from (3.42_k), it remains to prove the convergence

(3.43) $$F(\Psi_k) \to F(\Psi) \text{ for } k \to \infty .$$

To this purpose, note that

$$F^{(m)}(\mathbf{1} \otimes |(I-P_k)\cdot|\cdot\|\cdot\|) = \int_0^T \left| \int_N |(I-P_k)u|\cdot\|u\|\, d\mu_t^{(m)}(u) \right| dt \leq$$

$$\leq \int_0^T \left[\int_N |(I-P_k)u|^2\, d\mu_t^{(m)}(u) \right]^{1/2} \cdot \left[\int_N \|u\|^2\, d\mu_t^{(m)}(u) \right]^{1/2} dt \leq$$

$$\leq \left\{ \int_0^T \left[\int_N |(I-P_k)u|^2\, d\mu_t^{(m)}(u) \right] dt \right\}^{1/2} c'_{31} ,$$

whence

$$|F^{(m)}(\mathbf{1} \otimes |(I-P_k)\cdot|\cdot\|\cdot\|)| \leq c'_{31}\{F^{(m)}(\mathbf{1} \otimes |I-P_k)\cdot|^2)\}^{1/2}$$

for all $m, k=1, 2, \ldots$ Since

$$\mathbf{1} \otimes |(I-P_k)\cdot|\cdot\|\cdot\| \text{ and } \mathbf{1} \otimes |(I-P_k)\cdot|^2 \in \mathfrak{L}_{1,1}$$

we can infer that

$$F(\mathbf{1} \otimes |(I-P_k)\cdot|\cdot\|\cdot\|) \leq c'_{31}\{F(\mathbf{1} \otimes |I-P_k)\cdot|^2)\}^{1/2},$$

and this for all $k=1, 2, \ldots$ Taking into account ($3.42'_k$) we obtain

(3.43′) $$|F(\Psi)-F(\Psi_k)| \leq c'_{31}c_{33}\{F(\mathbf{1} \otimes |I-P_k)\cdot|^2)\}^{1/2}.$$

But $\mathbf{1} \otimes |(I-P_k)\cdot|^2 \in \mathfrak{L}_2$ thus by Lemma 4 we have

$$F(\mathbf{1} \otimes |(I-P_k)\cdot|^2) = \int_0^T \left| \int_N |(I-P_k)u|^2\, d\mu_t(u) \right| dt$$

which as we already remarked converges to 0 for $k \to \infty$. Thus (3.43′) yields (3.43) and this finishes the proof.

REMARK. It is easy to verif that the conclusions of Lemmas 3-4 and of the corollary above, remain valid if the sequence $\{F^{(m)}\}$ is replaced by a directed set $\{F^{(\alpha)}\}$.

c) We can now give the

PROOF OF THE EXISTENCE THEOREM (see Sec. 3.2.*a*)). We begin by considering the differential system

$$\frac{du}{dt}+Au_m+P_mB(u_m\,,\ u_m)=P_mf \text{ in } P_m\boldsymbol{N}. \tag{3.44_m}$$

This is one very wellknown Faedo-Galerkin approximation of (2.19), which plays a basic role in the proof of the existence of individual solutions for (2.19). For sake of completeness we shall give here the properties of (3.44_m) which we shall need in the sequel. A simple computation leads successively to

$$\frac{1}{2}\frac{d}{dt}|\,u_m\,|^2+\nu\,||\,u_m\,||^2=(f,\ u_m), \tag{$3.44'_m$}$$

$$|\,u_m(t)\,|^2+\nu\int_0^t||\,u_m(\tau)\,||^2\,d\tau\leq|\,u_m(0)\,|^2+c_{34}\int_0^t||\,f(\tau)\,||^2_{N^{-1}}d\tau, \tag{$3.44''_m$}$$

with a constant c_{34} dependent only on ν. In virtue of these relations, (3.44_m) has for any $u_{0m}\in P_m\boldsymbol{N}$ a solution defined on whole $[0,\ T]$ with initial data u_{0m}. Obviously this solution is uniquely determined by u_{0m} and satisfies ($3.44'_m-44''_m$) for all $t\in[0,\ T]$. Let $S^{(m)}(t)u_{0m}$ denote the value in t of this solution. It is easy to verify that $S^{(m)}(t)u_{0m}$ is continuous in $(t,\ u_{0m})$ as function from $[0,\ T]\times P_m\boldsymbol{N}$. Put

$$\mu^{(m)}(\omega)=\mu(P_m^{-1}(\omega\cap P_m\boldsymbol{N}))$$

for every Borel set $\omega\subset\mathbf{N}$. We obtain a Borel probability with support in $P_m\boldsymbol{N}$. By (3.1′) we have for any $\Phi(\cdot)\in\mathcal{C}_2$ that

$$(3.45)\qquad \int_N \Phi(u)d\mu^{(m)}(u)=\int_N \Phi(P_m u)d\mu(u).$$

Put, for any $m=1, 2, \ldots$ and $t\in(0, T)$,

$$\mu_t^{(m)}(\omega)=\mu^{(m)}((S^{(m)}(t))^{-1}(\omega\cap P_m\boldsymbol{N}))$$

for every Borel set $\omega\subset\boldsymbol{N}$. It is plain that one obtains a (Borel) probability on $\boldsymbol{N}$ with the $\operatorname{supp}\mu_t^{(m)}$ in $P_m\boldsymbol{N}$ and such that (for all $t\in[0, T]$)

$$(3.45')\qquad \int_N \Phi(u)d\mu_t^{(m)}(u)=\int_N \Phi(S^{(m)}(t)P_m u)d\mu(u)$$

for every $\Phi\in\mathcal{C}_0$; actually, using (3.1′), ($3.44_m''$) and $\operatorname{supp}\mu_t^{(m)}\subset P_m\boldsymbol{N}$ one can verify that (3.45′) is valid also for $\Phi\in\mathcal{C}_2$ or $\Phi\in\mathcal{C}_{1,1}$. Since the right term in (3.45′) is continuous in t if for instance $\Phi\in\mathcal{C}_0$ we can easily infer that $\{\mu_t^{(m)}\}_{0<t<T}$ satisfies condition (3.16″). Moreover integrating ($3.44_m''$) with respect to μ and taking into account (3.45′), we obtain (using also Fubini's theorem)

$$\begin{aligned}\int_N |u|^2 d\mu_t^{(m)}(u)+\nu\int_0^T\left|\int_N \|u\|^2 d\mu_t^{(m)}(u)\right|dt&\le\\ \le\int_N |u|^2 d\mu^{(m)}(u)+c_{35}&\le\\ \le\int_N |u|^2 d\mu(u)+c_{35}&=c_{36},\end{aligned}$$

where $c_{35}-c_{36}$ are constants independent of $t\in(0, T)$ and $m=1, 2, \ldots$ In this manner we verified that $\{\mu_t^{(m)}\}_{0<t<T}$ is a basic family of probabilities on $\boldsymbol{N}$, and moreover that the sequence $\{\{\mu_t^{(m)}\}_{0<t<T}\}_{m=1}^{\infty}$ satisfies the conditions in Lemma 3. We shall show now that actually it satisfies also the supplementary condition of Lemma 4.

Indeed, since by ($3.44_m''$)

$$(3.44_m''')\qquad |S^{(m)}(t)u_{0m}|^2\le|u_{0m}|^2+c_{37}$$

where c_{37} is a constant depending only on $f(\cdot)$ and ν, for $r\ge(c_{37})^{1/2}$

in the same way as that one which led us from the Navier-Stokes equations (2.19), in the case of dimension $n=2$, to (3.13_{I}). For Φ fixed, if m is enough large we have $P_m\Phi'_u=\Phi'_u$ in virtue of the condition (3.10) (namely this m has to be $\geq$ than that one, depending on Φ, which occurs in (3.10)). Thus we can write ($3.13_{\mathrm{I},m}$) in the following manner

$$F^{(m)}(\Psi_\Phi)=\int_N \Phi(0,\ P_m u)d\mu(u) \qquad \text{for all } m\geq m_\Phi\,, \tag{$3.13_{\mathrm{I,appr.}}$}$$

where m_Φ is the integer depending on $\Phi\in\mathcal{C}_0$ which occurs in (3.10) (or equivalently in (3.7)). Since $\Phi(0,\ P_m u)\to\Phi(0,\ u)$ for $m\to\infty$, by (3.5) and (3.25) we can infer that

$$\int_N \Phi(0,\ P_m u)d\mu(u)\to\int_N \Phi(0,\ u)d\mu(u),$$

for $m\to\infty$; consequently ($3.13_{\mathrm{I,appr.}}$) implies

$$F^{(m)}(\Psi_\Phi)\to\int_N \Phi(0,\ u)d\mu(u)$$

which on its turn implies ($3.13'_{\mathrm{II}}$). This finished the proof of the existence theorem.

Before finishing this section let us mention that to obtain statistical solutions with supplementary properties, we shall in the sequel return to this proof and analyse it more completely.

3. *Elementary properties of statistical solutions. a*) We start by noting that if $\Phi(t,\ u)=r(t)\Phi(u)$ where $r(\cdot)\in C^\infty([0,\ T))$ and $\Phi(\cdot)$ is a real functional defined on $\boldsymbol{N}^1$, then $\Phi(\cdot,\ \cdot)\in\mathcal{C}$, for all $r(\cdot)$, if and only if $\Phi(\cdot)$ satisfies the following conditions.

(i_c) $|\Phi(u)|\leq c_{38}+c_{39}|u|$ for any $u\in\boldsymbol{N}^1$ and some suitable constants c_{38}-c_{39} (depending on $\Phi(\cdot)$);

(ii_c) $\Phi(\cdot)$ is Frechet $\boldsymbol{N}$-differentiable in the direction of $\boldsymbol{N}^1$;

we have

$$|S^{(m)}(t)u_{0m}| \geq r \text{ only if } |u_{0m}| \geq \sqrt{r^2 - c_{37}}$$

so that for all $m=1, 2, \ldots$

$$\int_{\{u:|u|\geq r\}} |u|^2 d\mu_t^{(m)}(u) = \int_{\{u:|S^{(m)}(t)P_m u|\geq r\}} |S^{(m)}(t)P_m u|^2 d\mu(u) \leq \int_{\{u:|S^{(m)}(t)P_m u|\geq r\}} (c_{37} + |P_m u|^2) d\mu(u) \leq$$

$$\leq \int_{\{u:|P_m u|\geq\sqrt{r^2-c_{37}}\}} (c_{37} + |P_m u|^2) d\mu(u) \leq \int_{\{u:|u|\geq\sqrt{r^2-c_{37}}\}} (c_{37} + |u|^2) d\mu(u),$$

which in virtue of (3.5), converges to 0 for $r \to \infty$, and is independent of m. So we can apply Lemmas 3 and 4 as well as their Corollary to the present situation. We obtain a basic family of probabilities $\{\mu_t\}_{0<t<T}$ [1]) and a functional $F \in \mathfrak{L}^*_{1,1;2}$ connected as in Lemmas 3-4 and the Corollary, and such that F is a w^*-cluster point in $\mathfrak{L}^*_2$ of the sequence $\{F^{(m)}\}_{m=1}^{\infty} \subset \mathfrak{L}^*_{1,1;2}$ corresponding to our sequence of basic families $\{\mu_t^{(m)}\}_{0\leq t\leq T}$ $(m=1, 2, \ldots)$. In virtue of the Remark 3° in Sec. 3.1.e) and the Corollary to Lemma 4 we have only to show that the functional F satisfies

$$(3.13'_{\mathrm{II}}) \qquad F(\Psi_\Phi) = \int_N \Phi(0, u) d\mu(u)$$

for all $\Phi \in \mathfrak{T}_0$ satisfying (3.8′). For a fixed $m=1, 2, \ldots$ and for any $\Phi(\cdot) \in \mathfrak{T}_0$ satisfying (3.8′), we can pass from (3.44_m) to

$$(3.13_{\mathrm{I},m}) \qquad \int_0^T \left\{ \int_N [-\Phi'_t(t, u) + \nu((u, \Phi'_u(t, u))) + \right.$$

$$\left. + b(u, u, P_m\Phi'_u(t, u))] d\mu_t^{(m)}(u) \right\} dt = \int_N \Phi(0, P_m u) d\mu(u) +$$

$$+ \int_0^T \left| \int_N (P_m f(t), \Phi'_u(t, u)) d\mu_t^{(m)}(u) \right] dt$$

[1]) For those t for which μ_t is not a probability replace μ_t by μ. This alterates the family of a set of Lebesgue measure 0 in $(0, T)$ so all the other properties are conserved.

(iii$_c$) $\Phi'(\cdot)$ is continuous from $\boldsymbol{N}^1$ to $\boldsymbol{N}^1$ and $\|\Phi'(\cdot)\|$ is bounded [1]).

Such a functional $\Phi(\cdot)$ will be called a *time-independent test functional* and the family of these functional will be denoted by $\mathcal{T}^{\text{ind}}$.

If moreover

$$\Phi(u)=\Phi(P_m u) \qquad \text{for all } u\in\boldsymbol{N}^1$$

and a fixed suitable integer $m=1, 2, \ldots$, $\Phi(\cdot)$ will again be called elementary. The corresponding class will be denoted by $\mathcal{T}_0^{\text{ind}}$.

Plaintly the Remarks 5, and 6, in Sec. 3.1.*e*) can be also applied to a $\Phi(\cdot)\in\mathcal{T}^{\text{ind}}$, obtaining readily that $\Phi(\cdot)$ can be extended to a Lipshitz function to whole $\boldsymbol{N}^{-1}$ whose restriction to $\boldsymbol{N}$ is weakly continuous; $\Phi(u)$ can not be 0 for all $\|u\|$ large enough unless $\Phi(\cdot)$ is identical 0.

LEMMA 5. *Let* $\{\mu_t\}_{0<t<T}$ *be a basic family of probabilities on* $\boldsymbol{N}$, *let* μ *be a probability on* $\boldsymbol{N}$ *satisfying* (3.5) *and let* $f(\cdot)\in L^1(0, T; \boldsymbol{N}^{-1})$. *Then the following three conditions are equivalent*:

(i) $\{\mu_t\}_{0<t<T}$ *is a statistical solution of the Navier-Stokes equations with right term* $f(\cdot)$ *and initial data* μ (i.e. $\{\mu_t\}$ satisfies (3.13$_{\text{I}}$) for all test functionals which verify (3.8')).

(ii) $\{\mu_t\}_{0<t<T}$ *satisfies the equation*

$$(3.13_{\text{III}}) \qquad \int_{\boldsymbol{N}}\Phi(\tau, u)d\mu_\tau(u)+$$

$$+\int_0^\tau\left\{\int_{\boldsymbol{N}}[-\Phi'_t(t, u)+\nu((u, \Phi'_u(t, u)))+b(u, u, \Phi'_u(t, u))]d\mu_t(u)\right\}dt=$$

$$=\int_{\boldsymbol{N}}\Phi(0, u)d\mu(u)+\int_0^\tau\left\{\int_{\boldsymbol{N}}(f(t), \Phi'_u(t, u))d\mu_t(u)\right\}dt$$

a.e. on $(0, T)$, *for every* $\Phi(\cdot, \cdot)\in\mathcal{T}$.

(iii) $\{\mu_t\}_{0<t<T}$ *satisfies the equation* (3.13$_{\text{I}}$) *for all functionals*

---

1) Obviously the condition (i_c) is implied by the others.

$\Phi(\cdot,\ \cdot)$ *of the form* $\Phi(t,\ u)=r(t)\Phi(u)$ *with* $r\in C_0^\infty([0,\ T))$ *and* $\Phi(\cdot)\in$ $\in\mathcal{T}^{\text{ind}}$.

(iv) $\{\mu_t\}_{0<t<T}$ *satisfies the equation*

$$(3.13_{\text{IV}})\quad \int\limits_N \Phi(u)d\mu_\tau(u)+\int\limits_0^\tau\left\{\int\limits_N[\nu((u,\Phi'(u)))+b(u,u,\Phi'(u))]d\mu_t(u)\right\}dt=$$

$$\int\limits_N \Phi(u)d\mu(u)+\int\limits_0^\tau\left\{\int\limits_N(f(t),\ \Phi'(u))d\mu_t(u)\right\}dt$$

a.e. on $(0,\ T)$, *for every* $\Phi(\cdot)\in\mathcal{T}^{\text{ind}}$.

PROOF. It is obvious that

$$(3.46)\qquad \begin{array}{c}(\text{ii})\Rightarrow(\text{i})\Rightarrow(\text{iii})\\ \Downarrow\\ (\text{iv})\end{array}$$

Multiplying (3.13_{IV}) with $-r'(\tau)$ ($r(\cdot)$ belonging to $C_0^\infty([0,\ T))$), integrating in τ the resulting relation between 0 and T and efectuating an integration by part in those terms which involve the integrals from 0 to τ, we obtain easily (3.13_{I}) with $\Phi(t,\ u)=r(t)\Phi(u)$; thus (iv) $\Rightarrow$ (iii). On the other hand replacing in (3.13_I) the functional $\Phi(\cdot,\ \cdot)$ by its product with $r(\cdot)\in C_0^\infty([0,\ T))$ we get

$$-\int\limits_0^T r'(t)\left[\int\limits_N \Phi(t,\ u)d\mu_t(u)\right]dt+$$

$$+\int\limits_0^T r(t)\left\{\int\limits_N[-\Phi'_t(t,\ u)+\nu((u,\ \Phi'_u(t,\ u)))+b(u,\ u,\ \Phi'_u(t,\ u))]d\mu_t(u)\right.$$

$$=r(0)\int\limits_N \Phi(0,\ u)d\mu(u)+\int\limits_0^T r(t)\left\{\int\limits_N(f(t),\ \Phi'_u(u))d\mu_t(u)\right\}dt$$

and this is valid for all $r(\cdot)\in C_0^\infty([0,\ T))$ and $\Phi(\cdot,\ \cdot)\in\mathcal{T}$. But for a basic family $\{\mu_t\}_{0<t<T}$ and a functional $\Phi(\cdot, \cdot)\in\mathcal{T}$, the functions

$$\varphi_0(t)=\int_N \Phi(t, u)d\mu_t(u)$$

and

$$\varphi_1(t)=\int_N [-\Phi'_t(t, u)+\nu((u, \Phi'_u(t, u)))+$$
$$+b(u, u, \Phi'_u(t, u))-(f(t), \Phi'_u(t, u))]d\mu_t(u)$$

belong to $L^1(0, T)$, being related by the relation

$$-\int_0^T r'(t)\varphi_0(t)dt+\int_0^T r(t)\varphi_1(t)dt=r(0)\int\Phi(0, u)d\mu(u)$$

valid for all $r(\cdot)\in C_0^\infty([0, T))$. Plainly this gives

$$\text{(3.46}')\qquad -\int_0^T r'(t)\left[\varphi_0(t)+\int_0^t \varphi_1(\sigma)d\sigma\right|dt=r(0)\int_N \Phi(0, u)d\mu(u)$$

for all $r\in C_0^\infty([0, T))$. This shows firstly that the derivative (in the sense of the theory of distributions) of $\psi(\cdot)$, where

$$\psi(t)=\varphi_0(t)+\int_0^t \varphi_1(\sigma)d\sigma \text{ for } t\in(0, T),$$

is 0, thus $\psi(\cdot)$ coincides a.e. on $(0, T)$ with a constant which on account of (3.46′) must be $\int \Phi(0, u)d\mu(u)$. Thus we have verified that (i) $\Rightarrow$ (ii). The same argument applies for the implication (iii) $\Rightarrow$ (iv).

For the remaining part it is sufficient to prove the implication

$$\text{(3.46}'')\qquad \text{(iii)} \Rightarrow \text{(i)}.$$

Let $\Phi(\cdot, \cdot)\in T$ satisfy (3.8′) and let $r(\cdot)\in C_0^\infty(\mathbb{R})$. Define

(3.47) $$(\Phi * r)(t, u)=\int_0^T \Phi(s, u)r(t-s)ds, \text{ for all } (t, u)\in \mathbb{R}\times \boldsymbol{N}^1$$

and

$$(\Phi * r)_m(t, u)=\frac{T}{m}\sum_{k=0}^{m-1}\Phi\left(\frac{kT}{m}, u\right)r\left(t-\frac{kT}{m}\right) \text{for all } (t, u)\in \mathbb{R}\times \boldsymbol{N}^1.$$

Since for any fixed $t\in[0, T]$ we have $\Phi(t, \cdot)\in\mathcal{C}^{\text{ind}}$ (see Remark 6) in Sec. 3.1.*e*) for property (i_c) of time-independent test functions), if the condition (iii) of the lemma is satisfied, then (3.13_{I}) is valid for $(\Phi * r)_m$ for any $m=1, 2, \ldots,$ whenever the support of r verifies the condition

(3.47′) $$\operatorname{supp} r(\cdot)\subset[-\eta, \eta]$$

where $\eta>0$ is such that

$$\Phi(t, \cdot)=0 \text{ for } t>T-\eta.$$

Now

$$\| (\Phi * r)'_u(t, u)-[(\Phi * r)_m]'_u(t, u) \|\leq \sum_{k=0}^{m-1}\int_{\frac{kT}{m}}^{\frac{m}{k+1}T}\Big\| \Phi'_u(s, u)r(t-s)-$$

$$-\Phi'_u\left(\frac{m}{kT}, u\right)r\left(t-\frac{kT}{m}\right)\Big\| ds\leq$$

$$\leq\sum_{k=0}^{m-1}\int_{\frac{k}{m}T}^{\frac{k+1}{m}T}\Big\|\Phi'_u(t, u)-\Phi'_u\left(\frac{k}{m}T, u\right)\Big\|\cdot| r(t-s) | ds+$$

$$+\sum_{k=0}^{m-1}\int_{\frac{k}{m}T}^{\frac{k+1}{m}T}\Big\|\Phi'_u\left(\frac{k}{m}T, u\right)\Big\|\left|r(t-s)-r\left(t-\frac{k}{m}T\right)\right|ds\leq\theta_1(m, u)+\theta_0(m),$$

where

$$\theta_1(m, u) \text{ and } \theta_0(m)\to 0 \text{ for } m\to\infty$$

and

$$|\theta_1(m, u)| \leq c_{40}^0 \text{ for all } m \text{ and } u \in \boldsymbol{N}^1$$

the constant depending on Φ and r. Taking into account the fact that $\{\mu_t\}_{0<t<T}$ is a basic family, it is obvious that to obtain (3.13_I) for $\Phi * r$ from its validity for $(\Phi * r)_m$ by letting $m \to \infty$ we have to note the obvious fact that

$$[(\Phi * r)_m]'_t(t, u) \to (\Phi * r)'_t(t, u) \text{ for all } t, u$$

and

$$|[(\Phi * r)_m]'_t(t, u)| \leq c_{40} + c_{41}|u|,$$

where the constants c_{40}-c_{41} depend only on Φ (see (3.25)) and r; indeed these facts give also the convergence

$$(3.47'') \qquad \int_0^T \left\{ \int_N [(\Phi * r)_m]'_t(t, u) d\mu_t(u) \right\} dt \to$$

$$\to \int_0^T \left\{ \int_N (\Phi * r)'_t(t, u) d\mu_t(u) \right\} dt, \text{ for } m \to \infty.$$

In this manner, (3.13_I) is valid for any functional $\Phi * r$ given by (3.47) with $\Phi(\cdot, \cdot) \in \mathcal{T}$ satisfying (3.8′) and $r(\cdot) \in C_0^\infty(R)$ satisfying (3.47′).

Let now $\rho(\cdot) \in C_0^\infty(\mathbb{R})$ be such that

$$(3.48) \qquad \rho(t) = \rho(-t), \ \rho(t) \geq 0 \qquad \text{for all } t \in \mathbb{R}$$

$$(3.48'') \qquad \operatorname{supp} \rho \subset [-1, 1] \text{ and } \int_{\mathbb{R}} \rho(t) dt = 1.$$

Put $r_\varepsilon(t) = \frac{1}{\varepsilon} \rho\left(\frac{t}{\varepsilon}\right)$ for all $t \in \mathbb{R}$, $\varepsilon > 0$. Then for $\varepsilon \leq \eta$ $r_\varepsilon(\cdot)$ satisfies (3.47′), hence (3.13_I) is valid for $\Phi * r_\varepsilon$ for $0 < \varepsilon \leq \eta$. Letting $\varepsilon \to 0$, the convergences

$$\int_0^T \left\{ \int_N [((u, (\Phi * r_\varepsilon)'_u(t, u))) + b(u, u, (\Phi * r_\varepsilon)'_u(t, u))] d\mu_t(u) \right\} dt \to$$

$$\to\int_0^T\left\{\int_N [((u,\ \Phi'_u(t,\ u)))+b(u,\ u,\ \Phi'_u(t,\ u))]d\mu_t(u)\right\}dt,$$

$$\int_0^T\left[\int_N (f(t),\ (\Phi * r_\varepsilon)'_u(t,\ u))d\mu_t(u)\right]dt\to\int_0^T\left|\int_N (f(t),\ \Phi'_u(t,\ u))d\mu_t(u)\right|dt$$

are (as in the preceding case) direct consequences (via the Lebesgue dominated convergence theorem) of the fact that $\{\mu_t\}_{0<t<T}$ is basic and

$$(\Phi * r_\varepsilon)'_u(t,\ u)\to\Phi'_u(t,\ u)\ \text{ for } \varepsilon\to 0\ \ (t\in(0,\ T),\ u\in \boldsymbol{N}^1),$$

$$\|(\Phi * r_\varepsilon)'_u(t,\ u)\|\leq c'_{41}$$

$$(\varepsilon>0,\ t\in(0,\ T),\ u\in\boldsymbol{N}^1),$$

the constant c'_{41} depending only on $\Phi(\cdot,\ \cdot)$. On the other hand by the same Lebesgue theorem

$$\int_N (\Phi * r_\varepsilon)(0,\ u)d\mu(u)\to\frac{1}{2}\int_N \Phi(0,\ u)d\mu(u)$$

because (by (3.48-48′))

$$(\Phi * r_\varepsilon)(0,\ u)\to\frac{1}{2}\Phi(0,\ u)\ \text{ for all } u\in\boldsymbol{N}^1$$

$$|(\Phi * r_\varepsilon)(0,\ u)|\leq c_{42}+c_{43}|u|\ \text{ for all } u\in\boldsymbol{N}^1$$

with suitable constants c_{42}-c_{43} (depending only on Φ). It rests to obtain instead of the analogous relation to (3.47″) the relation

$$(3.47''')\qquad -\int_0^T\left[\int_N (\Phi * r_\varepsilon)'_t(t,\ u)d\mu_t(u)\right]dt\to$$

$$-\int_0^T\left[\int_N \Phi'_t(t,\ u)d\mu_t(u)\right]dt-\frac{1}{2}\int_N \Phi(0,\ u)d\mu(u).$$

Or

$$(\Phi * r_\varepsilon)'_t(t, u)=r_\varepsilon(t)\Phi(0, u)+\int_{t-T}^{t} r_\varepsilon(s)\Phi'_\square(\boxed{t-s}, u)ds=$$

$$=r_\varepsilon(t)\Phi(0, u)+\int_0^T r_\varepsilon(t-s)\Phi'_\square(\boxed{s}, u)ds,$$

where

$$\int_0^T r_\varepsilon(t-s)\Phi'_\square(\boxed{s}, u)ds \to \Phi'_t(t, u)$$

for all $t\in(0, T)$ and $u\in \boldsymbol{N}^1$. Moreover it is easy to verify that the above convergence is dominated in such a way that the Lebesgue dominated convergence theorem can be applied and henceforth deduce that (3.47‴) is valid in case

$$(3.49)\qquad \int_0^T r_\varepsilon(t)\left\{\int_N \Phi(0, u)d\mu_t(u)\right\}dt \to \frac{1}{2}\int_N \Phi(0, u)d\mu(u).$$

Here $\Phi(0, \cdot)\in\mathscr{C}^{\text{ind}}$. Since (iii) $\Rightarrow$ (iv) we have integrating (3.13_{IV}) with $\Phi(\cdot)=\Phi(0, \cdot)$,

$$\int_0^T r_\varepsilon(t)\left[\int_N \Phi(0, u)d\mu_t(u)\right]dt+$$

$$+\int_0^T r_\varepsilon(t)\left\{\int_0^t\left\{\int_N [\nu((u, \Phi'_u(0, u)))+\right.\right.$$

$$\left.\left.+ b(u, u, \Phi'_u(0, u))]d\mu_\sigma(u)\right\}d\sigma\right\}dt=$$

$$=\int_0^T r_\varepsilon(t)\left[\int_N \Phi(0, u)d\mu(t)\right]dt+$$

$$+\int_0^T r_i(t)\left\{\int_0^t\left[\int_N (f(\sigma),\ \Phi'_u(0,\ u))d\mu_\sigma(u)\right]d\sigma\right\}dt$$

from where, using again the fact that $\{\mu_t\}_{0<t<T}$ is basic and (3.48-48′), (3.49) follows readily.

This finished the proof of Lemma 5.

REMARK. In virtue of Lemma 5, for any solution $\{\mu_t\}_{0<t<T}$, the relation (3.13_{IV}) is, for every $\Phi(\cdot)\in\mathfrak{T}^{\mathrm{ind}}$, valid a.e. on $(0,\ T)$. However the exceptional set $E(\Phi)$ depends on the functional Φ. We shall prove now that *for the statistical solution constructed in Sec.* 3.2 *the equation* (3.13_{IV}) *is valid for all* $t\in(0,\ T)\setminus E$, *where E is a fixed set of Lebesgue measure* 0, *and all functionals* $\Phi(\cdot)\in\mathfrak{T}^{\mathrm{ind}}$ (i.e. the exceptional set in (3.13_{IV}) does not depend on $\Phi(\cdot)\in\mathfrak{T}^{\mathrm{ind}}$).

PROOF. Recall that except a set E of measure 0 we have (see the notations in Sec. 3.2)

$$F_t(\Phi)=\int_N \Phi(u)d\mu_t(u)$$

for all $\Phi\in\mathcal{C}_2\supset\mathfrak{T}^{\mathrm{ind}}$. The equation ($3.13_{IV}$) being for a fixed $\Phi\in\mathfrak{T}^{\mathrm{ind}}$ valid a.e. on (0, T), we have that

$$(3.13'_{IV})\qquad F_t(\Phi)=\int_0^t\left\{\int_N[-((u,\ \Phi'_u(u)))-b(u,\ u,\ \Phi'_u(u))+ +(f(t),\ \Phi'_u(u))]d\mu_t(u)\right\}dt+\int_N \Phi(u)d\mu(u)$$

a.e. on $(0,\ T)$, where the exceptional set depends a priori on Φ. But the right term is a continuous function of t and the left term satisfies (3.28″), so that applying the strong lifting property, ($3.13'_{IV}$) holds everywhere on $(0,\ T)$. Taking into account the remark made before ($3.13'_{IV}$) we infer readily that (3.13_{IV}) actually holds for any t outside E and this for every $\Phi\in\mathfrak{T}^{\mathrm{ind}}$.

b) It is clear that the definition of a statistical solution $\{\mu_t\}_{0<t<T}$ of the Navier-Stokes equation determines the measures $\{\mu_t\}$ a.e. on $(0, T)$, in the sense that any other family of measures $\{\tilde{\mu}_t\}_{0<t<T}$ for which $\tilde{\mu}_t=\mu_t$ for almost all $t\in(0, T)$, is also a solution with the same initial data μ of the Navier-Stokes equations with the same right term $f(\cdot)$.

In connection with this remark let us firstly prove the following:

LEMMA 6. *For any statistical solution* $\{\mu_t\}_{0<t<T}$ *with initial data* μ *of the Navier-Stokes equations the following conditions are equivalent*:

(i) *The equation* (3.13_{III}) *holds for all* $\Phi(\cdot, \cdot)\in\mathfrak{T}$ *and all* $t\in(0, T)$.

(ii) *The equation* (3.13_{IV}) *holds for all* $\Phi(\cdot)\in\mathfrak{T}^{\mathrm{ind}}$ *and all* $t\in(0, T)$.

(k) *For every* $\Phi(\cdot, \cdot)\in\mathfrak{T}$ *the function in* t

$$\int_N \Phi(t, u)d\mu_t(u) \tag{3.50}$$

is continuous on $(0, T)$ *and converges to* $\int_N \Phi(0, u)d\mu(u)$ *for* $t\to+0$

(kk) *For every* $\Phi(\cdot)\in\mathfrak{T}^{\mathrm{ind}}$, *the function in* t

$$\int_N \Phi(u)d\mu_t(u) \tag{3.50'}$$

is continuous on $(0, T)$ *and converges to* $\int_N \Phi(u)d\mu(u)$ *for* $t\to+0$.

PROOF. It is obvious that (i) $\Rightarrow$ (ii), (k) $\Rightarrow$ (kk) and that (in virtue of the preceding Lemma 5) (i) and (k), resp. (ii) and (kk) are equivalent. To finish the proof if will be sufficient to show that (kk) $\Rightarrow$(k). Therefore suppose that (kk) is true, and let $\varphi(\cdot)\in C^1(\mathbb{R})$ be bounded and such that $\varphi(r)=0$ for $|r|\le 1$. Then $\varphi(|P_m\cdot|)\in\mathfrak{T}^{\mathrm{ind}}$, so that

$$\int_N \varphi(|P_m u|)d\mu_t(u)\le\left\|\int_N \varphi(|P_m u|)d\mu_{\cdot}(u)\right\|_{L^\infty(0,T)} \quad \text{for all } t\in(0, T). \tag{3.51}$$

Suppose that $0 \leq \varphi(r) \leq r^2 - 1$ for $r \geq 1$. Then (3.51) implies readily that

$$\int_N \varphi(|P_m u|) d\mu_t(u) \leq \left\| \int_N |u|^2 d\mu_\cdot(u) \right\|_{L^\infty(0,T)} = c_{44}$$

and by an approximation process we can deduce that

$$\int_N |P_m u|^2 d\mu_t(u) \leq c_{44} + 2.$$

Making $m \to \infty$ we obtain finally

$$\int_N |u|^2 d\mu_t \leq c_{45} \text{ for all } t \in (0, T). \tag{3.51'}$$

Let now $\Phi(\cdot, \cdot) \in \mathcal{C}$ and remark that since we have

$$\limsup_{t \to t_0} \left| \int \Phi(t, u) d\mu_t(u) - \int \Phi(t_0, u) d\mu_{t_0}(u) \right| \leq \tag{3.52}$$
$$\leq \limsup_{t \to t_0} \int_N |\Phi(t, u) - \Phi(t_0, u)| d\mu_t(u) = \lambda,$$

where $t, t_0 \in [0, T]$ and for $t_0 = 0$ we put $\mu_{t_0} = \mu$. But, with suitable constants c_{46}-c_{47} (depending on $\Phi(\cdot, \cdot)$) and $\rho > 1$,

$$\int_{\{u : |u| \geq \rho\}} |\Phi(t, u) - \Phi(t_0, u)| d\mu_t(u) \leq \int_{\{u : |u| \geq \rho\}} (c_{46} + c_{47} |u|) d\mu_t(u) \leq$$
$$\leq \frac{c_{46} + c_{47}}{\rho} \int_{\{u : |u| \geq \rho\}} |u|^2 d\mu_t(u) \leq \frac{(c_{46} + c_{47}) c_{45}}{\rho}$$

so that

$$\lambda \leq \limsup_{t \to t_0} \int_{\{u : |u| \leq \rho\}} |\Phi(t, u) - \Phi(t_0, u)| d\mu_t(u) + \frac{c_{48}}{\rho}. \tag{3.52'}$$

Since $\Phi(\cdot, \cdot)$ is continuous from $[0, T] \times \boldsymbol{N}_{\text{weak}}$ to $\mathbb{R}$ and since

$\{u : |u| \leq \rho\}$ is compact in $\boldsymbol{N}_{\text{weak}}$, the integral in (3.52′) converges to 0. Thus $\lambda \leq \frac{c_{48}}{\rho}$ for all $\rho > 1$, hence $\lambda = 0$. This means that (k) is satisfied.

c) Let us now prove the main result of this section 3.3, namely the following

THEOREM 2. *Suppose that the initial data* μ *has a bounded support in* $\boldsymbol{N}$. *Then the statistical solution* $\{\mu_t\}_{0<t<T}$, *constructed in Sec.* 3.2.*c*), *has a uniformly (in* $t \in (0, T)$*) bounded support in* $\boldsymbol{N}$ *and moreover one can suppose that it satisfies the following condition*:

(j) *The function defined on* $[0, T)$ *by*

$$\int_{\boldsymbol{N}} \Phi(u) d\mu_t(u) \text{ for } t \in (0, T), \text{ and } \int_{\boldsymbol{N}} \Phi(u) d\mu(u) \text{ for } t = 0,$$

is continuous for every real functional $\Phi(\cdot)$ *weakly continuous on* $\boldsymbol{N}$ (i.e. continuous from N_{weak} to $\mathbb{R}$).

REMARK. In virtue of Lemma 6, the solution satisfies (3.13_{III}) and (3.13_{IV}) for all $t \in [0, T)$.

PROOF OF THE THEOREM. Suppose that the support

$$\operatorname{supp} \mu \subset \{u : u \in \boldsymbol{N}, \ |u| \leq r_0\}$$

for a certain $0 \leq r_0 < \infty$. It is obvious that then

$$\operatorname{supp} \mu^{(m)} \subset \{u : u \in \boldsymbol{N}, \ |u| \leq r_0\}$$

(see the proof of the existence theorem, given in Sec. 3.2.*c*)). On the other hand in virtue of ($3.44_m'''$), for $r_1 = \sqrt{c_{37} + r_0^2}$, we have

$$\begin{aligned} \mu_t^{(m)}(\{u : u \in \boldsymbol{N}, \ |u| \leq r_1\}) &= \mu^{(m)}(S^{(m)}(t)^{-1}\{u : u \in \boldsymbol{N}, \ |u| \leq r_1\}) = \\ &= \mu^{(m)}(\{u : u \in \boldsymbol{N}, \ |S^{(m)}(t)u| \leq r_1, \ |u| \leq r_0\}) = \\ &= \mu^{(m)}(\{u : u \in \boldsymbol{N}, \ |u| \leq r_0\}) = 1, \end{aligned}$$

so that for all $m = 1, 2, \ldots$ and $t \in (0, T)$ we have

$$\operatorname{supp} \mu_t^{(m)} \subset \{u : u \in \boldsymbol{N}, \ |u| \leq r_1\}.$$

Let now

$$\Phi_1(u)=0 \text{ for } |u|\leq r_1 \text{ and } =|u|^2-r_1^2 \text{ for } |u|\geq r_1 .$$

Then $\Phi_1\in\mathcal{C}_2$, $\Phi_1\geq 0$ and

$$F^{(m)}(\mathbf{1}\otimes\Phi_1)=\int_0^T\left|\int_{\boldsymbol{N}}\Phi_1(u)d\mu_t^{(m)}(u)\right|dt=0$$

so that, since $\mathbf{1}\otimes\Phi_1\in\mathfrak{L}_2$ we deduce

$$F(\mathbf{1}\otimes\Phi_1)=0$$

But, by Lemmas 3-4, we have

$$F(\mathbf{1}\otimes\Phi_1)=\int_0^T\left|\int_{\boldsymbol{N}}\Phi_1(u)d\mu_t(u)\right|dt,$$

whence

$$\int_{\boldsymbol{N}}\Phi_1(u)d\mu_t(u)=0 \quad \text{a.e. on } (0,\, T).$$

so that, since $\Phi_1(u)>0$ on $\boldsymbol{N}\backslash\{u : u\in\boldsymbol{N},\ |u|\leq r_1\}$, we deduce

(3.53) $\qquad \operatorname{supp}\mu_t\subset\boldsymbol{B}_1=\{u : u\in\boldsymbol{N},\ |u|\leq r_1\},\quad$ a.e. on $(0,\ T)$.

Let now $\mathcal{C}(\boldsymbol{B}_1)$ denote the space of all weakly continuous real functionals defined on $\boldsymbol{B}_1$ and let E be the exceptional set $\subset(0,\ T)$ occurring in the Remark folowing Lemma 5. For any $t\in(0,\ T)\backslash E$, let $G_t\in(\mathcal{C}(\boldsymbol{B}_1))^*$ be defined by

$$G_t(\Phi)=\int_{\boldsymbol{B}1}\Phi(u)d\mu_t(u)=\int_{\boldsymbol{N}}\Phi(u)d\mu_t(u)$$

and for any $s\in E$ let G_s be any functional belonging to

$$\bigcap_{\varepsilon>0}(\text{weak}^* \text{ closure in } (C(\boldsymbol{B}_1))^* \text{ of } \{G_t\,;\ t\in(0,\ T)\backslash E,\ |t-s|<\varepsilon\}).$$

Since the ball of radius 1 is weakly-compact in $(\mathcal{C}(\boldsymbol{B}_1))^*$, G_s exists. By the Riesz-Kakutani representation theorem (see Dunford-Schwartz [1], Ch. IV, 6.3) there exists a Borel measure μ_s on the metric compact space $\boldsymbol{B}_1$ endowed the $\boldsymbol{N}_{\text{weak}}$-topology such that $G_s(\Phi)=\int_{\boldsymbol{B}_1}\Phi d\mu_s$. However on $\boldsymbol{B}_1$ the Borel sets with respect to the $\boldsymbol{N}_{\text{weak}}$-topology coincide with those with respect to the usual $\boldsymbol{N}$-topology [1]), so that μ_s is a Borel measure on $\boldsymbol{N}$. Obviously μ_s *must* be a probability. For $s=0$ put finally $\mu_0=\mu$. Let now $\Phi(\cdot)\in\mathcal{T}^{\text{ind}}$. Then (3.13_{IV}) is valid for all $t\in 0, T)\backslash E$ (see the Remark following Lemma 5), therefore for $t\to s$, $t\notin E$, $G_t(\Phi)$ must converge. By the definition of G_s, this limit is $G_s(\Phi)$. In this way we proved that for our new definition for $t\in E$ of the probability μ_t we obtained that

$$\int_{\boldsymbol{N}}\Phi(u)d\mu_t(u)$$

is continuous on $[0, T]$ for every $\Phi(\cdot)\in\mathcal{T}^{\text{ind}}$.

To finish the proof it will be sufficient to show that

$$\mathcal{Q}=\{\Phi(\cdot)\mid\boldsymbol{B}_1:\Phi(\cdot)\in\mathcal{T}^{\text{ind}}\}$$

is dense in $C(\boldsymbol{B}_1)$. This follows directly from the following useful

LEMMA 7. *Let $\boldsymbol{B}_1=\{u;\ u\in\boldsymbol{N},\ |u|\leq r_1\}$ (with $0<r_1<\infty$) and let $C(\boldsymbol{B}_1)$ denote the algebra of all real weakly continuous functionals on $\boldsymbol{B}_1$. Let moreover*

$$\mathcal{Q}_0=\{\Phi(\cdot)\mid\boldsymbol{B}_1:\Phi(\cdot)\in\mathcal{T}_0^{\text{ind}}\}.$$

Then $\mathcal{Q}_0$ is dense in $\mathcal{C}(\boldsymbol{B}_1)$ (normed by the usual sup-norm, i.e.

$$\|\Phi\|_{\mathcal{C}(\boldsymbol{B}_1)}=\sup_{u\in\boldsymbol{B}_1}|\Phi(u)|).$$

---

[1]) Firstly since the identical map $\boldsymbol{N}\to\boldsymbol{N}_{\text{weak}}$ is continuous, every « weakly » Borel set (i.e. Borel set with respect to the $\boldsymbol{N}_{\text{weak}}$-topology) is a « strongly » Borel set (i.e. with respect to the usual $\boldsymbol{N}$-topology). Conversely it is obvious that any ball in $\boldsymbol{N}$ is weakly closed, hence « weakly » Borel set. From here follows readily that any strongly Borel set is also weakly Borel set. (See PRODI [4].)

PROOF. In virtue of the Stone-Weierstrass theorem it will be sufficient to prove that $\mathcal{C}_0$ is an algebra containing $\mathbf{1}$ and separating the points of $\boldsymbol{B}_1$. Obviously $\mathbf{1}\in\mathcal{C}_0$. Moreover if $\Phi_m(u)=(u, w_m)$ then $\Phi_m(\cdot)\in\mathcal{T}_0^{\text{ind}}$ since the eigenvector w_m of D obviously belongs to $\boldsymbol{N}^1$. If for $u', u''\in\boldsymbol{B}_1$ we have $\Phi_m(u')=\Phi_m(u'')$ for all m, since $\{w_m\}$ is a basis of $\boldsymbol{N}$ it results $u'=u''$. Thus $\mathcal{C}_0$ separates the points of $\boldsymbol{B}_1$. Let now $\Phi(\cdot)\in\mathcal{T}_0^{\text{ind}}$ and let m be large enough that

$$\Phi(u)=\Phi(P_m u),\ u\in\boldsymbol{N}.$$

There exists a C^1-function $\Psi(\xi_1, \xi_2, ..., \xi_m)$ defined on $\mathbb{R}^m$ such that

$$\|\operatorname{grad}\Psi\|=\left(\Sigma\left(\frac{\partial\Psi}{\partial\xi_j}\right)^2\right)^{1/2}$$

is bounded on $\mathbb{R}^m$ and

$$\Phi(u)=\Psi((u, w_1), ..., (u, v_m)), \qquad u\in\boldsymbol{N}.$$

Let $\psi\in C^1(\mathbb{R}^m)$ be 1 for $\|\xi\|=\sqrt{\Sigma_i\xi_i^2}\leq r_1+\frac{1}{2}$ and $=0$ for $\|\xi\|\geq r_1+1$. Put

$$\Phi_1(u)=\Phi(u)\psi((u, w_1), ..., (u, w_m)) \quad \text{for all } u\in\boldsymbol{N}.$$

The functional Φ_1 belongs to $\mathcal{T}_0^{\text{ind}}$ and coincides with Φ on $\boldsymbol{B}_1$. Thus if Θ is any other functional $\in\mathcal{T}_0^{\text{ind}}$, we have

$$\Phi(u)\Theta(u)=\Phi_1(u)\Theta(u) \text{ for all } u\in\boldsymbol{B}_1$$

and $\Phi_1(\cdot)\Theta(\cdot)\in\mathcal{T}_0^{\text{ind}}$. Since plainly $\mathcal{C}_0$ is a linear set, the preceding remark shows that it is an algebra. This finishes the proof of Lemma 7 and thus also that of the theorem.

4. *Individual solutions as statistical solutions. a)* Let μ be a Dirac measure δ_{u_0}, i.e.

$$\mu(A)=\begin{cases}1 \text{ if } u_0\in A,\\ 0 \text{ if } u_0\notin A.\end{cases}$$

Suppose that the statistical solution given by the Theorem in Sec. 3.3.*c*) is formed by Dirac measures, i.e.

$$\mu_t(A)=\delta_{u(t)} \quad (0\leq t\leq T). \tag{3.54}$$

Then, for any $\Phi\in\mathcal{T}^{\text{ind}}$,

$$\Phi(u(t))+\int_0^t [\nu((u(\tau),\ \Phi'_\square(\overline{|u(\tau)|})))+b(u(\tau),\ u(\tau),\ \Phi'_\square(\overline{|u(\tau)|}))]d\tau=$$

$$=\Phi(u_0)+\int_0^t (f(\tau),\ \Phi'_\square(\overline{|u(\tau)|}))d\tau.$$

Taking $\Phi(u)=(u,\ v)$ with $v\in\boldsymbol{N}^1$, we obtain that

$$(u(t),\ v)+\int_0^t [\nu((u(\tau),\ v))+b(u(\tau),\ u(\tau),\ v)]d\tau=\int_0^t (f(\tau),\ v)d\tau+(u_0,\ v)$$

(for all $v\in\boldsymbol{N}^1$),

from where we deduce readily that $\{u(t)\}_{0<t<T}$ is an individual solution of the Navier-Stokes equations with initial value u_0. We shall prove now also the converse fact, that is the following

PROPOSITION. *The individual solutions of the Navier-Stokes are the statistical solutions which are Dirac-measure valued.*

PROOF. Let $\{u(t)\}_{0<t<T}$ be an individual solution of the Navier-Stokes equations. We know (see Sec. 2.5) that $u(\cdot)$ is absolutely continuous if regarded as function with values in $\boldsymbol{N}^{-2}$ and that (see (2.9-9′))

$$\frac{d}{dt}(u(t),\ v)+((u(t),\ v))+b(u(t),\ u(t),\ v)=(f(t),\ v) \tag{3.55}$$

for all t outside a set $\varepsilon\subset(0,\ T)$ of Lebesgue measure 0, and this for all $v\in\boldsymbol{N}^2$. Let now

$$\Theta(u)=\varphi_1((u,\ v_1))\varphi_2((u,\ v_2))\ \dots\ \varphi_k((u,\ v_k)) \tag{3.56}$$

where $\varphi_j(\cdot)\in C_0^1(\mathbb{R})$, $v_j\in\boldsymbol{N}^2$ $(j=1,\ 2,\ \dots,\ k)$. Plainly $\Theta\in\mathcal{T}^{\text{ind}}$. An easy computation shows that, outside ε, we have

$$\frac{d}{dt}\Theta(u(t))+\nu((u(t),\ \Theta'_{\square}(\underline{\overline{|u(t)|}})))+ \tag{3.56'}$$

$$+b(u(t),\ u(t),\ \Theta'_{\square}(\underline{\overline{|u(t)|}}))=(f(t),\ \Theta'_{\square}(\underline{\overline{|u(t)|}}))$$

and obviously

$$\Theta(u(0))=\Theta(u_0)\ \text{(where } u_0=u(0)\text{)}. \tag{3.56''}$$

Now it is easy to verify that $\Theta(u(\cdot))$ is absolutely continuous thus (3.56'-56'') yield

$$\Theta(u(\tau))+\int_0^\tau[\nu((u(t),\ \Theta'_{\square}(\underline{\overline{|u(t)|}}))+ \tag{3.56'''}$$

$$+b(u(t),\ u(t),\ \Theta'_{\square}(\underline{\overline{|u(t)|}})]dt=\Theta(u_0)+$$

$$+\int_0^\tau(f(t),\ \Theta'_{\square}(\underline{\overline{|u(t)|}}))dt \text{ for all } 0\le\tau\le T.$$

It is clear that if (3.13_{IV}) holds for all $\Phi\in\mathfrak{T}_0^{\mathrm{ind}}$ then it holds also for all $\mathfrak{T}^{\mathrm{ind}}$. We have thus to prove that (3.13_{IV}) holds for all $\Phi\in\mathfrak{T}_0^{\mathrm{ind}}$. Suppose therefore that

$$\Phi(u)=\Phi(P_k u) \quad \text{for all } u\in\boldsymbol{N}. \tag{3.57}$$

Let moreover ρ be such that (see Sec. 2)

$$|u(t)|\le\rho \quad \text{for all } 0\le t\le T. \tag{3.58}$$

By (3.57) we have a function φ defined on $\mathbb{R}^k$ of class C^1 such that

$$\Phi(u)=\varphi((u,\ w_1),\ (u,\ w_2),\ \ldots,\ (u,\ w_k)) \tag{3.57'}$$

and such that

$$\sup_{r\in\mathbb{R}^k}|\varphi'_{\square}(r_1,\ r_2,\ \ldots,\ \underline{\overline{|r_j|}},\ \ldots,\ r_k)|<\infty$$

all $j=1, 2, \dots, k$. On

$$K_\rho=\{r \mid r\in \mathbb{R}^k, \ |r_j|\le\rho, \ j=1, 2, \dots, k\}$$

there exists a sequence of polynomials $\{p_m(r_1, \dots, r_k)\}_{m=1}^{\infty}$ such that

$$\sup_{K_\rho} |p_m(r_1, \dots, r_k)-\varphi(r_1, \dots, r_k)| \to 0 \text{ for } m\to\infty \tag{3.59}$$

$$\sup_{K_\rho} |(p_m)'_\square(r_1, \dots, \boxed{r_j}, \dots, r_k)-\varphi'_\square(r_1, \dots, \boxed{r_j}, \dots, r_k)| \to \to 0 \text{ for } m\to\infty, \tag{3.59'}$$

and for any $j=1, 2, \dots, k$. Consider

$$\Phi_m(u)=p_m(\psi((u, w_1)), \psi((u, w_2)), \dots, \psi((u, w_k))) \tag{3.57''}$$

where $\psi\in C_0^1(\mathbb{R})$ and $\psi(r)=r$ for $|r|\le\rho$. Then since $\Phi_m(u)$ is a sum of functions $\Theta(\cdot)$ of the type occuring in (3.56‴) we deduce by linearity that

$$\Phi_m(u(t))+\int_0^\tau [\nu((u(t), (\Phi_m)'_\square(\boxed{u(t)}))) + b(u(t)), u(t), (\Phi_m)'_\square(\boxed{u(t)}))]dt=\Phi_m(u_0)+ \int_0^\tau (f(t), (\Phi_m)'_\square(\boxed{u(t)}))dt \quad \text{for all } 0\le\tau\le T. \tag{3.60}$$

Now in virtue of (3.58), (3.57′-57″) and (3.59-59′) we can pass to the limit in (3.60) obtaining

$$\Phi(u(\tau))+\int_0^\tau [\nu((u(t), \Phi'_\square((\boxed{u(t)}))) + b(u(t), u(t), \Phi'_\square(\boxed{u(t)}))]dt=\Phi(u_0)+\int_0^\tau (f(t), \Phi'_\square(\boxed{u(t)}))dt \tag{3.60'}$$

for all $0 \leq \tau \leq T$). This relation (3.60′) coincides with (3.13_{IV}) if we define μ_t, for $0<t<T$ by $\mu_t = \delta_{u(t)}$ and μ by $\mu = \delta_{u_0}$. This finishes the proof of our proposition.

REMARKS. 1) If for a given initial determined data u_0 there exists two distinct individual solutions say $\{u_1(t)\}_{0<t<T}$ and $\{u_2(t)\}_{0<t<T}$, then the statistical solution with initial data $\mu = \delta_{u_0}$ is on account of the preceding proposition also not unique, namely we have the distinct statistical solutions $u_t' = \delta_{u_1(t)}$ and $u_t'' = \delta_{u_2(t)}$ ($t \in (0, T)$). However in this case we have already the phenomenon which in the introduction was called *intrinsic turbulence*. Indeed, for any fixed $0<\theta<1$,

$$\mu_t = \theta\delta_{u_1(t)} + (1-\theta)\delta_{u_2(t)} \qquad (0<t<T)$$

will be a statistical solution with initial data $\mu = \delta_{u_0}$, but which has not an one-point support for all t, since otherwise $u_1(t) = u_2(t)$ for all $t>0$.

2) It would be very interesting if intrinsic turbulence is always connected with non-uniqueness of the individual solutions.

§ 4. The energy inequality and consequences.

1. For statistical solutions the useful analogue of the energy intquality (2.12) for individual solutions is not that one which can be obtained by a « formal » integration of (2.12), i.e.

$$(4.1) \qquad \frac{1}{2}\int_N |u|^2 d\mu_t(u) + \nu \int_0^t \left[\int_N \|u\|^2 d\mu_t(u) \right] dt \Bigg\{ =$$

$$= \int_N \left[\frac{1}{2}|u(t)|^2 + \nu \int_0^t \|u(\tau)\|^2 d\tau \right] d\mu(u) \leq \int_N \left[\frac{1}{2}|u|^2 + \right.$$

$$\left. + \int_0^t (f(\tau),\ u(\tau))d\tau \right] d\mu(u) = \Bigg\} \frac{1}{2}\int_N |u|^2 d\mu(u) + \int_0^t \left[\int_N (f(\tau),\ u) d\mu_\tau(u) \right] d\tau,$$

where in the brackets $\{...\}$ is indicate the formal integration leading, via

(3.2), to the two extremal terms of (4.1). (Note that in case the space dimension n is 2 this is already a proof that (4.1) is valid for the solution given by (3.2)). For many reasons some of which will be seen below, the statistical analogue of (4.1) seems to be the following relation

$$(4.2)\qquad \frac{1}{2}\int_N \psi(|u|^2)d\mu_t(u)+\nu\int_0^t\left[\int_N \psi'(|u|^2)\,\|u\|^2\,d\mu_\tau(u)\right]d\tau\le$$

$$\le\frac{1}{2}\int_N \psi(|u|^2)d\mu(u)+\int_0^t\left[\int_N \psi'(|u|^2)(f(\tau),\ u)d\mu_\tau(u)\right]d\tau$$

which must be satisfied a.e. on $(0, T)$ for any real-valued function $\psi(\cdot)$ of class C^1 on $[0, \infty)$ such that

$$(4.2')\qquad 0\le\psi'(x)\le c_{49}<\infty\quad\text{for all } x\in[0,\ \infty);$$

here c_{49} is a constant depending on ψ.

We shall call (4.1) *the energy inequality for statistical solutions,* while (4.2) *will be called the strengthened energy inequality for statistical solutions.*

PROPOSITION 1. *The statistical solutions constructed in* §3 *satisfy the strengthened energy inequality.*

PROOF. Let us return to the proof of the existence theorem of Sec. 3.2.*a*) given in Sec. 3.2.*c*). Let ψ be as in (4.2-2′). Then using (3.44_m) we have

$$\frac{1}{2}\frac{d}{dt}\psi(|u_m(t)|^2)+\psi'(|u_m(t)|^2)(u_m(t),\ Au_m(t))+$$
$$+\psi'(|u_m(t)|^2)(P_mB(u_m(t),\ u_m(t)),\ u_m(t))=$$
$$=\psi'(|u_m(t)|^2)(P_mf(t),\ u_m(t))$$

whence (by 2.17))

$$\frac{1}{2}\frac{d}{dt}\psi(|u_m(t)|^2)+\nu\psi'(|u_m(t)|^2)\,\|u_m(t)\|^2=$$
$$=\psi'(|u_m(t)|^2)(f(t),\ u_m(t))\ \text{on}\ [0,\ T),$$

so that integrating with respect to t and to $\mu^{(m)}$ we obtain

$$(4.3)\qquad \frac{1}{2}\int\limits_{N}\psi(|u|^2)d\mu_t^{(m)}(u)+\nu\int\limits_0^t\left[\int\limits_N\psi'(|u|^2)\,\|u\|^2\,d\mu_\tau^{(m)}(u)\right]d\tau=$$

$$=\frac{1}{2}\int\limits_N\psi(|u|^2)d\mu^{(m)}(u)+\int\limits_0^t\left[\int\limits_N(f(\tau),\ u)d\mu_\tau^{(m)}(u)\right]d\tau$$

for all $t\in[0,\ T)$. Let $\rho(\cdot)$ be any function ≥ 0 in $L^1(0,\ T)$. Then (4.3) gives

$$(4.3')\qquad \frac{1}{2}\int\limits_0^T\left[\int\limits_N\rho(t)\psi(|u|^2)d\mu_t^{(m)}(u)\right]dt+$$

$$+\nu\int\limits_0^T\left[\int\limits_N\left(\int\limits_\tau^T\rho(s)ds\right)\psi'(|u|^2)\,\|P_ku\|^2\,d\mu_\tau^{(m)}(u)\right]d\tau\leq$$

$$\leq\frac{1}{2}\left(\int\limits_0^T\rho(t)dt\right)\left(\int\limits_N\psi(|u|^2)d\mu^{(m)}(u))+\right)+$$

$$+\int\limits_0^T\left[\int\limits_N\left(\int\limits_\tau^T\rho(s)ds\right)\psi'(|u|^2)(f(\tau),\ u)d\mu_\tau^{(m)}(u)\right]d\tau$$

in which $k,\ m=1,\ 2,\ \ldots$ and all functions (on $\boldsymbol{N}\times(0,\ T)$) belong to $\mathfrak{L}_2$. Therefore the relation (4.3′), written for the functionals $F^{(m)}$ $(m=1,\ 2,\ \ldots)$ given by (3.18) with $\{\mu_t^{(m)}\}$ instead of $\{\mu_t\}$, will be conserved for any w^*-cluster point $F\in\mathfrak{L}^*_2$ of $\{F^{(m)}\}$. But such a cluster point is of the form (3.18) with a statistical solution $\{\mu_t\}$. Therefore we conclude that any statistical solution $\{\mu_t\}_{0<t<T}$ obtained in Sec. 3.2 will satisfy the relation

$$(4.3'')\qquad \frac{1}{2}\int\limits_0^T\left[\int\limits_N\rho(t)\psi(|u|^2)d\mu_t(u)\right]dt+$$

$$+\nu\int_0^T\left[\int_N\left(\int_\tau^T(\rho(s)ds\right)\psi'(|u|^2)\,\|P_k u\|^2\,d\mu_\tau(u)\right]d\tau\le$$

$$\le\frac{1}{2}\left(\int_0^T\rho(t)dt\right)\left(\int_N\psi(|u|^2)d\mu(u)\right)+$$

$$+\int_0^T\left|\int_N\left(\int_\tau^T(\rho(s)ds\right)\psi'(|u|^2)(f(\tau),\ u)d\mu_\tau(u)\right|d\tau$$

for all $k=1, 2, \ldots$ and $\rho(\cdot)\in L^1(0, T)$, $\rho(\cdot)\ge 0$. Letting $k\to\infty$, we obtain

$$\int_0^T\rho(t)\left\{\frac{1}{2}\int_N\psi(|u|^2)d\mu_t(u)+\nu\int_0^t\left[\int_N\psi'(|u|^2)\,\|u\|^2\,d\mu_\tau(u)\right]d\tau-\right.$$

$$\left.-\frac{1}{2}\int_N\psi(|u|^2)d\mu(u)-\int_0^t\left|\int_N(f(\tau),\ u)\psi'(|u|^2)d\mu_\tau(u)\right]dt\right\}\le 0,$$

from which, the inequality (4.2) results readily.

REMARK. If in (4.2) we put $\psi(x)=x$ for $x\ge 0$ we obtain (4.1); thus if $\{\mu_t\}_{0<t<T}$ satisfies the strengthened energy inequality (4.2) it satisfies also the energy inequality (4.1).

The statistical solutions satisfying the energy inequality or its strengthened form have a certain number of interesting and useful supplementary properties:

COROLLARY 1. *For any statistical solution* $\{\mu_t\}_{0<t<T}$ *with initial data* μ, *satisfying the energy inequality* (i.e. (4.1)), *we have*

$$\left\|\int_N|u|^2\,d\mu_\cdot(u)\right\|_{L^\infty(0,T)}+\nu\left\|\int_N\|u\|^2\,d\mu_\cdot(u)\right\|_{L^1(0,T)}\le \tag{4.4}$$

$$\le 2\int_N|u|^2\,d\mu(u)+c_{50},$$

where c_{50} *is a constant depending only on* $f(\cdot)$, ν *and* Ω.

PROOF. The right term of (4.1) is less than

$$\frac{1}{2}\int_N |u|^2 d\mu(u)+\left(\int_0^t |f(t)|^2 d\tau\right)^{\frac{1}{2}}\left[\int_0^t\left(\int_N |u|\, d\mu_\tau(u)\right)^2 d\tau\right]^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2}\int_N |u|^2 d\mu(u)+c_{51}\left[\int_0^t\left(\int_N |u|^2 d\mu_\tau(u)\right)d\tau\right]^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2}|u|^2 d\mu(u)+\frac{c_{51}}{\lambda_1^{1/2}}\left[\int_0^t\left(\int_N \|u\|^2 d\mu_\tau(u)\right)d\tau\right]^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2}\int_N |u|^2 d\mu(u)+\frac{\nu}{2}\int_0^t\left|\int_N \|u\|^2 d\mu_\tau(u)\right|d\tau+\frac{c_{51}^2}{2\nu\lambda_1}$$

from where (4.4) can be easily obtained with $c_{50}=c_{51}^2\ \nu^{-1}\lambda_1^{-1}$.

COROLLARY 2. *Let $f(\cdot)\in L^\infty(0,\ T;\ \boldsymbol{N}^{-1})$. Then, for any statistical solution $\{\mu_t\}_{0<t<T}$, with initial data μ, satisfying the strengthened energy inequality* (i.e. (4.2)), *we have a.e. on* $(0,\ T)$

$$(4.5)\qquad \int_{\{u:|u|>r\}} |u|^2 d\mu_t(u)+\nu\int\left|\int_{\{u:|u|>r\}} |u|^2\, d\mu_\tau(u)\right|d\tau\leq \int_{\{u:|u|>r\}} |u|^2 d\mu(u)$$

for all $r\geq c_{52}$, where c_{52} is a constant independent of μ and $\{\mu_t\}_{0<t<\infty}$ (i.e. depending only on Ω, ν and $f(\cdot)$).

PROOF. Let $\psi(\cdot)$ satisfy (4.2′) and such that $\psi(x)=0$ for $x\in[0,\ r^2]$. Then (4.2) gives a.e.

$$\int_N \psi(|u|^2)d\mu_t(u)+2\nu\int_0^t\left[\int_N \psi'(|u|^2)\,\|u\|^2 d\mu_\tau(u)\right|d\tau\leq$$

$$\leq\int_N \psi(|u|^2)d\mu(u)+2c_{53}\int_0^t\left|\int_N \|u\|\,\psi'(|u|^2)d\mu_\tau(u)\right|d\tau\leq$$

$$\leq \int_N \psi(|u|^2)d\mu(u)+\frac{2c_{53}}{r}\int_0^t\left|\int_N |u|\cdot\|u\|\,\psi'(|u|^2)d\mu_\tau(u)\right|d\tau\leq$$

$$\leq \int_N \psi(|u|^2)d\mu(u)+\frac{2c_{53}}{r\lambda_1^{1/2}}\int_0^t\left[\int_N \|u\|^2\,\psi'(|u|^2)d\mu_\tau(u)\right]d\tau$$

where $c_{53}=\|f(\cdot)\|_{L^\infty(0,T;N^{-1})}$ and λ_1 is the first eigenvalue of D (see Sec. 2.1). Putting $c_{52}=2\nu^{-1}\lambda_1^{1/2}c_{53}$, we obtain that, for $r\geq c_{52}$,

$$\int_N \psi(|u|^2)d\mu_t(u)+\nu\int_0^t\left[\int_N \psi'(|u|^2)\,\|u\|^2\,d\mu_\tau(u)\right]d\tau\leq \tag{4.6}$$
$$\leq \int_N \psi(|u|^2)d\mu(u)$$

a.e. on $(0, T)$. In particular (4.6) shows that

$$\int_N \psi(|u|^2)d\mu_t(u)\leq \int_N \psi(|u|^2)d\mu(u) \qquad \text{a.e. on } (0, T) \tag{4.7}$$

for any $\psi(\cdot)$ satisfying (4.2′) and such that $\psi(x)=0$ for $x\in[0, r^2]$, $r\geq c_{52}$. Let $r\geq c_{52}$ be fixed, and put

$$\psi_k(x)=k\int_x^{x+1/k}\theta_\varepsilon(y)dy \qquad (0\leq x\leq\infty)$$

where $\varepsilon>0$ and

$$\theta_\varepsilon(y)=\begin{cases}0 & \text{if } \leq y\leq r^2\\ \text{linear} & \text{if } r^2\leq y\leq (r+\varepsilon)^2\\ 1 & \text{if } (r+\varepsilon)^2\leq y.\end{cases}$$

Then in (4.7) we can replace $\psi(\cdot)$ by $\psi_k(\cdot)$; therefore letting $k\to\infty$ we obtain also

$$(4.7') \qquad \int_N \theta_\varepsilon(|u|^2)d\mu_t(u) \leq \int_N \theta_\varepsilon(|u|^2)d\mu(u) \qquad \text{a.e. on } (0, T).$$

Taking $\varepsilon=h^{-1}$ ($h=1, 2, \ldots$) and letting $h \to \infty$ we deduce easily that

$$(4.8) \qquad \mu_t(\{u : |u|>r\}) \leq \mu(\{u : |u|>r\}) \qquad \text{a.e. on } (0, T).$$

However the exceptional set ω of those $t \in (0, T)$ for which (4.8) is not valid depends on r, i.e. $\omega=\omega_r$. Taking $\omega=\cup\,\omega_r$ where the union is taken for all rationals $r \geq c_{52}$ we conclude that for $t \notin \omega$, the inequality (4.8) is valid for all rational $r \geq c_{52}$ thus (using the fact that μ_t is a measure) also for all $r \geq c_{52}$. Let now

$$\theta(x)=\begin{cases} 0 & \text{if } x \leq r^2 \\ x-r^2 & \text{if } x \geq r^2 \end{cases}$$

and let $\psi_k(\cdot)$ be defined as above with $\theta_\varepsilon(\cdot)$ replaced by $\theta(\cdot)$. Introducing these $\psi(\cdot)$ ($k=1, 2, \ldots$) in (4.6) and letting $k \to \infty$ we deduce that

$$\int_N \theta(|u|^2)d\mu_t(u)+\nu\int_0^t \left[\int_N \theta'(|u|^2)\,\|u\|^2\,d\mu_t(u)\right]d\tau \leq$$
$$\leq \int_N \theta(|u|^2)d\mu(u) \qquad \text{a.e. on } (0, T).$$

From this relation and (4.8) we obtain plainly the desired relation (4.5).

REMARK. It is obvious that the corollary 2 yields also the conclusion that if the initial data μ has a bounded support in $\boldsymbol{N}$ then the supports of $\mu_t(0<t<T)$ are uniformity (in t) bounded in $\boldsymbol{N}$; this result was already obtained in the theorem of Sec. 3.3, without the hypothesis that $f(\cdot) \in L^\infty(0, T; \boldsymbol{N}^{-1})$. However this last assumption is obviously verified if $f(t) \equiv f$ does not depend on t. In this case one can consider those statistical solutions which do not depend on t, i.e. the *stationary statistical solutions*. For them, Corollary 2 will yield an interesting property (see Sec. 6.1).

2. *a*) We shall now give the main consequence of (4.1) and (4.2). For this recall that to any basic family of measures $\{\mu_t\}_{0<t<T}$, formula (3.18) associates a functional $F \in \mathfrak{L}^*_{1,1;2}$. We shall endow the set of all statistical solutions with the topology induced, on the set of the corresponding functionals $\in \mathfrak{L}^*_{1,1;2}$, by the w^*-topology on $\mathfrak{L}^*_{1,1}$. Obviously in this way the set of all statistical solutions becomes a subset of a locally convex space. Our main result in this section is given by the following

THEOREM 1. *Let $f(\cdot) \in L^\infty(0, T; \boldsymbol{N})$. Then for given initial data μ, the set of all statistical solutions satisfying the strengthened energy inequality is a convex compact set.*

PROOF. Let $\mathfrak{S}^{(\mu)}$ denote the set of all statistical solutions with initial data μ satisfying the strengthened energy inequality. It is obvious that this set is convex (as subset of $\mathfrak{L}^*_{1,1;2}$). Thus the only fact to be proved is that it is compact (as subset of $\mathfrak{L}^*_{1,1}$).

We start the proof with the conclusions of Corollary 1 above, namely that

$$(4.9)\qquad \begin{cases} \left\| \int_{\boldsymbol{N}} |\, u \,|\, d\mu_\cdot(u) \right\|_{L^\infty(0,T)} \le c_{54} \\ \left\| \int_{\boldsymbol{N}} \|\, u \,\|^2\, d\mu_\cdot(u) \right\|_{L^1(0,T)} \le c_{55}, \end{cases}$$

for all $\{\mu_t\}_{0<t<T} \in \mathfrak{S}^{(\mu)}$, where $c_{54\text{-}55}$ are constants depending only on μ (and naturally on Ω, ν and $f(\cdot)$). By Lemma 1, the functionals F corresponding by (3.18) to $\{\mu_t\}_{0<t<T} \in \mathfrak{S}^{(\mu)}$ form a bounded set (denoted again by $\mathfrak{S}^{(\mu)}$) in $\mathfrak{L}^*_{1,1}$, thus a subset of a compact set (the ball centred in origin and of an enough large radius in $\mathfrak{L}^*_{1,1}$) in the w^*-topology of $\mathfrak{L}^*_{1,1}$. Therefore we have only to show that any F belonging to the w^*-closure in $\mathfrak{L}^*_{1,1}$ of $\mathfrak{S}^{(\mu)}$ actually belongs to $\mathfrak{S}^{(\mu)}$. For this we firstly note that, as remarked at the end of Sec. 3.2.*b*), the Lemmas 3-4 and their Corollary (in Sec. 3.2.*b*)) remain valid if the sequence $\{F^{(m)}\}$ of functionals considered in Lemma 3 is replaced by a directed set $\{F^{(\alpha)}\}$. Now let α label the w^*-neighbourhoods of F in $\mathfrak{L}^*_{1,1}$. For any α, choose an $F^{(\alpha)} \in \mathfrak{S}^{(\mu)}$ belonging to that w^*-neighbourhood of F which is labelled by α. Applying the extended version of Lemma 3 (Sec. 3.2), we obtain a basic family $\{\mu_t\}_{0<t<T}$ of Borel probabilities on $\boldsymbol{N}$ such that (3.18)

is valid for all $\Phi(\cdot, \cdot) \in L^1(0, T; \mathcal{C}_0)$. To apply Lemma 4 (Sec. 3.2) we have to show that for our family $\{\{\mu_t^{(\alpha)}\}_{0<t<T}\}_\alpha$ the function $u \to |u|^2$ on $\boldsymbol{N}$ is almost uniformly integrable. To this purpose let $\varepsilon > 0$ be given and let r_ε be such that

$$\int_{\{u:|u|\geq r_\varepsilon\}} u|^2 \, d\mu(u) \leq \varepsilon.$$

Obviously we may suppose that $r_\varepsilon \geq c_{52}$ (see Corollary 2 in Sec. 4.1). Then in virtue of (4.5), for $\{\mu_t\} = \{\mu_t^{(\alpha)}\}$ we have

$$\int_{\{u:|u|\geq r_\varepsilon\}} |u|^2 d\mu_t^{(\alpha)}(u) \leq \varepsilon \qquad \text{a.e. on } (0, T);$$

this shows that the function $|\cdot|^2$ on $\boldsymbol{N}$ is almost uniformly integrable with respect to $\{\{\mu_t^{(\alpha)}\}_{0<t<T}\}_\alpha$. Corollary 1 in Sec. 4.1, shows (see also Lemma 1 in Sec. 3.1.*c*)) that $\{F^{(\alpha)}\}_\alpha$ is bounded in $\mathfrak{L}^*_2$, hence, since $\mathfrak{L}_{1,1} \cap \mathfrak{L}_2$ is dense in $\mathfrak{L}_2$ that F is also a cluster point in the w^*-topology of $\mathfrak{L}^*_2$. The Lemmas 4,5 and their Corollary (in Sec. 3.2.*a*)-*b*)) can be applied to F, yielding a basic family of probabilities $\{\mu_t\}_{0<t<T}$ on $\boldsymbol{N}$, connected with F by the formula (3.18), valid for any $\Phi \in \mathfrak{L}_2$ or Φ of the form Ψ_φ with $\varphi \in \mathfrak{T}_0$. Now $\Psi_\varphi \in \mathfrak{L}_{1,1}$ for any $\varphi \in \mathfrak{T}_0$ and

$$F^{(\alpha)}(\Psi_\varphi) = \int_{\boldsymbol{N}} \varphi(0, u) d\mu(u) \qquad \text{for all } \alpha,$$

so that (since F is a w^*-cluster point of $\{F^\alpha\}_\alpha$ in $\mathfrak{L}^*_{1,1}$)

$$\mathrm{F}(\Psi_\varphi) = \int_{\boldsymbol{N}} \varphi(0, u) d\mu(u),$$

which is valid for all $\varphi \in \mathfrak{T}_0$, whence $\{\mu_t\}_{0<t<\infty}$ is a statistical solution of the Navier-Stokes equations with initial data μ (see Remarks 1) and 3) in Sec. 3.1.*e*)). It remains to prove that $\{\mu_t\}_{0<t<T}$ satisfies the strengthened energy inequality. For this, let $\rho \in L^1(0, T)$, $\rho \geq 0$. Then the relation (4,2) for $\{\mu_t\} = \{\mu_t^{(\alpha)}\}$ gives

(4.10)
$$\int_0^T \rho(t)\left[\frac{1}{2}\int_N \psi(|u|^2)d\mu_t^{(\alpha)}(u)\right]dt+$$
$$\int_0^T \nu\left(\int_t^T \rho(\sigma)d\sigma\right)\left[\int_N \psi'(|u|^2)\,\|P_k u\|^2\,d\mu_t^{(\alpha)}(u)\right]dt\leq$$
$$\leq\left(\int_0^T \rho(t)dt\right)\left(\frac{1}{2}\int_N \psi(|u|^2\,d\mu(u)\right)+$$
$$+\int_0^T\left(\left(\int_t^T \rho(\sigma)d\sigma\right)\int_N (f(t),\ u)d\mu_t^{(\alpha)}(u)\right]dt$$

where $k=1,\ 2,\ \ldots$ is arbitrary. Introducing the functional

$$\Theta_{k,\rho}(t,\ u)=\frac{1}{2}\rho(t)\psi(|u|^2)+\nu\left(\int_t^T \rho(\sigma)d\sigma\right)\psi'(|u|^2)\,\|P_k u\|^2-$$
$$-\left(\left(\int_t^T \rho(\sigma)d\sigma\right)f(t),\ u\right)$$

which obviously (on account of the hypothesis on $f(\cdot)$) belongs to $\mathfrak{L}_2$, the relation (4.10) can be written in the following form

(4.10′)
$$F^{(\alpha)}(\Theta_{k,\rho})\leq\left(\int_0^T \rho(t)dt\right)\left(\frac{1}{2}\int_N \psi(|u|^2)d\mu(u)\right).$$

Since (4.10) is valid for all α, and since F is a w^*-cluster point of $\{F^\alpha\}_\alpha$ in $\mathfrak{L}^*_2$ too, we deduce

(4.10″)
$$F(\Theta_{k,\rho})\leq\left(\int_0^T \rho(t)dt\right)\left(\frac{1}{2}\int_N \psi(|u|^2)d\mu(u)\right).$$

The relation (4.10″) being valid for all $\rho\in L^1(0,\ T)$, $\rho\geq0$, it results (via

Lemma 4, Sec. 3.2.*b*)) that

$$(4.10''') \qquad \frac{1}{2}\int_N \psi(|u|^2)d\mu_t(u)+\nu\int_0^t\left[\int_N \psi'(|u|^2)\,\|P_k u\|^2\,d\mu_\tau(u)\right]d\tau\leq$$

$$\leq\frac{1}{2}\int_N \psi(|u|^2)\,d\mu(u)+\int_0^t\left[\int_N (f(\tau),\,u)d\mu_\tau(u)\right]d\tau$$

a.e. on $(0, T)$, $k=1, 2, \ldots$ Letting $k\to\infty$ in (4.10‴) we obtain readily (4.2). This finishes the proof of the Theorem.

The same proof yields the following

PROPOSITION 2. *Let* $\{\mu^{(\alpha)}\}$ *be a directed set of probabilities on* $\boldsymbol{N}$ *such that*

$$(4.11) \qquad \int_N \Phi(u)d\mu^{(\alpha)}(u)\to\int_N \Phi(u)d\mu(u) \quad \textit{for all } \Phi(\cdot)\in\mathcal{C}_0^{\text{ind}},$$

$$(4.11') \qquad \int_{\{u:\,u\in N,\ |u|\geq r_\varepsilon\}} |u|^2\,d\mu^{(\alpha)}(u)\leq\varepsilon \quad \textit{for all } \alpha,$$

where $\varepsilon>0$, *and* r_ε *is adequately chosen* (i.e. $|\cdot|^2$ is uniformly integrable with respect to $\{\mu^{(\alpha)}\}$). *For any* α *let* $\{\mu_t^{(\alpha)}\}_{0<t<T}$ *be a statistical solution of the Navier-Stokes equations with initial data* $\mu^{(\alpha)}$, *satisfying the strengthened energy inequality, and let* $F^{(\alpha)}$ *be the corresponding* (via (3.18)) *functional in* $\mathfrak{L}^*_{1,1;2}$. *Let* F *be a* w^**-cluster point in* $\mathfrak{L}^*_{1,1}$ *of* $\{F^{(\alpha)}\}$. *Then* F *corresponds* (via (3.18)) *to a statistical solution* $\{\mu_t\}_{0<t<T}$ *of the Navier-Stokes equations with initial data* μ, *and satisfying the strengthened energy inequality.*

REMARKS. 1) If all the measures $\mu^{(\alpha)}$ have their supports $\subset \boldsymbol{B}_0=\{u : u\in\boldsymbol{N},\ |u|\leq r_0\}$, it is clear that (4.11′) is satisfied. Moreover, in virtue of the Lemma in Sec. 3.3.*c*), (4.11) is equivalent with the convergence $\mu^{(\alpha)}\to\mu$ in the w^*-topology of $\mathcal{C}(\boldsymbol{B}_0)^*$. Therefore the preceding Proposition yields in this case a kind of continuity of the (not necessarily uniquely determined) map $\mu\to\{\mu_t\}_{0<t<T}$ from $\mathcal{C}(\boldsymbol{B}_0)^*$ to $\mathfrak{L}^*_{1,1}$ both endowed with the w^*-topology.

2) Let the initial data μ in the Theorem be the Dirac measure δ_{u_0}, where $u_0 \in \boldsymbol{N}$. It is obvious that any Dirac measure valued statistical solution of the Navier-Stokes equations with initial data $\mu_0 = \delta_{u_0}$, is an extremal point of $\mathcal{S}^{(\mu_0)}$ (i.e. the set of all statistical solutions with initial data $\mu_0 = \delta_{u_0}$, considered as a subset of $\mathcal{L}^*_{1,1}$). Is the converse also true, i.e. is any extremal point of $\mathcal{S}^{(\mu_0)}$, a Dirac-measure valued solution? If the answer to this question is « Yes », then the intrinsic turbulence occurs if and only if there is no uniqueness for the individual solutions of the Navier-Stokes equations with initial value u_0.

b) Let us sketch the difficulties which occur in the study of the converse question raised in the preceding Remark.

Let $u_0 \in \boldsymbol{N}$ be fixed and let $\mathcal{S}$ denote the set of all (functionals in $\mathcal{L}^*_{1,1}$ corresponding to the) statistical solutions of the Navier-Stokes equations with initial data $\mu = \delta_{u_0}$ and satisfying the strengthened energy inequality; $\mathcal{S}$ is considered as subset of $\mathcal{L}^*_{1,1}$. If $\{\mu_t\}_{0<t<T}$ is such a solution, then

$$\text{supp}\,\mu_t \subset \{u : u \in \boldsymbol{N},\ |u| \leq r_0\} \qquad \text{a.e. on } (0, T) \tag{4.12}$$

where r_0 depends only on u_0 (and of course on Ω, ν and $f(\cdot)$), but not on the solution. In virtue of Sec. 3.3.*c*), we can suppose that (3.13$_{\mathrm{III}}$) and (3.13$_{\mathrm{IV}}$) are satisfied for all $t \in [0, T]$ and that the continuity property (j) given in the Theorem 2, Sec. 3.3.*c*), is valid. Arguments used in Sec. 3.3 show that we can suppose that (4.2) is also satisfied for all $t \in [0, T]$. If $\{\mu_t\}_{0<t<T}$ is not an extremal point of $\mathcal{S}$, then $F = \frac{1}{2} F' + \frac{1}{2} F''$ where F corresponds to $\{\mu_t\}_{0<t<T}$, while F', F'' to other two different statistical solutions $\{\mu_t'\}_{0<t<T}$, $\{\mu_t''\}_{0<t<T}$. Now taking $\Phi \in \mathcal{T}_0^{\mathrm{ind}}$ and $r \in L^1(0, T)$ we have

$$\int_0^T r(t) \left[\int_N \Phi(u) d\mu_t(u) - \frac{1}{2} \int_N \Phi(u) d\mu_t'(u) - \frac{1}{2} \int_N \Phi(u) d\mu_t''(u) - \right| dt =$$

$$= \left(F - \frac{1}{2} F' - \frac{1}{2} F'' \right)(r \otimes \Phi) = 0,$$

whence ($r \in L^1(0, T)$ being arbitrary)

$$\int_N \Phi(u)d\mu_t(u) = \int_N \frac{1}{2}\Phi(u)d\mu_t'(u) + \int_N \frac{1}{2}\Phi(u)d\mu_t''(u) \tag{4.12'}$$

a.e. on $(0, T)$. By the continuity condition (j) (already quoted) (4.12') holds for all $t \in [0, T)$ and all $\Phi \in \mathcal{T}_0^{\text{ind}}$ But the integrals in (4.12') are taken in fact only over $\boldsymbol{B}_0$, so that in virtue of Lemma 7 (in Sec. 3.3.*c*)) we have

$$\mu_t = \frac{1}{2}\mu_t' + \frac{1}{2}\mu_t'', \quad t \in [0, T], \tag{4.13}$$

with adequate solutions $\{\mu_t'\}_{0<t<T}$ and $\{\mu_t''\}_{0<t<T}$. It results $\mu_t' \leq 2\mu_t$ for all $t \in [0, T]$ so that

$$\mu_t'(\omega) = \int_\omega \delta(t, u)d\mu_t(u) \qquad (\omega \text{ Borel set} \subset \boldsymbol{N}) \tag{4.13'}$$

with a certain density $\delta(t, u) \geq 0$, which must also verify

$$\int_N \delta(t, u)d\mu_t(u) = 1 \qquad \text{for all } t \in [0, T]. \tag{4.13''}$$

Taking into account (4.13), we can infer that

$$0 \leq \delta(t, u) \leq 2 \qquad \text{a.e. on } N. \tag{4.13'''}$$

for all $t \in [0, T]$. Actually δ is subjected to a very strong condition of different nature, intimately related to the Navier-Stokes equations. To justify this assertion let us suppose that for any test functional $\Phi(\cdot, \cdot)$, the function $(t, u) \to \Phi(t, u)\ \delta(t, u)$ is also a test functional. Writing (3.13_{III}) for this functional, we obtain (for all $\tau \in [0, T]$)

$$\int_N \Phi(\tau, u)\delta(\tau, u)d\mu_\tau(u) + \int_0^\tau \left\{ \int_N \Big[[-\Phi_t'(t, u) + \nu((u, \Phi_u'(t, u))) + \right.$$
$$\left. + b(u, u, \Phi_u'(t, u))]\delta(t, u)d\mu_t(u) \right\} dt + \int_0^\tau \left\{ \int_N [-\delta_t'(t, u) + \right.$$

$$+\nu((u,\ \delta'_u(t,\ u)))+b(u,\ u,\ \delta'_u(t,\ u))]\Phi(t,\ u)d\mu_t(u)\Big\}dt=$$

$$=\Phi(0,\ u_0)\delta(0,\ u_0)+\int_0^\tau\Big\{\int(f(t),\ \Phi'_u(t,\ u))\delta(t,\ u)d\mu_t(u)\Big\}dt+$$

$$+\int_0^\tau\Big\{\int_N(f(t),\ \delta'_u(t,\ u))\Phi(t,\ u)d\mu_t(u)\Big\}dt.$$

Taking into account that $\{\mu_t'\}_{0<t<T}$ is also a solution with the same initial data δ_{u_0}, it results

$$\int_0^\tau\Big\{\int_N[-\delta'_t(t,\ u)+\nu((u,\ \delta'_u(t,\ u)))+b(u,\ u,\ \delta'_u(t,\ u))-$$
$$-(f(t),\ \delta'_u(t,\ u))]\Phi(t,\ u)d\mu_t(u)\Big\}dt=0 \tag{4.14}$$

for all test functionals $\Phi(\cdot,\ \cdot)$ and all $\tau\in[0,\ T]$. Obviously (4.14) implies

$$\int_N[-\delta'_t(t,\ u)+\nu((u,\ \delta'_u(t,\ u)))+b(u,\ u,\ \delta'_u(t,\ u))-$$
$$-(f(t),\ \delta'_u(t,\ u))]\Phi(t,\ u)d\mu_t(u)=0 \tag{4.14'}$$

a.e. on $(0,\ T)$. Let $E_{\Phi(.,.)}$ denote the exceptional set in (4.14′), i.e. such that (4.14′) is valid for all $t\in(0,\ T)\setminus E_{\Phi(.,.)}$. The space $\mathcal{C}(\boldsymbol{B}_0)$, where $\boldsymbol{B}_0=\{u : u\in\boldsymbol{N},\ |u|\leq r_0\}$ is endowed with the weak topology, is separable (since $\boldsymbol{B}_0$ is metric and compact), therefore, in virtue of the Lemma 7 in Sec. 3.3.*c*), there exists a sequence $\{\Phi_m(\cdot)\}_{m=1}^{\infty}\subset\mathcal{T}_0^{\text{ind}}$ dense in $\mathcal{C}(\boldsymbol{B}_0)$. Let $E=\bigcup_{m=1}^{\infty}E_{1\Phi_m(.)}$. Then for any $t\in(0,\ T)\setminus E$ we have

$$\int_N[-\delta'_t(t,\ u)+\nu((u,\ \delta'_u(t,\ u)))+b(u,\ u,\ \delta'_u(t,\ u))-$$
$$-(f(t),\ \delta'_u(t,\ u))]\Phi_m(u)d\mu_t(u)=0,\ m=1,\ 2,\ \ldots$$

In virtue of (3.16′) we may suppose that $\int_{N} \| u \|^2 d\mu_t(u) < \infty$ for $t \in [0, T] \setminus E$, so that the function in the brackets [...] is μ_t-integrable. It results that

$$\int_{N} [\ldots] \Phi(u) d\mu_t(u) = 0$$

for all $\Phi \in \mathcal{C}(\boldsymbol{B}_0)$, so that

$$(4.14_{\mathrm{I}}) \qquad -\delta'_t(t, u) + ((u, \delta'_u(t, u))) + b(u, u, \delta'_u(t, u)) = \\ = (f(t), \delta'_u(t, u))$$

μ_t-almost everywhere in $\boldsymbol{N}$.

In this manner δ is subjected to satisfy (4.14_{I}). To solve (4.14_{I}) in functionals δ satisfying also the conditions (4.13″-13‴) seems to be extremely hard (if possible?).

3. The function $u \mapsto \frac{1}{2} | u |^2$ on $\boldsymbol{N}^1$ is obviously not a time independent test functional; therefore we cannot replace the functional Φ in (3.13_{IV}) by it. If however (3.13_{IV}) would be valid also for this particular functional, then since its Frechet derivative is $u \mapsto u$ and $b(u, u, u) = 0$, we will receive

$$(4.1') \qquad \int_{N} \frac{1}{2} | u |^2 d\mu_\tau(u) + \nu \int_0^\tau \Big[\int_{N} \| u \|^2 d\mu_t(u) \Big] d\tau = \frac{1}{2} \int_{N} | u |^2 d\mu(u) + \\ + \int_0^\tau \Big[\int_{N} (f(t), u) d\mu_t(u) \Big] d\tau \qquad \text{a.e. on } (0, T),$$

that is, the energy equation instead of (4.1). Unhappily this deduction of the energy equation is not rigorous in our approach. Therefore we shall now give some supplementary conditions for the validity of (4.1′).

PROPOSITION 3. *Let* $\{\mu_t\}_{0<t<T}$ *be a statistical solution of the Navier-Stokes equations with initial data* μ, *satisfying the strengthened energy inequality. Let n denote the space dimension* (i.e. *the dimension*

of Ω), and let γ denote the function

(4.15)
$$\gamma(u)=\begin{cases} |u|\cdot\|u\|^2 & \text{if } n=2,\\ |n|^{\frac{1}{2}}\cdot\|u\|^{\frac{5}{2}} & \text{if } n=3,\\ \|u\|^3 & \text{if } n=4.\end{cases}$$

Then if

(4.15′)
$$\int\left[\int\gamma(u)d\mu_t(u)\right]dt<\infty,$$

in the strengthened energy inequality the equality holds, i.e. *the strengthened energy equation*

(4.2″)
$$\frac{1}{2}\int_N\psi(|u|^2)d\mu_t(u)+\nu\int_0^t\left[\int_N\psi'(|u|^2)\,\|u\|^2\,d\mu_\tau(u)\right]d\tau=$$
$$=\frac{1}{2}\int_m\psi(|u|^2)d\mu(u)+\int_0^t\left[\int_N\psi'(|u|^2)(f(\tau),\,u)d\mu_\tau(u)\right]d\tau$$

is valid (a.e. on $(0, T)$) for any function ψ of class C^1 on $[0, \infty)$ satisfying

(4.2‴)
$$|\psi'(\xi)|\leq c_{56} \text{ for all } \xi\in[0, \infty),$$

and for a suitable constant c_{56} (of course, depending on ψ).

PROOF. It is plain that for any function ψ occurring in (4.2″) there exists a sequence $\{\psi_m\}_{m=1}^{\infty}\subset C_0^1([0, \infty))$ converging pointwise (on $[0, \infty)$) to ψ and such that

(4.16)
$$|\psi_m(\xi)|\leq c_{57}(1+\xi),\ |\psi'_m(\xi)|\leq c_{58}$$

for all $\xi\in[0, \infty)$ and $m=1, 2, \ldots$, where c_{67}, c_{58} are some suitable constants (i.e. independent of m and ξ). In virtue of (3.16-16′), (4.16) and of Lebesgue's dominated convergence theorem, if (4.2″) holds for all ψ_m (instead of ψ), $m=1, 2, \ldots$, then it holds also for ψ. Therefore it is sufficient to prove (4.2″) under the supplementary assumption that $\psi\in C_0^\infty([0, \infty))$. For such ψ, the functional $\Phi_k : u\mapsto\frac{1}{2}\psi(|P_ku|^2)$ belongs to $\mathcal{T}_0^{\mathrm{ind}}$ for all $k=1, 2, \ldots$ Therefore taking $\Phi=\Phi_k$ in (3.13_{IV}) we obtain a.e. on $(0, T)$

$$\frac{1}{2}\int_N \psi(|P_ku|^2)d\mu_\tau(u)+\int_0^\tau\left\{\int_N \psi'(|P_ku|^2)[\nu\,\|P_ku\|^2+\right.$$

$$\left.+b(u, u, P_ku)]d\mu_t(u)\right\}dt=$$

$$=\frac{1}{2}\int_N \psi(|P_ku|^2)d\mu(u)+\int_0^\tau\left\{\int_N \psi'(|P_ku|^2)(f(t), P_ku)d\mu_t(u)\right|d\tau.$$

Here, except

$$I_k=\int_0^\tau\left|\int_N \psi'(|P_ku|^2)b(u, u, P_ku)d\mu_t(u)\right|dt \tag{4.17}$$

all the other terms obviously converge (a.e. on $(0, T)$) to the corresponding terms in (4.2″) and this without any use of the assumption (4.15-15′). However to infer that (4.2″) is valid we must show that the integral (4.17) tends to 0 while $k\to\infty$. For this we shall use (4.15-15′). In this aim, note that since $b(u, u, u)=0$ we have

$$|I_k|\le c_{59}\int_0^\tau\left[\int_N |b(u, u, (I-P_k)u)|\,d\mu_t(u)\right]dt \tag{4.17′}$$

with a suitable constant c_{59} (i.e., independent of τ and $k=1, 2, \ldots$). In virtue of (2.3) we have

$$|b(u, u, (1-P_k)u)|\le c_1\|u\|^2\;\|(I-P_k)u\|^2\le c_1\|u\|^3= \tag{4.18_4}$$

$$=c_1\gamma(u) \qquad\qquad \text{in case } n=4.$$

If $n=2$, then by (2.3″) and (2.4) we have

$$(4.18_2)\qquad |b(u,\ u,\ (1-P_k)u)|\le 2^{1/2}|u|\cdot\|u\|\cdot\|(1-P_k)u\|\le$$
$$\le 2^{1/2}|u|\cdot\|u\|^2=2^{1/2}\gamma(u).$$

Finally if $n=3$, then using first (2.4) and Hölder inequality

$$|b(u,\ u,\ (1-P_k)u)|\le c_{60}|u|_6\cdot\|(1-P_k)\|\ \ |u|_3\le c_{60}|u|_6\cdot|u|_3\cdot\|u\|,$$

where on account of Sobolev's inequality (2.2) $|u|_6\le c_{61}\|u\|$ so that

$$(4.18'_3)\qquad |b(u,\ u,\ (1-P_k)u)|\le c_{62}\|u\|^2\cdot|u|_3$$

where $c_{60\text{-}62}$ are suitable constants. To estimate $|u|_3$ note that $\boldsymbol{N}^1=$ $=\boldsymbol{D}_{D^{1/2}}\subset\boldsymbol{L}^6$ and $\boldsymbol{N}=\boldsymbol{D}_{D^0}\subset\boldsymbol{L}^2$ so that the identical map will imbed continuously $\boldsymbol{D}_{D^{1/4}}$ (in virtue of Interpolation Theory; see Lions-Peetre [1]) in $\boldsymbol{L}^q$, where $q^{-1}=\frac{1}{2}\cdot 6^{-1}+\frac{1}{2}2^{-1}=3^{-1}$, i.e. $q=3$. Thus for $u\in\boldsymbol{N}^1$

$$(4.19)\qquad |u|_3\le c_{63}|D^{\frac{1}{4}}u|=c_{63}(D^{\frac{1}{4}}u,\ D^{\frac{1}{4}}u)^{\frac{1}{2}}=$$
$$=c_{63}(D^{\frac{1}{2}}u,\ u)^{\frac{1}{2}}\le c_{63}\|u\|^{\frac{1}{2}}\cdot|u|^{\frac{1}{2}}$$

so that (4.18′₃) and (4.19) give

$$(4.18_3)\qquad |b(u,\ u(1-P_k)u)|\le c_{64}|u|^{\frac{1}{2}}\cdot\|u\|^{\frac{5}{2}}=c_{64}\gamma(u)$$

where $c_{64}=c_{62}c_{63}$ and c_{63} is a constant (i.e. independent of u). The relations (4.18₂₋₄) being valid for all $u\in\boldsymbol{N}^1$ and all $k=1,\ 2,\ \ldots$, a successive application of Lebesgues dominated convergence theorem (which is permitted in virtue of (4.15′)) shows that the integral in (4.17′) tends to 0 for $k\to\infty$, since $|b(u,\ u(1-P_k)u)|\le c_1\|u\|^2\ \|(I-P_k)u\|\to 0$ for $k\to\infty$. This finishes the proof of the Proposition.

COROLLARY. *Let the space dimension n be $=2$, and let $\{\mu_t\}_{0<t<T}$ be a statistical solution of the Navier-Stokes equations satisfying the strengthened energy inequality. Then, if the initial data μ is with bounded support in $\boldsymbol{N}$, the solution $\{\mu_t\}_{0<t<T}$ satisfies the strengthened energy equation.*

Indeed in this case, the support of μ_t is (a.e. on $(0, T)$) uniformly bounded in $\boldsymbol{N}$ (see the Remark at the end of Corollary 2 in Sec. 4.1), so that with a suitable constant c_{65} we have, by (3.16′),

$$\int_0^T \left[\int_{\boldsymbol{N}} \gamma(u) d\mu_t(u) \right] dt \leq c_{65} \int_0^T \left[\int_{\boldsymbol{N}} \| u \|^2 \, d\mu_t(u) \right] dt < \infty ;$$

therefore the preceding Proposition yields the Corollary.

4. The energy inequality for individual solutions of the Navier-Stokes equations obviously gives some *à priori* estimations for these solutions. However the relations (2.16-16′) are in fact also *à priori* estimations. We shall give now an analogue of these relations for the statistical solutions.

THEOREM 2. *Let* $\{\mu_t\}_{0<t<T}$ *be the statistical solution of the Navier-Stokes equations constructed in Sec. 3.2. Suppose that its initial data* μ *is carried by a bounded set in* $\boldsymbol{N}^1$ *and that the right term* $f(\cdot)$ *of the equations belongs to* $L^\infty(0, T; \boldsymbol{N})$. (Recall that actually we note by $f(\cdot)$ the « projection » on $\boldsymbol{N}$ of the external body forces in the classical expression of the Navier-Stokes equations). *Then there exists a constant* r_2 *such that*

$$\mu_t(\{u : u \in \boldsymbol{N}^1, \ \| u \| \leq r_2\}) = 1 \tag{4.20}$$

for

$$t \in [0, T] \ \textit{if} \ n=2 \ \textit{and} \ t \in [0, t_2] \ \textit{if} \ n=3, \tag{4.20′}$$

where $t_2 > 0$ *depends on* μ.

PROOF. We have, for some real r_2^0,

$$\mu(\{u : u \in \boldsymbol{N}^1, \ \| u \| \leq r_2^0\}) = 1. \tag{4.21}$$

Therefore for the measure $\mu^{(m)}$ (see Sec. 3.2.*c*)), we have also

$$\mu^{(m)}(\{u : u \in \boldsymbol{N}^1, \ \| u \| \leq r_2^0\}) = \mu(\{u : u \in \boldsymbol{N}, \ \| P_m u \| \leq r_2^0\}) \geq$$

$$\geq \mu(\{u : u \in \boldsymbol{N}^1,\ \| u \| \leq r_2^0\}) = 1,$$

i.e.

$$\text{(4.21}_m) \qquad \operatorname{supp} \mu^{(m)} \subset P_m \boldsymbol{N} \cap \{u : u \in \boldsymbol{N}^1,\ \| u \| \leq r_2^0\}.$$

The inequalities (2.16-16′) conserve, of course, their validity for the equation (3.44_m) instead of the Navier-Stokes equations (see for instance Prodi [6] and Foiaş-Prodi [1]). For sake of completeness let us sketch their proof: From (3.44_m) by scalar multiplication in $\boldsymbol{N}$ with Du_m we deduce readily

$$\text{(3.44}_m^{\mathrm{IV}}) \qquad \frac{1}{2}\frac{d}{dt} \| u_m \|^2 + \nu \,| Du_m |^2 \leq | f | \cdot | Du_m | + | b(u_m,\ u_m,\ Du_m) | \leq$$
$$\leq c_{66} \,| Du_m | + | u_m |_p \ | \operatorname{grad} u_m |_{\frac{2p}{p-2}} | Du_m |$$

where $c_{66} = \| f(\cdot) \|_{L^\infty(0,T;\ \boldsymbol{N})}$ and the bound for $b(\cdot,\ \cdot,\ \cdot)$ was taken from (2.3′), where $p \geq 2$ is arbitrary. Take $p=4$ if $n=2$ and $p=6$ if $n=3$. Then

$$\text{(4.22)} \qquad | u |_p \leq \begin{cases} c_{67} \,| u |^{\frac{1}{2}} \| u \|^{\frac{1}{2}} & \text{for } u \in \boldsymbol{N}^1, \text{ if } n=2, \\ c_{67} \,\| u \| & \text{for } u \in \boldsymbol{N}^1, \text{ if } n=3, \end{cases}$$

(see (2.2) and (2.2′)) and

$$\text{(4.22}') \qquad | \operatorname{grad} u |_{\frac{2p}{p-2}} \leq c_{68} \| u \|^{\frac{1}{2}} | Du | \qquad \text{for } u \in \boldsymbol{D}_D$$

where $c_{67\text{-}68}$ are some suitable constants; therefore (3.44_m^{IV}) yields

$$\text{(4.23)} \qquad \frac{1}{2}\frac{d}{dt} \| u_m \|^2 + \nu \,| Du_m |^2 \leq c_{66} \,| Du_m | +$$
$$+ c_{67} c_{68} \,| u_m |^{\frac{3-n}{2}} \cdot \| u_m \|^{\frac{n}{2}} \cdot | Du_m |^{\frac{3}{2}} \leq$$
$$\leq \gamma (1 + \| u_m \|^2)^n + \nu \,| Du_m |^2$$

where γ is constant (with respect to $t \in [0, T]$ and $m = 1, 2, \ldots$) and is a non decreasing function $\gamma(\cdot)$ of $| u_m(0) |$. In case $n=2$, we deduce from (4.23)

$$(4.24_2)\qquad 1+\| u_m(t) \|^2 \leq (1+\| u_m(0) \|^2) \exp\left[\gamma \int_0^t (1+\| u_m(\tau) \|^2)\right] \leq$$

$$\leq (1+\| u_m(0) \|^2)\gamma_1 \qquad \text{for all } t\in[0, T],$$

where, by (3.44$''_m$), γ_1 is a non decreasing function $\gamma_1(\cdot)$ of $| u_m(0) |$. In case $n=3$, we deduce from (4.23) that

$$(4.24_3)\qquad \| u_m(t) \|^2 \leq 1+2 \| u_m(0) \|^2 \qquad \text{for } 0\leq t\leq t_0$$

where

$$(4.24'_3)\qquad t_0=\gamma^{-1} \frac{3}{16} \frac{1}{(1+\| u_m(0) \|^2)^2}\,.$$

Put

$$(4.25)\qquad r_2=\max\{((r_2^0)^2+1)^{\frac{1}{2}}[\gamma_1(r_2^0\lambda_1^{-\frac{1}{2}})]^{\frac{1}{2}},\ [1+2(r_2^0)^2]^{\frac{1}{2}}\}$$

and

$$(4.25')\qquad t_2=\gamma(r_2^0\lambda_1^{-\frac{1}{2}})^{-1} \frac{3}{16} \frac{1}{[1+(r_2^0)^2]^2}\,.$$

Then

$$(4.26)\qquad \| S^{(m)}(t)u \| = \| u_m(t) \| \leq r_2$$

whenever

$$(4.26')\qquad \| u \| = \| u_m(0) \| \leq r_2^0$$

and

$$(4.26'')\qquad \begin{cases} 0\leq t\leq T \text{ if } n=2, \\ 0\leq t\leq t_2 \text{ if } n=3. \end{cases}$$

In virtue of (4.26-26$''$), if t is as in (4.26$''$) we have (see Sec. 3.3.c)) [1])

---

[1]) We will make the convention that $\| u \| = \infty$ if $u\in \boldsymbol{N}\setminus \boldsymbol{N}^1$.

$$\mu_t^{(m)}(\{u : u \in \boldsymbol{N},\ \| u \| > r_2\}) =$$
$$= \mu_t^{(m)}(\{u : u \in P_m \boldsymbol{N},\ \| u \| > r_2\}) =$$
$$= \mu^{(m)}(\{u : u \in P_m \boldsymbol{N},\ \| u \| \leq r_2^0,\ \| S^{(m)}(t) u \| > r_2\} =$$
$$= \mu^{(m)}(\emptyset) = 0,$$

i.e.

$$\mu^{(m)}(\{u : u \in \boldsymbol{N},\ \| u \| > r_2\}) = 0 \tag{4.27}$$

whenever

$$\begin{aligned} &0 \leq t \leq T \text{ if } n=2, \\ &0 \leq t \leq t_2 \text{ if } n=3. \end{aligned} \tag{4.27'}$$

Let now $\psi : [0, \infty) \mapsto [0, 1]$ be a nondecreasing continuous function such that

$$\psi(\xi) = 0 \text{ for } 0 \leq \xi \leq r_2^2, \tag{4.28}$$

and let $\chi : [0, T] \mapsto \mathbb{R}$ be the characteristic function of $[0, T]$ if $n=2$ (i.e. $\chi = \mathbf{1}$) or of $[0, t_2]$ if $n=3$. Then $\chi \otimes \Phi_k$, where $\Phi_k : u \mapsto \psi(\| P_k u \|^2)$, belongs to $\mathfrak{L}_2$ for all $k=1, 2, \ldots$ Moreover we have

$$F^{(m)}(\chi \otimes \Phi_k) = \int_0^T \left[\int_{\boldsymbol{N}} \chi(t) \Phi_k(u) d\mu_t^{(m)}(u) \right] dt =$$
$$= \int_0^T \chi(t) \left[\int_{\boldsymbol{N}} \psi(\| P_k u \|^2) d\mu_\tau^{(m)}(u) \right] dt =$$
$$= \int_0^T \chi(t) \left[\int_{\{u : \| P_k u \| > r_2^2\}} \psi(\| P_k u \|^2) d\mu_\tau^{(m)}(u) \right] dt \leq$$
$$\leq \int_0^T \chi(t) \left[\int_{\{u : u \in \boldsymbol{N},\ \| u \| > r_2\}} \psi(\| P_k \|^2) d\mu_\tau^{(m)}(u) \right] dt = 0$$

on account of (4.27-27'). Since this conclusion is valid for all $m=1$, 2, ... it results

$$\int_0^T \left[\int_N \chi(t)\Phi_k(u)d\mu_t(u) \right] dt = 0 \tag{4.29}$$

for any solution $\{\mu_t\}_{0<t<T}$ constructed in Sec. 3.2.c). Therefore

$$\int_N \Phi_k(u)d\mu_t(u) = 0 \quad \text{a.e. on} \quad \begin{cases} [0, T] & \text{if } n=2, \\ [0, t_2] & \text{if } n=3. \end{cases} \tag{4.29'}$$

But $\Phi_k \in \mathfrak{C}_0^{\text{ind}}$ and thus in virtue of the Theorem in Sec. 3.3, we may suppose that the integral in (4.29′) is a continuous function of t, so that

$$\int_N \Phi_k(u)d\mu_t(u) = 0 \quad \text{on} \quad \begin{cases} [0, T] & \text{if } n=2, \\ [0, t_2] & \text{if } n=3. \end{cases}$$

Letting $k \to \infty$, we obtain finally that

$$\int_N \psi(\| u \|^2)d\mu_t(u) = 0 \quad \text{on} \quad \begin{cases} [0, T] & \text{if } n=2, \\ [0, t_2] & \text{if } n=3. \end{cases} \tag{4.30}$$

and for all non decreasing continuous functions $\psi : [0, \infty) \mapsto [0, 1]$ satisfying (4.28). Since there exists a non decreasing sequence $\{\psi_j\}_{j=1}^{\infty}$ of such functions converging to the characteristic function of (r_2, ∞), the relation (4.30) yields

$$\mu_t(\{u : u \in \boldsymbol{N}^1, \ \| u \| \leq r_2\}) = 1$$

for all $t \in [0, T]$ if $n=2$ and all $t \in [0, t_2]$ if $n=3$. This concludes the proof of the Theorem.

REMARKS. 1) In case $n=2$, T can be taken ∞ without altering the preceding theorem.

2) In case $n=3$, the relations (4.25-25′) yield some (rough) estimations for r_2 and t_2 in function of r_2^0 (see (4.21)).

§ 5. Uniqueness theorems for statistical solutions.

1. *a*) We shall devote this paragraph to the study of the uniqueness of our statistical solutions. To avoid some technical in essential difficulties *we shall suppose in the sequel that the right term* $f(\cdot)$ *of the Navier-Stokes equations does not depend on* t, *i.e.* $f(t)\equiv f$ *for all* $t\in[0, T]$ *where* f *is an element of* $\boldsymbol{N}$ (i.e. in particular such that all preceding assumptions on $f(\cdot)$ hold). Moreover we shall also often suppose in this paragraph that the initial data μ has a bounded support in $\boldsymbol{N}$. This assumption avoids a difficulty which seems rather deep and is therefore of quite a different nature than the first one. With this last assumption we know that our statistical solutions satisfy the additional properties of Theorem 2 in Sec. 3.3, as well as the strengthened energy inequality, so that by the Corollary in Sec. 4.3, they satisfy also the strengthened energy equation [1]). In particular we have also

$$\text{(5.1)} \qquad \operatorname{supp}\mu_t \subset \{u : u\in\boldsymbol{N},\ |u|\leq r_1\} \quad \text{for all } t\in[0, T]$$

with some convenient constant r_1 and any solution $\{\mu_t\}_{0<t<T}$.

Fix $\tau\in(0, T)$. Let $\Phi\in\mathfrak{T}_0^{\text{ind}}$ and put

$$\text{(5.2)} \qquad \Psi_m(t, w)=\Phi(S_{\tau-t}^{(m)}w)$$

for

$$\text{(5.2')} \qquad w\in P_m\boldsymbol{N} \text{ and } 0\leq t\leq\tau.$$

Here $S_t^{(m)}=S^{(m)}(t)$ was defined in Sec. 3.2.*c*) and $m=1, 2, \ldots$ Note that since $P_m f(t)\equiv P_m f$ for all t we have obviously the semi-group property

$$\text{(5.3)} \qquad S_\alpha^{(m)}\circ S_\beta^{(m)}=S_{\alpha+\beta}^{(m)}$$

in fact for all α, $\beta\geq 0$. Therefore

$$\text{(5.4)} \qquad \Psi_m(t, S_t^{(m)}w)=\Phi(S_\tau^{(m)}w) \quad \text{for all } w\in P_m\boldsymbol{N}$$

---

[1]) One can easily verify in this case the strengthened energy equation holds for all $t\in[0, T]$ not only a.e. on $[0, T]$.

and all $t \in [0, \tau]$. Differentiating with respect to t we obtain

$$(\Psi_m)'_{\square}(\boxed{t}, S_t^{(m)}v) + \left((\Psi_m)'_{\square}(t, \boxed{S_t^{(m)}v}), \frac{dS_t^{(m)}v}{dt}\right) = 0,$$

whence, by (3.44_m),

$$(\Psi_m)'_t(t, v) - ((\Psi_m)'_v(t, v), Av + P_mB(v, v) - P_mf) = 0$$

where $v = S_t^{(m)}u$ runs over whole $P_m\boldsymbol{N}$. In this manner

$$(\Psi_m)'_t(t, v) - \nu((v, (\Psi_m)'_v(t, v))) - \\ - b(v, v, (\Psi_m)'_v(t, v)) + (f, (\Psi_m)'_v(t, v)) = 0 \tag{5.5}$$

for all

$$0 \leq t \leq \tau \text{ and } v \in P_m\boldsymbol{N}. \tag{5.5'}$$

(Here above we have used the fact that $P_m(\Psi_m)'_v = (\Psi_m)'_v$). We can supplement (5.5-5′) with

$$\Psi_m(\tau, w) = \Phi(w) \quad \text{for } w \in P_m\boldsymbol{N}. \tag{5.5''}$$

Let us finally define the functional $[\Phi]_m$ by

$$[\Phi]_m(u) = \Psi_m(0, P_mu) = \Phi(S_\tau^{(m)}P_mu) \quad \text{for all } u \in \boldsymbol{N}. \tag{5.6}$$

Our main aim in this paragraph will be to find some sufficient conditions on $\{\mu_t\}_{0<t<T}$ for the convergence

$$\int_{\boldsymbol{N}} \Phi(P_mu)\,d\mu_t(u) - \int_{\boldsymbol{N}} [\Phi]_m(u)\,d\mu(u) \to 0$$

for $m \to \infty$. We begin by putting

$$\Phi_m(t, u) = \Psi_m(t, P_mu) \quad \text{for } t \in [0, \tau],\ u \in \boldsymbol{N}. \tag{5.7}$$

Obviously

$$\Phi_m(\tau, u) = \Phi(P_mu) \text{ and } \Phi(0, u) = [\Phi]_m(u),\ u \in \boldsymbol{N}. \tag{5.7'}$$

Moreover it is clear that $\Phi_m(\cdot, \cdot) \in \mathscr{T}_0$, i.e. is an elementary test functional. Therefore we can write (3.13_{III}) for $\Phi(\cdot, \cdot) = \Phi_m(\cdot, \cdot)$:

$$\int_N \Phi(P_m u) d\mu_\tau(u) + \int_0^\tau \left\{ \int_N [-(\Phi_m)'_t(t, u) + \right.$$
$$\left. + \nu((u, (\Phi_m)'_u(t, u))) + b(u, u, (\Phi_m)'_u(t, u))] d\mu_t(u) \right\} dt =$$
$$= \int_N [\Phi]_m(u) d\mu(u) + \int_0^\tau \left\{ \int_N (f, (\Phi_m)'_u(t, u)) d\mu_t(u) \right\} dt. \tag{5.8}$$

ιn (5.8) we can replace $(\Phi_m)'_t(t, v) = (\Psi_m)'_t(t, P_m v)$ with its value given by (5.5), obtaining

$$\int_N \Phi(P_m u) d\mu_\tau(u) - \int_N [\Phi]_m(u) d\mu(u) = \tag{5.8'}$$
$$= \int_0^\tau \left\{ \int_N [(f, (\Phi_m)'_u(t, u)) - \nu((u, (\Phi_m)'_u(t, u))) - \right.$$
$$\left. - b(u, u, (\Phi_m)'_u(t, u))] d\mu_t(u) \right\} dt +$$
$$+ \int_0^\tau \left\{ \int_N [-(P_m f, (\Psi_m)'_\square(t, \boxed{P_m u})) + \nu((u, (\Psi_m)'_\square(t, \boxed{P_m u}))) + \right.$$
$$\left. + b(P_m u, P_m u, \Psi'_\square(t, \boxed{P_m u}))] d\mu_t(u) \right\} dt.$$

But

$$(\Phi_m)'_u(t, u) = P_m(\Psi_m)'_\square(t, \boxed{P_m u}) = (\Psi_m)'_\square(t, \boxed{P_m u}) \tag{5.7''}$$

so that (5.8′) reduces to the relation

$$\int_N \Phi(P_m u) d\mu_\tau(u) - \int_N [\Phi]_m(u) d\mu(u) =$$

$$=\int_0^\tau\left\{\int_N [b(P_mu,\ P_mu,\ (\Phi_m)'_u(t,\ u))-b(u,\ u,\ (\Phi_m)'_u(t,\ u))]d\mu_t(u)\right\}dt,$$

whence

$$\left|\int_N \Phi(P_mu)d\mu_\tau(u)-\int_N [\Phi]_m(u)d\mu(u)\right|\leq \tag{5.8''}$$

$$\leq\int_0^\tau\left\{\int|\ b(P_mu,\ P_mu,\ (\Phi_m)'_u(t,\ u))-b(u,\ u,\ (\Phi_m)'_u(t,\ u))\ |\ d\mu_t(u)\right\}dt$$

$$\leq\int_0^\tau\left\{\int_N[|\ b((I-P_m)u,\ P_mu,\ (\Phi_m)'_u(t,\ u))\ |+\right.$$

$$\left.+|\ b(u,\ (I-P_m)u,\ (\Phi_m)'_u(t,\ u))\ |]d\mu_t(u)\right\}dt\leq$$

$$\leq\int_0^\tau\left\{\int_N[|\ b((I-P_m)u,\ (\Phi_m)'_u(t,\ u),\ P_mu)\ |+\right.$$

$$\left.+|\ b(u,\ (\Phi_m)'_u(t,\ u),\ (I-P_m)u)\ |]d\mu_t(u)\right\}dt.$$

If n=2 by (2.1) and (2.2′), we have

$$\beta(u)=|\ b((I-P_m)u,\ (\Phi_m)'_u(t,\ u),\ P_mu)\ |+$$

$$+|\ b(u,\ (\Phi_m)'_u(t,\ u),\ (I-P_m)u)\ |\leq$$

$$\leq 2^{\frac{1}{2}}|\ (I-P_m)u\ |^{\frac{1}{2}}\cdot\|\ (I-P_m)u\ \|^{\frac{1}{2}}\cdot\|\ (\Phi_m)'_u\ \|\ (|\ P_mu\ |^{\frac{1}{2}}\ \|\ P_mu\ \|^{\frac{1}{2}}+$$

$$+|\ u\ |^{\frac{1}{2}}\cdot\|\ u\ \|^{\frac{1}{2}})\leq$$

$$\leq 2^{\frac{3}{2}}|\ u\ |\cdot\|\ u\ \|^{\frac{1}{2}}\|\ (I-P_m)u\ \|^{\frac{1}{2}}\|\ (\Phi_m)'_u(t,\ u)\ \|,$$

while for $n=3$ we have, by (2.2) and (4.19),

$$\beta(u)\leq|\ (I-P_m)u\ |_3\cdot\|\ (\Phi_m)'_u(t,\ u)\ \|\ (|\ P_mu\ |_6+|\ u\ |_6)\leq$$

$$c_{69}\ |\ u\ |^{\frac{1}{2}}\cdot\|\ (I-P_m)u\ \|^{\frac{1}{2}}\cdot\|\ (\Phi_m)'_u(t,\ u)\ \|\cdot\|\ u\ \|$$

with a convenient constant c_{69}. Using these inequalities as well as (5.1), we obtain from (5.8″) the relation

$$(5.9)\qquad \left|\int_{N}\Phi(P_m u)d\mu_\tau(u)-\int_{N}[\Phi]_m(u)d\mu(u)\right|\leq$$

$$\leq c_{70}\int_0^\tau\left[\int_N \|(I-P_m)u\|^{\frac{1}{2}}\cdot\|u\|^{\frac{n-1}{2}}\|(\Phi_m)'_u(t,u)\|\,d\mu_t(u)\right]dt,$$

valid for any $\tau\in[0,T]$ for which ($3.13_{\rm III}$) holds for all Φ_m, $m=1,2,\ldots$; in particular for any $\tau\in[0,T]$ if $\operatorname{supp}\mu$ is bounded in $\boldsymbol{N}$. To exploit (5.9) we must estimate

$$(5.9')\qquad \|(\Phi_m)'_u(t,u)\|=\sup_{\substack{|v|\leq 1\\ v\in N}}|(D^{\frac{1}{2}}(\Phi_m)'_u(t,u),v)|=$$

$$=\sup_{\substack{|v|\leq 1\\ v\in N^1}}|(D^{\frac{1}{2}}(\Phi_m)'_u(t,u),v)|=\sup_{\substack{|v|\leq 1\\ v\in N^1}}|((\Phi_m)'_v(t,v),D^{\frac{1}{2}}v)|=$$

$$=\sup_{\substack{|v|\leq 1\\ v\in N^1}}|(\Phi'_\square(\underline{|S^{(m)}_{\tau-t}P_m u|}),(S^{(m)}_{\tau-t}\underline{|P_m u|})'_\square P_m D^{\frac{1}{2}}v)|$$

where we used (5.2) and (5.7), as well as the chain rule for differentiations. Since $\Phi\in\mathfrak{T}_0^{\rm ind}\subset\mathfrak{T}^{\rm ind}$, we have

$$\|\Phi'_v(t,v)\|\leq c_{71}\qquad\text{for all } v\in \boldsymbol{N}^1$$

and a suitable constant c_{71}. Therefore from (5.9′) we can infer

$$(5.9'')\quad \|(\Phi_m)'_u(t,u)\|\leq c_{71}\sup_{\substack{|v|\leq 1\\ v\in N^1}}|D^{-\frac{1}{2}}[(S^{(m)}_{\tau-t}\underline{|P_m u|})'_\square P_m D^{\frac{1}{2}}v]|\,.$$

In this manner we arrived to the study of

$$(5.10)\qquad \eta(s)=(S^{(m)}_s\underline{|w|})'_\square z$$

where $s\geq 0$, w, $z\in P_m\boldsymbol{N}$ are fixed. Differentiating ($3.44_{\rm m}$) with respect to w, we obtain

$$\frac{d\eta(s)}{ds}+A\eta(s)+P_mB(S_s^{(m)}w,\ \eta(s))+P_mB(\eta(s),\ S_s^{(m)}w)=0,$$

so that, by taking the scalar product (in $\boldsymbol{N}$) with $D^{-1}\eta(s)$ we arrive to

$$(5.11)\qquad \frac{1}{2}\frac{d}{ds}|D^{-\frac{1}{2}}\eta(s)|^2+\nu|\eta(s)|^2\leq|b(S_s^{(m)}w,\ \eta(s),\ D^{-1}\eta(s))|+$$

$$+|b(\eta(s),\ S_s^{(m)}w,\ D^{-1}\eta(s))|=|b(S_s^{(m)}w,\ D^{-1}\eta(s),\ \eta(s))|+$$

$$+|b(\eta(s),\ D^{-1}\eta(s),\ S_s^{(m)}w)|\leq 2|\eta(s)|\cdot|\operatorname{grad}(D^{-1}\eta(s))|_{\frac{2p}{p-2}}\cdot|S_s^{(m)}w|_p,$$

by (2.3′), where $p\geq 2$ is arbitrary. By (2.15′-15‴), the last term is (with $p=4$ if $n=2$ and $p=6$ if $n=3$)

$$\leq c_{72}|\eta(s)|\cdot|D^{\frac{1}{2}}(D^{-1}\eta(s))|^{\frac{1}{2}}\cdot|D(D^{-1}\eta(s))|^{\frac{1}{2}}\cdot|S_s^{(m)}w|^{\frac{3-n}{2}}\cdot||S_s^{(m)}w||^{\frac{n-1}{2}}$$

$$=c_{72}|\eta(s)|^{\frac{3}{2}}\cdot|D^{-\frac{1}{2}}\eta(s)|^{\frac{1}{2}}\cdot|S_s^{(m)}w|^{\frac{3-n}{2}}\cdot||S_s^{(m)}w||^{\frac{n-1}{2}}\leq$$

$$\leq\nu|\eta(s)|^2+c_{73}|D^{-\frac{1}{2}}\eta(s)|^2\cdot|S_s^{(m)}w|^{2(3-n)}\cdot||S_s^{(m)}w||^{2(n-1)}.$$

In this manner, (5.11) can be given the form

$$(5.11')\qquad \frac{d}{ds}|D^{-\frac{1}{2}}\eta(s)|^2\leq 2c_{73}||S_s^{(m)}w||^{2(n-1)}|S_s^{(m)}w|^{2(3-n)}|D^{-\frac{1}{2}}\eta(s)|^2$$

where $s\leq 0$ and c_{73} is a suitable constant [1]). Integrating (5.11′) we obtain

$$(5.12)\qquad |D^{-\frac{1}{2}}\eta(s)|^2\leq|D^{-\frac{1}{2}}\eta(0)|^2\cdot$$

$$\cdot\exp\left(2c_{73}\int_0^s|S_\sigma^{(m)}w|^{2(3-n)}\cdot||S_\sigma^{(m)}w||^{2(n-1)}d\sigma\right)$$

Putting $w=P_mu$, $z=P_mD^{\frac{1}{2}}v$ and $s=\tau-t$ in (5.12) we arrive to the

---

[1]) Depending only on ν, f and Ω.

estimation

$$(5.9''') \qquad \sup_{\substack{|v|\leq 1\\ v\in N^1}} |D^{-\frac{1}{2}}[(S_{\tau-t}\overline{|P_m u|})'_{\sqsubset} P_m D^{\frac{1}{2}} v]| \leq$$

$$\sup_{\substack{v\in N^1\\ |v|\leq 1}} |P_m v|\cdot \exp\left(c_{73}\int_0^{\tau-t} |S_\sigma^{(m)}P_m u|^{2(3-n)}\cdot \| S_\sigma^{(u)}P_m u\|^{2(n-1)} d\sigma\right)$$

which together with (5.9″) give

$$(5.9^{\mathrm{IV}}) \quad \|(\Phi_m)'_u(t, u)\| \leq c_{71}\cdot \exp\left(c_{73}\int_0^{\tau-t} | S_\sigma^{(m)}P_m u|^{2(3-n)}\cdot \| S_s^{(m)}P_m u\|^{2(n-1)}\, d\sigma\right).$$

Introducing (5.9^{IV}) in (5.9), we finally obtain

$$(5.13) \qquad \left|\int_N \Phi(P_m u) d\mu_\tau(u) - \int_N [\Phi]_m(u) d\mu(u)\right| \leq$$

$$\leq c_{70}c_{71}\int_0^\tau \left\{\int_N [\| (I-P_m)u\|^{\frac{1}{2}}\cdot \| u\|^{\frac{n-1}{2}}\cdot\right.$$

$$\left.\cdot\exp\left(c_{73}\cdot\int_0^{\tau-t} | S_\sigma^{(m)}P_m u|^{2(3-n)}\cdot \| S_\sigma^{(m)}P_m u\|^{2(n-1)}\, d\sigma\right)\right] d\mu_t(u)\Bigg\} dt.$$

We are now able to conclude these considerations with the following basic

LEMMA. *Let* μ *be with bounded support in* $\boldsymbol{N}$ *and let* $\{\mu_t\}_{0<t<T}$ *be any statistical solution of the Navier-Stokes equations with initial data* μ, *satisfying the supplementary properties given in the Theorem of Sec.* 3.3. *Let, moreover,* $\Phi\in\mathcal{C}_0^{\mathrm{ind}}$ *and let* $[\Phi]_m$ *be defined by* (5.6). *Then the convergence*

$$(5.14) \qquad \int_N \Phi(P_m u) d\mu_\tau(u) - \int_N [\Phi]_m(u) d\mu(u) \to 0, \quad \textit{for } m\to\infty,$$

holds whenever

$$n=2, \tag{5.14$_2$}$$

or

$$n=3, \tag{5.14$_3$}$$

the initial data μ *is carried by a bounded set in* $\boldsymbol{N}^1$ *and* τ *is sufficiently small* (i.e. $0\leq\tau\leq\tau(\mu)$, where $\tau(\mu)>0$ is a non increasing function of $r_4=\inf\{r: r>0,\ \mu(\{u: \in\boldsymbol{N},\ \|u\|>r\})=0\}$).

PROOF. By (5.1) and (3.44$_m$), $|S_\sigma^{(m)}u|^2\leq c_{37}+r_1^2$ for all $u\in\operatorname{supp}\mu_t$, $\sigma,\ t\in[0, T]$. Now, consider first the case $n=2$. Then

$$\left|\int_N \Phi(P_m u)d\mu_\tau(u)-\int_N[\Phi]_m(u)d\mu(u)\right|\leq \tag{5.13$_2$}$$

$$\leq c_{74}\int_0^\tau\left\{\int_N \|(I-P_m)u\|^{\frac{1}{2}}\cdot\|u\|^{\frac{1}{2}}\cdot\exp\left[c_{73}(c_{37}+r_1^2)\cdot\right.\right.$$

$$\left.\left.\cdot\int_0^{\tau-t}\|S_\sigma^{(m)}P_m u\|^2\,d\sigma\right]d\mu_t(u)\right\}dt$$

where this last term is, by (3.44$''_m$), less than

$$c_{74}\int_0^\tau\left\{\int_N \|(I-P_m)u\|^{\frac{1}{2}}\cdot\|u\|^{\frac{1}{2}}\cdot\exp\left[c_{75}(|P_m u|^2+c_{76})\right]d\mu_t(u)\right\}dt\leq$$

$$\leq c_{74}\int_0^\tau\left\{\int \|(I-P_m)u\|^{\frac{1}{2}}\cdot\|u\|^{\frac{1}{2}}\cdot\exp\left[c_{75}(r_1^2+c_{76})\right]d\mu_t(u)\right\}dt\leq$$

$$\leq c_{77}\int_0^T\left[\int_N \|(I-P_m)u\|^{\frac{1}{2}}\cdot\|u\|^{\frac{1}{2}}d\mu_t(u)\right]dt$$

the constants c_{74}-c_{77} depending only on f, ν, Ω and μ, i.e. being independent of m. For $u \in \boldsymbol{N}^1$ we have $\| (I-P_m)u \| \to 0$ for $m \to \infty$, thus the last integral tends (in virtue of (3.16) and Lebesgue dominated convergence theorem) to 0 for $m \to \infty$. Therefore (5.14) follows from (5.13_2).

Consider now the case $n=3$. Then

$$(5.13_3) \qquad \left| \int_{\boldsymbol{N}} \Phi(P_m u) d\mu_\tau(u) - \int_{\boldsymbol{N}} [\Phi]_m(u) d\mu(u) \right| \leq$$

$$\leq c_{74} \int_0^\tau \left\{ \int_{\boldsymbol{N}} \| (I-P_m)u \|^{\frac{1}{2}} \cdot \| u \|^{\frac{1}{2}} \cdot \exp \left[c_{73} \int_0^{\tau - t} \| S_\sigma^{(m)} P_m u \|^4 \, d\sigma \right] d\mu_t(u) \right\} dt.$$

Take $\tau(\mu)$ equal to the value t_2 given by (4.25′). Then in virtue of (4.24_3) we have

$$\| S_\sigma^{(m)} P_m u \|^4 \leq (1+2 \| u \|^2)^2 \quad \text{for all } 0 \leq \sigma \leq \tau(\mu)$$

so that the last term in (5.13_3) is, for $0 \leq \tau \leq \tau(\mu)$, less than

$$c_{74} \int_0^\tau \left| \int_{\boldsymbol{N}} \| (I-P_m)u \|^{\frac{1}{2}} \cdot \| u \|^{\frac{1}{2}} \cdot \exp \left[\int_0^{\tau-t} (1+2 \| u \|^2)^2 d\sigma \right] d\mu_t(u) \right\} dt \leq$$

$$\leq c_{74} \int_0^\tau \left\{ \int_{\boldsymbol{N}} \| (I-P_m)u \|^{\frac{1}{2}} \cdot \| u \|^{\frac{1}{2}} \cdot \exp [\tau(\mu)(1+2 \| u \|^2)^2] d\mu_t(u) \right\} dt \leq$$

$$\leq c_{74} \int_0^\tau \left\{ \int_{\boldsymbol{N}} \| (I-P_m)u \|^{\frac{1}{2}} \cdot \| u \|^{\frac{1}{2}} \cdot e^{c_{77}} d\mu_t(u) \right\} dt,$$

where in the last inequality we used the conclusion (4.20) of Theorem 2 in Sec. 4.4. Now we can conclude the proof as in the case $n=2$.

b) The preceding Lemma allows us to obtain two uniqueness theorems for the statistical solutions of the Navier-Stokes equations.

THEOREM 1. *Let $n=2$ and let μ be with bounded support in $\boldsymbol{N}$. Then the statistical solution of the Navier-Stokes equations, with initial data μ, satisfying the supplementary properties given in the Theorem of Sec. 3.3 is uniquely determined.*

REMARKS. 1) The statistical solution given by the formula (3.2) (see Remark 2) in Sec. 3.1.*e*)) plainly satisfies also the supplementary properties fo the Theorem in Sec. 3.3. Therefore Theorem 1 above shows, in fact, that (3.2) yields the unique statistical solution within the indicated class, with initial data μ.

2) It is clear that Theorem 1 is a global (in t) uniqueness theorem for the statistical solution. This is in perfect correspondence with the case of individual solutions.

PROOF OF THEOREM 1. Let $\{\mu_t\}_{0<t<T}$ and $\{\mu'_t\}_{0<t<T}$ two statistical solutions of the Navier-Stokes equations satisfying both the properties given in the statement. By the preceding Lemma we have for any $\Phi \in \mathcal{T}_0^{\text{ind}}$ and any $\tau \in [0, T]$:

$$\int_N \Phi(u) d\mu_\tau(u) - \int_N \Phi(u) d\mu'_\tau(u) =$$

$$= \lim_{m\to\infty} \left[\int_N \Phi(P_m u) d\mu_\tau(u) - \int_N \Phi(P_m u) d\mu'_\tau(u) \right] =$$

$$= \lim_{m\to\infty} \left| \int_N [\Phi]_m(u) d\mu(u) - \int_N [\Phi]_m(u) d\mu(u) \right| = 0$$

that is

$$\int_N \Phi(u) d\mu_\tau(u) = \int_N \Phi(u) d\mu'_\tau(u) \tag{5.15}$$

for all $\tau \in [0, T]$ and

$$\text{all } \Phi \in \mathcal{T}_0^{\text{ind}}. \tag{5.15'}$$

But the support of μ_τ is contained in a ball $\boldsymbol{B}_1 = \{u : u \in \boldsymbol{N},\ |u| \le r_1\}$ for some suitable real r_1. Therefore in virtue of Lemma 7, Sec. 3.3, we infer from (5.15) that

$$\int_{\boldsymbol{B}_1} \varphi(u) d\mu_\tau(u) = \int_{\boldsymbol{B}_1} \varphi(u) d\mu'_\tau(u) \tag{5.15''}$$

for all $\tau\in[0, T]$ and

(5.15‴) all $\varphi\in\mathcal{C}(\boldsymbol{B}_1)$.

Plainly (5.15″)-(5.15‴) imply that, as measures on the metric compact space $\boldsymbol{B}_1$ ($\boldsymbol{B}_1$ being endowed with the weak topology of $\boldsymbol{N}$) μ_τ and μ'_τ coincide. By the remark made in the foot note at page 285, $\mu_\tau(\omega)=\mu'_\tau(\omega)$ for ani Borel set ω of $\boldsymbol{N}$ included in $\boldsymbol{B}_1$ .Since the supports of both measures are also included in $\boldsymbol{B}_1$ we have finally that $\mu_\tau=\mu'_\tau$ for all $\tau\in[0, T]$. This concludes the proof.

THEOREM 2. *Let $n=3$ and let μ be carried by a bounded set of $\boldsymbol{N}^1$. Then the statistical solution of the Navier-Stokes equations with initial data μ, satisfying the supplementary properties given in the Theorem of Sec.* 3.3, *is uniquely determined on an interval* $[0, \tau(\mu)]$ *where $\tau(\mu)>0$ is sufficiently small (i.e. μ_τ is uniquely determined for all $0\le\tau\le\tau(\mu)$; here $\tau(\mu)$ is as in the preceding Lemma in Sec.* 5.1.*a*)).

The proof of this theorem is identical with that of Theorem 1, therefore we pass to the following:

REMARKS. 3) Theorem 2 is a local (in t) uniqueness theorem for the statistical solution. This is also in perfect correspondence with the case of individual solutions.

4) Both Theorems 1 and 2 contain as particular cases (on account of Sec. 3.4) the basic (not the most elaborate) uniqueness theorems for the individual solutions of the Navier-Stokes equations.

5) We can complete the preceding remark, by observing that Theorem 1 has the following consequence: *In the case of space dimension $n=2$ there is no intrinsic turbulence.* Indeed in this case for the measure δ_{u_0} (*whose support is* $=\{u_0\}$) there exists a unique statistical solution which has δ_{u_0} as initial data, namely (see Remark 1) above) that given by formula (3.2), i.e. $\mu_t=\delta_{S(t)u_0}$ for all $0\le t\le T$. In case $n=3$, this argument is no more valid because Theorem 2 is a local uniqueness theorem.

c) In dimension $n=2$, as pointed out in Remark 1) in the preceding section, there is a direct relation between μ and μ_t namely, that given by (3.2). Our purpose in this section is to exibit a similar connection in dimension $n=3$ but in the particular case considered in Theorem 2.

First for the smoothness of the exposition, we shall discuss in more details some features of the individual solutions of the Navier-Stokes equations in dimension $n=3$. (Remember that we are concerned with the case when $\Omega\subset\mathbb{R}^3$ is bounded with boundary of class C^2 and that in the abstract form of the Navier-Stokes equations the right term f does not depend on t and belongs to $\boldsymbol{N}$). An individual solution $u(\cdot)$ will be called *regular* on $[0, t_0]$ if its restriction $u\,|\,[0, t_0]$ to this integral belongs to $C([0, t_0]; \boldsymbol{N}^1)$. Since $\boldsymbol{N}^1\subset\boldsymbol{L}^6$ (by Sobolev's imbedding theorem) we have $C([0, t_0]; \boldsymbol{N}^1)\subset C([0, t_0]; \boldsymbol{L}^6)\subset L^\infty(0, t_0; \boldsymbol{L}^6)$ so that by a uniqueness theorem for « some » individual solution (se Prodi [1]), any other individual solution with initial data $u(0)$ coincides with $u(\cdot)$ on $[0, t_0]$. Therefore we can define the map $T(t_0)$ on those $u_0\in\boldsymbol{N}^1$ for which there exists an individual solution $u(\cdot)$ with initial value u_0, regular on $[0, t_0]$, by $T(t_0)u_0=u(t_0)$.

We shall prove now *that for* $r_2^0>0$ *there exists an* t_2 (for instance that one determined by (4.25′)) *such that for any* $0\leq\tau\leq t_2$, $T(\tau)$ *is an* $\boldsymbol{N}^1$*-continuous map from* $\boldsymbol{B}_0^1=\{u : u\in\boldsymbol{N}^1,\ \|u\|\leq r_2^0\}$ *to* $\boldsymbol{N}^1$ *and* $\|S_\tau^{(m)}P_mu-T(\tau)u\|\to 0$ *for* $m\to\infty$, *uniformly on* $[0, t_2]$ *for any fixed* $u\in\boldsymbol{B}_0^1$.

To this purpose note first that in virtue of (4.26-26″) we get from (4.23) the relation

$$\frac{1}{2}\frac{d}{dt}\|S_t^{(m)}P_mu\|^2+\nu\,|DS_t^{(m)}P_mu|^2\leq$$
$$\leq c_{66}\,|DS_t^{(m)}P_mu|+c_{67}c_{68}(r_2)^{\frac{3}{2}}\ DS_t^{(m)}P_mu\,|^{\frac{3}{2}}$$

where we used the fact that (see (4.26′-26″))

(4.26‴) $\quad\|S_t^{(m)}P_mu\|\leq r_2$ for all $t\in[0, t_2]$, $u\in\boldsymbol{B}_0^1$, $m=1, 2, \ldots$

It results

$$\frac{d}{dt}\|S_t^{(m)}P_mu\|^2+\nu\,|DS_t^{(m)}P_mu|^2\leq c'_0$$

where, as above, $m=1, 2, \ldots, t\in[0, t_2]$ and $u\in\boldsymbol{B}_0^1$ are arbitrary, while

c'_0 is a constant (depending on r_2, thus on r_2^0). The last relation implies

$$(5.26^{\mathrm{IV}}) \qquad \int_0^{t_2} |DS_t^{(m)}P_m u|^2\, dt \leq c_0''$$

where m and u are again arbitrary and c_0'' is a constant (depending on c'_0, r_2 and ν, that is on r_2^0 and ν). But for a fixed u, the sequence $\{S_\cdot^{(m)}P_m u\}_{m=1}^\infty$ contains a subsequence $\{S_\cdot^{(m_j)}P_{m_j}u\}_{j=1}^\infty$ which converges (strongly) in $L^2(0, T; \boldsymbol{N})$ to an individual solution $u(\cdot)$ on $(0, T)$ with initial value u (see Foiaş [2]; see also Lions [1], Ch. I, or Prodi [4]). Since for $u \in \boldsymbol{B}_0^1$ (4.26$^{\mathrm{IV}}$) implies

$$\int_0^{t_2} |DP_k S_t^{(m_j)}P_{m_j}u|^2\, dt \leq c_0'' \text{ for all } k=1, 2, \ldots$$

and DP_k is a bounded operator in $\boldsymbol{N}$, we can easily infer that

$$\int_0^{t_2} |DP_k u(t)|^2\, dt \leq c_0'' \text{ for all } k=1, 2, \ldots$$

This implies (since $|DP_k u(t_0)|$ tends to ∞, for $k \to \infty$, if $u(t_0) \notin \boldsymbol{D}_D$ and tends to $|Du(t_0)|$ if $u(t_0) \in \boldsymbol{D}_D$)

$$(4.26^{\mathrm{V}}) \qquad \int_0^{t_2} |Du(t)|^2\, dt \leq c_0'', \text{ if } u(0)=u \in \boldsymbol{B}_0^1$$

Replacing $\{S_\cdot^{(m_j)}P_{m_j}u\}_{j=1}^\infty$ by a suitable subsequence we can also suppose that $S_t^{(m_j)}P_{m_j}u \to u(t)$ in $\boldsymbol{N}$, a.e. on $(0, T)$. Using (4.26‴) instead (4.26$^{\mathrm{IV}}$) we can deduce in a similar way as above that

$$(4.26^{\mathrm{VI}}) \qquad \| u(t) \| \leq r_2 \quad \text{a.e. on } (0, t_2), \text{ if } u(0)=u \in \boldsymbol{B}_0^1.$$

Now $u(\cdot)$ is a function in $\boldsymbol{N}^{-1}$ (see Sec. 2.3) an absolutely continuous function and satisfies in $\boldsymbol{N}^{-1}$ equation

$$\frac{du}{dt}+\nu Du+B(u,\ u)=f \tag{2.9'}$$

a.e. on $(0,\ T)$. But on $(0,\ t_2)$ we have (on account of $(4.26^{\mathrm{V}}\text{-}26^{\mathrm{VI}})$ that a.e.

$$\left|\frac{du}{dt}\right|\leq|\,f\,|+\nu\,|\,Du\,|+c_{67}\,\|\,u\,\|\,c_{68}\,\|\,u\,\|^{\frac{1}{2}}\,|\,Du\,|^{\frac{1}{2}},$$

hence that $\frac{du}{dt}\in L^2(0,\ t_2\ ;\ \boldsymbol{N})$; therefore $u\,|\,[0,\ t_2]$ is absolutely continuous as function in $\boldsymbol{N}$ and the above equation (2.9) holds in $\boldsymbol{N}$, a.e. on $(0,\ t_2)$. It results easily that for any $u\in\boldsymbol{B}_0^1$ and $l,\ k=1,\ 2,\ \ldots$ the function $\|\,P_k(u(\cdot)-S_{\cdot}^{(l)}P_lu)\,\|^2$ is absolutely continuous on $(0,\ t_2)$ and

$$\begin{aligned}
\frac{1}{2}\cdot\frac{d}{dt}\|\,P_k(u(t)-S_t^{(l)}P_lu\,\|^2&=\Big(\frac{d}{dt}(u(t)-S_t^{(l)}P_lu),\\
DP_k(u(t)-S_t^{(l)}P_lu)\Big)&=-\nu\,|\,DP_k(u(t)-S_t^{(l)}P_lu))\,|^2-\\
&-b(u(t),\ u(t),\ DP_k(u(t)-S_t^{(l)}P_lu))+\\
&+b(S_t^{(l)}P_lu,\ S_t^{(l)}P_lu,\ P_lDP_k(u(t)-S_t^{(l)}P_lu))+\\
&+(f-P_lf,\ DP_k(u(t)-S_t^{(l)}P_lu)), \qquad \text{a.e. on } (0,\ t_2)
\end{aligned}$$

Integrating this equation and letting $k\to\infty$ (which is permitted since $u(\cdot)\in L^\infty(0,\ t_2;\ \boldsymbol{N}^1)\cap L^2(0,\ t_2;\ \boldsymbol{N}^2)$ (see $(4.26^{\mathrm{V}}\text{-}26^{\mathrm{VI}})$ above)) we finally obtain

$$\begin{aligned}
\frac{1}{2}\|\,u(t)-S_t^{(l)}P_lu\,\|^2&=\frac{1}{2}\|\,u-P_lu\,\|^2+\\
+\int_0^t\Big[-\nu\,|\,D(u(\tau)-S_\tau^{(l)}P_lu)\,|^2&-b(u(\tau),\ u(\tau),\ D(u(\tau)-S_\tau^{(l)}P_lu))+\\
&+b(S_\tau^{(l)}P_lu,\ S_\tau^{(l)}P_lu,\ P_lD(u(\tau)-S_\tau^{(l)}P_lu))+\\
&+(f-P_lf,\ D(u(\tau)-S_\tau^{(l)}P_lu))\Big]d\tau,\quad \text{for all } t\in[0,\ t_2],
\end{aligned}$$

Putting $\xi(t)=u(t)-v(t)$, where $v(t)=S_t^{(l)}\,P_lu$, $t\in[0,\ t_2]$, the absolutely continuous function $\| \xi(\cdot) \|^2$ will satisfy the following differential inequality (a.e. on $(0, t_2)$)

$$\frac{1}{2}\frac{d}{dt}\| \xi \|^2 \leq -\nu \,| D\xi |^2 + | b(\xi, u, D\xi) | + | b(v, \xi, D\xi) | +$$
$$+ | b(v, v, (I-P_l)D\xi) | + | f - P_l f | \cdot | D\xi | \leq -\frac{\nu}{2} | D\xi |^2 +$$
$$+ \frac{1}{\nu} | f - P_l f |^2 + c_{67} \| \xi \| \cdot c_{68} \| u \|^{\frac{1}{2}} \cdot | Du |^{\frac{1}{2}} \cdot | D\xi | +$$
$$+ c_{67} \| v \| \, c_{68} \| \xi \|^{\frac{1}{2}} | D\xi |^{\frac{3}{2}} + c_{67} c_{68} \| v \|^{\frac{3}{2}} \cdot | Dv |^{\frac{1}{2}} \cdot | (I-P_l)D\xi | \leq$$
$$\leq c_{67}^{\mathrm{I}} (1 + | Du |) \| \xi \|^2 + c_{67}^{\mathrm{II}} | Dv |^{\frac{1}{2}} \cdot | (I-P_l)D\xi | +$$
$$+ \frac{1}{\nu} | f - P_l f |^2 = c_{67}^{\mathrm{I}} (1 + | Du |) \| \xi \|^2 + c_{67}^{\mathrm{II}} | Dv |^{\frac{1}{2}} \cdot$$
$$\cdot | (I-P_l)Du | + \frac{1}{\nu} | f - P_l f |^2,$$

where (4.26^{III} and 26^{VI}) were used. It results

$$\| \xi(t) \|^2 \leq e^{2c_{67}^{\mathrm{I}} \int_0^t (1 + | Du(\tau) |) d\tau} \cdot \Big[\| \xi(0) \|^2 +$$
$$+ \frac{2t}{\nu} | f - P_l f |^2 + 2 c_{67}^{\mathrm{II}} \int_0^t | Dv(\tau) |^{\frac{1}{2}} | (I-P_l)Du(\tau) | \, d\tau \Big],$$

whence using also (4.26^{V})

$$\| \xi(t) \|^2 \leq c_{67}^{\mathrm{III}} \left[\| \xi(0) \|^2 + | f - P_l f |^2 + \left(\int_0^{t_2} | (I-P_l)Du(\tau) |^2 \, d\tau \right)^{\frac{1}{2}} \right]$$

for all $t \in [0, t_2]$. In this manner we have obtained

$$(4.26^{\mathrm{VII}}) \qquad \| u(t) - S_t^{(l)} P_l u \|^2 \leq c_{67}^{\mathrm{III}} \Big[\| u - P_l u \|^2 +$$
$$+ | f - P_l f |^2 + \left(\int_0^{t_2} | (I-P_l)Du(\tau) |^2 \, d\tau \right)^{\frac{1}{2}} \Big]$$

for all $t\in[0, t_2]$ and $l=1, 2, \ldots$ Here c_{67}^{III} is a constant independent of t, and $u\in \boldsymbol{B}_0^1$ (obviously depending on r_2^0). Letting $l\to\infty$ and using the Lebesgue's dominated convergence theorem for the integral in (4.26$^{\mathrm{VII}}$), we finally conclude with

$$(4.26^{\mathrm{VIII}}) \qquad \sup_{0\le t\le t_2} \| u(t)-S_t^{(l)}P_l u \| \to 0 \text{ for } l\to\infty .$$

Since $S_{\cdot}^{(l)}P_l u$ is continuous as function in $\boldsymbol{N}^1$, (4.26$^{\mathrm{VIII}}$) implies that $u(\cdot)\,|\,[0, t_2]\in C([0, t_2]; \boldsymbol{N}^1)$. This proves that $u(t)=T(t)u$ for all $t\in[0, t_2]$. The conclusion being valid for all $u\in\boldsymbol{B}_0^1$, it remains only to prove that $T(t)$ is continuous from $\boldsymbol{B}_0^1$ (endowed with the topology of $\boldsymbol{N}^1$) to $\boldsymbol{N}^1$. To this aim repeating the argument used to arrive from (2.9′) to (4.26$^{\mathrm{VII}}$), we will firstly obtain for $u(t)=T(t)u$, $v(t)=T(t)v$ where $t\in[0, t_2]$, $u, v\in\boldsymbol{B}_0^1$, the relation

$$\frac{1}{2}\cdot\frac{d}{dt}\| u-v \|^2+\nu\, | D(u-v) |^2=-b(u, u, D(u-v))+$$

$$+b(v, v, D(u-v))\le | b(u-v, u, D(u-v)) |+| b(v, u-v, D(u-v)) |\le$$

$$\le c_{67}\| u-v \|\, c_{68}\| u \|^{\frac{1}{2}} | Du |^{\frac{1}{2}} | D(u-v) |+$$

$$+c_{67}\| v \|\, c_{68}\| u-v \|^{\frac{1}{2}} | D(u-v) |^{\frac{3}{2}}\le$$

$$\le c_{67}^{\mathrm{IV}}\| u-v \|^2 (| Du |+1)+\nu\, | D(u-v) |^2,$$

whence

$$\| u(t)-v(t) \|^2\le c_{67}^{\mathrm{V}}\| u(0)-v(0) \|^2$$

for all $t\in[0, t_2]$. Finally this means

$$(4.26^{\mathrm{IX}}) \qquad \| T(t)u-T(t)v \|\le \sqrt{c_{67}^{\mathrm{V}}}\cdot\| u-v \|$$

for

$$(4.26^{\mathrm{X}}) \qquad \text{all } t\in[0, t_2] \text{ and } u, v\in\boldsymbol{B}_0^1.$$

This concludes the proof of the above underlined statement. We are now in state to complete Theorem 2 by the following

THEOREM 2′. *In the conditions of Theorem 2, we have* $\mu_t(\omega)= =\mu(T(t)^{-1}\omega)$ *for all Borel sets* $\omega\subset\boldsymbol{N}$, *whenever* t *is sufficiently small.*

PROOF. Let $\mu(\boldsymbol{B}_0^1)=1$ and let $\tau_0\in(0, t_2]$ be enough small for the validity of (5.14) for all $\tau\in(0, \tau_0]$. Thus letting $m\to\infty$ in (5.14) we obtain in virtue of the underlined properties of $\{S_t^{(m)}\}$ and $\{T(t)\}$

$$\int_{\boldsymbol{N}}\Phi(u)d\mu_\tau(u)=\int_{\boldsymbol{N}}\Phi(T(t)u)d\mu(u)$$

for all $\Phi\in\mathcal{C}_0^{\mathrm{ind}}$. Here the integrations actually can be restricted to $\boldsymbol{B}_0^1$. Now for any Borel set $\omega\subset\boldsymbol{N}$ put

$$\begin{aligned}\nu_\tau(\omega)=\mu(T(t)^{-1}\omega)=\mu(\{u: u\in\boldsymbol{N},\ T(\tau)\text{ is defined and } T(\tau)u\in\omega\})=\\ =\mu(\{u: u\in\boldsymbol{B}_0^1,\ T(\tau)u\in\omega\}).\end{aligned}$$

Obviously $\boldsymbol{B}_0^1$ is a Borel set in $\boldsymbol{N}$ and $T(\tau)$ is a Borel map from $\boldsymbol{B}_0^1$ (as Borel subset of $\boldsymbol{N}$) to $\boldsymbol{N}$. Indeed since closed balls in $\boldsymbol{N}^1$ are Borel sets in $\boldsymbol{N}$ so are also open balls in $\boldsymbol{N}^1$ (hence any open set in $\boldsymbol{N}^1$ is a Borel set in $\boldsymbol{N}$), $T(t)^{-1}\omega$ is a Borel set in $\boldsymbol{N}$, for any set ω open in $\boldsymbol{N}$. Therefore the above formula defines a Borel measure in $\boldsymbol{N}$. It is plain that $\nu_\tau(\{u: u\in\boldsymbol{N}^1,\ \|u\|\le r_2\})=1$. In this manner for a sufficiently large r_1 we have

$$\operatorname{supp}\mu_\tau\cup\operatorname{supp}\nu_\tau\subset\boldsymbol{B}_1=\{u: u\in\boldsymbol{N},\ |u|\le r_1\}.$$

Since

$$\int_{\boldsymbol{B}_1}\Phi(u)d\mu_\tau(u)=\int_{\boldsymbol{B}_1}\Phi(u)d\nu_\tau(u)$$

for all $\Phi\in\mathcal{C}_0^{\mathrm{ind}}$ it results in the same way, as in the proof of Theorem 1 in Sec. 5.1.*b*), that $\mu_\tau=\nu_\tau$. This conclusion holding for all $\tau\in[0, \tau_0]$ we finished the proof.

REMARK. It is clear that the conclusions of Theorems 2-2′ hold for any t such that μ_τ is carried for all $\tau\in[0, t]$, by a bounded set in $\boldsymbol{N}^1$. In this way the possible pecularity in dimension $n=3$ may occur only at that time t when any bounded set of $\boldsymbol{N}^1$ is not of probability μ_t equal to 1.

2. Let us return to the case $n=2$, but suppose that μ has perhaps an unbounded support in $\boldsymbol{N}$. Then a estimate more accurate that (5.9) is obviously

$$\left|\int_{\boldsymbol{N}} \Phi(P_m u)d\mu_\tau(u) - \int_{\boldsymbol{N}} [\Phi]_m(u)d\mu(u)\right| \leq \tag{5.16}$$

$$\leq 2^{\frac{3}{2}} \int_0^\tau |u|\cdot\|u\|^{\frac{1}{2}}\cdot\|(I-P_m)u\|^{\frac{1}{2}}\|(\Phi_m)'_u(t,\ u)\|\ d\mu_t(u)\Big\}dt$$

for τ outside a set $E(\Phi)$ of measure 0 in $[0, T]$. Now in virtue of $(3.44_m''')$ and (3.44_m^{II}) we obtain from (5.9^{IV})

$$\|(\Phi_m)'_u(t,\ u)\| \leq c_{71}\cdot\exp[c_{75}(\tau-t)(1+|u|^2)^2] \tag{5.16'}$$

so that (5.16) becomes

$$\left|\int_{\boldsymbol{N}} \Phi(P_m u)d\mu_\tau(u) - \int_{\boldsymbol{N}} [\Phi]_m(u)d\mu(u)\right| \leq \tag{5.16''}$$

$$\leq c_{76}\cdot \int_0^\tau \left|\int_{\boldsymbol{N}} |u|\cdot\|u\|^{\frac{1}{2}}\cdot\|(I-P_m)u\|^{\frac{1}{2}}\exp[c_{77}(\tau-t)|u|^4]d\mu_t(u)\right\}dt \leq$$

$$\leq c_{76}\cdot\left\{\int_0^\tau\left|\int_{\boldsymbol{N}}\|u\|\cdot\|(I-P_m)u\|\ d\mu_t(u)\right|dt\right\}^{\frac{1}{2}}\cdot$$

$$\cdot\left\{\int_0^\tau\left|\int_{\boldsymbol{N}}|u|^2\cdot\exp[2c_{77}(\tau-t)|u|^4]d\mu_t(u)\right|dt\right\}^{\frac{1}{2}}$$

for all $\tau\in[0, T]\backslash E(\Phi)$ and some convenient constants $c_{76\text{-}77}$ (independent of μ, $\{\mu_t\}$ and Φ). Let us suppose that for some $\delta>0$

$$\int_0^T\left|\int_{\boldsymbol{N}}|u|^2\cdot\exp(\delta|u|^4)d\mu_t(u)\right|dt<\infty. \tag{5.17}$$

Then for $\tau \in (0, \delta(2c_{77})^{-1}) \backslash E(\Phi)$ we infer easily from (5.16″) that for $m \to \infty$,

$$\int_N \Phi(P_m u) d\mu_\tau(u) - \int_N [\Phi]_m(u) d\mu(u) \to 0. \tag{5.18}$$

Clearly from (5.18) it results that if $\{\mu'_t\}_{0<t<T}$ is another statistical solution (with the same initial data μ) which satisfies (5.17), then

$$\int_N \Phi(u) d\mu_\tau(u) = \int_N \Phi(u) d\mu'_\tau(u) \tag{5.19}$$

for all $\tau \in [0, \tau_0] \backslash E_0(\Phi)$, where $\tau_0 > 0$ (being independent of Φ), Φ is arbitrary in $\mathcal{T}_0^{\text{ind}}$ and $E_0(\Phi) \subset [0, \tau_0]$ is of Lebesgue measure 0. Let $\rho_0 > 0$. By an argument used in the proof of Lemma 7 in Sec. 3.3, for any $\Phi \in \mathcal{T}_0^{\text{ind}} \cap \mathcal{C}_0$ there exists a $\Phi_0 \in \mathcal{T}_0^{\text{ind}}$ such that $\Phi(u) = \Phi_0(u)$ for $u \in \boldsymbol{B}_0 = \{u : u \in \boldsymbol{N}, \ |u| \leq \rho_0\}$ and $\|\Phi_0\|_{\mathcal{C}_0} \leq 2 \|\Phi \mid \boldsymbol{B}_0\|_{\mathcal{C}(\boldsymbol{B}_0)}$. For any rational $\rho_0 > 0$ choose a sequence $\{\Phi_{0j}\} \subset \mathcal{T}^0_{\text{ind}} \cap \mathcal{C}_0$ verifying the last relations (i.e.

$$\|\Phi_{0j}\|_{\mathcal{C}_0} \leq 2 \|\Phi \mid B_0\|_{\mathcal{C}(\boldsymbol{B}_0)})$$

such that $\{\Phi_{0j}\}_{j=1}^\infty$ is dense in $\mathcal{C}(\boldsymbol{B}_0)$. The union of all these sequences is countable, so that the union E of all $E_0(\Phi_{0j})$ will be of Lebesgue measure 0 in $[0, \tau_0]$. Suppose moreover that $\{\mu_t\}$ and $\{\mu'_t\}$ satisfy the relation (4.5), so that if $\rho_0 \geq c_{52}$ (see Corollary 2 in Sec. 4.1) we will have for all $\tau \in [0, \tau_0] \backslash E_0$ the relation

$$\left| \int_{N \backslash B_0} \Phi_{0j}(u) d[\mu_\tau(u) + \mu'_\tau(u)] \right| \leq \frac{1}{\rho_0^2} 2 \|\Phi_{0j} \mid \boldsymbol{B}_0\|_{\mathcal{C}(\boldsymbol{B}_0)} \cdot \tag{5.20}$$

$$\cdot 2 \int_N |u|^2 d\mu(u).$$

From (5.19-20) we can infer rather easily that

$$\left| \int_{\boldsymbol{B}_0} \varphi(u) d\mu_\tau(u) - \int_{\boldsymbol{B}_0} \varphi(u) d\mu'_\tau(u) \right| \leq \frac{1}{\rho^2} 4 \|\varphi\|_{\mathcal{C}(\boldsymbol{B}_0)} \cdot$$

$$\cdot \int_N |u|^2 \, d\mu(u)$$

for all

(5.21′) $\quad \varphi \in \mathcal{C}(\boldsymbol{B}_0)$, $\tau \in [0, \tau_0] \backslash E_0$ and rational $\rho_0 \geq c_{52}$.

Let ω be a Borel subset of $\boldsymbol{N}$. Then for a fixed $\tau \in [0, \tau_0] \backslash E_0$ there exists a sequence $\{\varphi_j\} \subset \mathcal{C}(\boldsymbol{B}_0)$ such that $0 \leq \varphi_j \leq 1$ and

$$\int_{\boldsymbol{B}_0} | \varphi_j(u) - \chi_{\omega \cap \boldsymbol{B}_0}(u) | \, d(\mu_\tau(u) + \mu'_\tau(u)) \rightarrow 0.$$

Taking $\varphi = \varphi_j$ in (5.21) and making $j \rightarrow \infty$ we obtain

$$|\mu_\tau(\omega \cap \boldsymbol{B}_0) - \mu'_\tau(\omega \cap \boldsymbol{B}_0)| \leq \frac{4}{\rho_0^2} \int_{\boldsymbol{N}} |u|^2 \, d\mu(u). \tag{5.22}$$

Finally for $\rho_0 \rightarrow \infty$, (5.22) yields $\mu_\tau(\omega) = \mu'_\tau(\omega)$. In this manner we obtained that $\mu_\tau = \mu'_\tau$ for all $\tau \in [0, \tau_0] \backslash E_0$. Obviously we may suppose that $\tau_0 \notin E_0$. Repeting the same argument for the interval $[\tau_0, 2\tau_0]$ and and so on we finally arrive to the conclusion that $\mu_\tau = \mu'_\tau$, a.e. on $[0, T]$ so that, by the convention made in Sec. 3.3, the two solutions $\{\mu_t\}$ and $\{\mu'_t\}$ coincide.

In this manner we obtained the main part of the proof of the following

THEOREM 3. *Let $n=2$ and let μ satisfy*

$$\int_{\boldsymbol{N}} \exp(\vartheta |u|^4) \, d\mu(u) < \infty \tag{5.23}$$

for some $\vartheta > 0$. Then the statistical solution of the Navier-Stokes equations with initial data μ and satisfying the strengthened energy inequality is uniquely determined (i.e. it is given by the formula (3.2); see the Remarks 1) in Sec. 5.1.*b*)).

To conclude the proof we have to show that (5.23) implies (5.17) with some suitable $\delta > 0$. In this aim note that (4.2) yields readily that

$$\int_0^\tau \left[\int_N \psi'(|u|^2) \|u\|^2 d\mu_t(u) \right] dt \leq \frac{1}{2\nu} \int_N \psi(|u|^2)\, d\mu(u) +$$

$$+|f| \int_0^\tau \left[\int_N \psi'(|u|^2)|u|^2 d\mu_t(u) \right] dt \leq \frac{1}{2\nu} \int_N \psi(|u|^2)\, d\mu(u) +$$

$$+|f|\tau\psi'(c_{79})c_{79} + \frac{|f|}{\lambda_1 c_{79}} \int_0^\tau \left[\int_N \psi'(|u|^2)\|u\|^2 d\mu_t(u) \right] dt,$$

whence

$$\int_0^\tau \left[\int_N \psi'(|u|^2)\|u\|^2 d\mu_t(u) \right] dt \leq \frac{1}{\nu} \int_N \psi(|u|^2)\, d\mu(u) + \tag{5.24}$$

$$+c_{78}\psi'(c_{79}), \text{ a.e. on } [0, T],$$

where $c_{78\text{-}79}$ are constants large enough (independent of τ and ψ). It is easy to infer from (5.24) that actually (5.24) holds for $\tau = T$ and for any $\psi \in C^1([0, \infty))$ such that $\psi' \geq 0$. In particular for

$$\psi(\xi) = \int_0^\xi e^{\frac{\vartheta}{2}\eta^2} d\eta \leq c_{80} e^{\vartheta \xi^2}$$

we obtain

$$\int_0^T \left[\int_N e^{\frac{\vartheta}{2}|u|^4} |u|^2 d\mu_t(u) \right] dt \leq \lambda_1^{-1} \int_0^T \left[\int_N e^{\frac{\vartheta}{2}|u|^4} \|u\|^2 d\mu_t(u) \right] dt \leq$$

$$\leq \lambda_1^{-1}\nu^{-1} \int_N \psi(|u|^2) d\mu(u) + \lambda_1^{-1} c_{78} e^{\frac{\vartheta}{2} c_{79}} \leq$$

$$\leq c_{80}\lambda_1^{-1}\nu^{-1} \int_N e^{\vartheta |u|^4} d\mu(u) + c_{81} < \infty,$$

where $c_{80\text{-}81}$ are some convenient constants.

This concludes the proof of Theorem 3.

REMARK. The Corollary in Sec. 4.3 conserves its interest only in that case which is not covered by Theorems 1 and 3. Indeed, since for $n=2$ the energy equation holds for the individual solutions (see (2.13)), an integration with respect to μ yields the energy equation for the statistical solution defined by (3.2). But in the cases involved in Theorems 1 and 3, this it the unique statistical solution (with initial data μ). (Of course the quoted Corollary concerns the strengthened equation but in this discussion this is not an essential fact.)

Concluding this section let us mention that unfortunately in dimension $n=3$ there seems not to exist an analogue to Theorem 3.

3. For individual solution in dimension $n=3$ there is a global (in t) uniqueness theorem in case $f=0$ and the initial data $u_0 \in \boldsymbol{N}^1$ has a sufficient small norm $\| u_0 \|$. There is a perfect correspondent of this theorem for statistical solutions, namely:

THEOREM 4.[1]) *Let $n=3$ and $f=0$. There exists a constant c_{82} (depending only on Ω and ν) such that if*

$$\mu(\{u : u \in \boldsymbol{N}^1,\ \| u \| \leq c_{82}\})=1, \tag{5.25}$$

then the statistical solution $\{\mu_t\}$ of the Navier-Stokes equations (with initial data μ), constructed in Sec. 3.2, is uniquely determined (for all $t \in [0, T]$).

PROOF. In virtue of Theorem 2′ and the Remark following it, we have only to show that if c_{82} is sufficiently small then there exists another constant c'_{82} such that

$$T(t)\{u : u \in \boldsymbol{N}^1,\ \| u \| \leq c_{82}\} \subset \{u : u \in \boldsymbol{N}^1,\ \| u \| \leq c'_{82}\} \tag{5.26}$$

for all $t \geq 0$ (see the notations in Sec. 5.1.c)). By (4.26‴) we have

$$\int_0^{t_2} | DT(t)u |^2 \, dt \leq c''_0 \tag{5.27}$$

for sufficiently small $t_2 > 0$.

1) Suggested to us by M. SHINBROT.

so that if $u(\cdot)$ is a regular solution on $[0, t_0]$ we can easily deduce that $u(\cdot)\in L^2(0, t_0; \boldsymbol{D}_D)$. Using this fact we can prove without difficulty that as function with values in $\boldsymbol{N}$, $u(\cdot)$ is absolutely continuous and

$$\frac{du}{dt}+\nu Du+B(u, u)=0,$$

whence (see the discussion preceding Theorem 2′ in Sec. 5.1.*c*))

$$\frac{1}{2}\frac{d\,\|u\|^2}{dt}+\nu\,|Du|^2=-b(u, u, Du)\leq$$
$$\leq c_{67}\,\|u\|\cdot c_{68}\,\|u\|^{\frac{1}{2}}|Du|^{\frac{1}{2}}|Du|\leq$$
$$\leq c_{67}c_{68}\,\|u\|^{\frac{3}{2}}|Du|^{\frac{3}{2}}\leq c_{83}\,\|u\|^6+\nu\,|Du|^2.$$

Integrating this differential inequality we obtain

$$\frac{1}{\|u(0)\|^2}-\frac{1}{\|u(t)\|^2}\leq 2c_{83}\int_0^t\|u(\tau)\|^2\,d\tau\leq$$
$$\leq c_{84}\,|u(0)|^2\leq c_{84}\lambda_1^{-1}\|u(0)\|^2,$$

where c_{84} is a constant depending only on Ω and ν. Putting $c_{82}=\left(\dfrac{\lambda_1}{2c_{84}}\right)^{\frac{1}{4}}$ we have

$$\|u(t)\|^2\leq\frac{\|u(0)\|^2}{1-\dfrac{1}{2c_{82}^4}\|u(0)\|^4}\qquad\text{for } 0\leq t\leq t_0. \tag{5.28}$$

Hence if $\|u\|\leq c_{82}$ then for any t for which $T(t)u$ is defined we shall shave

$$\|T(t)u\|\leq\frac{(c_{82})^2}{1-\dfrac{1}{2}}=2(c_{82})^2=c'_{82} \tag{5.28′}$$

But $T(t)$ is defined for $t\in[0, \tau]$ if τ is sufficiently small. In virtue of (5.28′), $\|T(\tau)u\|\leq c'_{82}$ thus $T(t)\,T(\tau)u$ is defined for $t\in[0, \tau']$ where

τ' can be chosen dependent only on c'_{82}. But $T(t)T(\tau)u=T(t+\tau)u$. Since $\| T(\tau'+\tau)u \| \leq c'_{82}$, $T(t)T(\tau'+\tau)u$ will be defined on $[0, \tau]$, that is $T(t)u$ is also defined on $[\tau+\tau', \tau+2\tau']$. Continuing this process, we finally obtain (5.26), finishing our proof.

REMARK. The preceding proof concerning individual solutions is well known (see for instance Lions [1], Ch. I, § 6.7); we have given it for sake of completeness.

4. *a*) Let again $n=3$ and μ be with bounded support in $\boldsymbol{N}$. Then there exists an r such that

$$\operatorname{supp} \mu_t \subset \{u : u \in \boldsymbol{N},\ |u| \leq r\} = \boldsymbol{B}$$

for any statistical solution $\{\mu_t\}_{0<t<T}$ with initial data μ, constructed in Sec. 3.2, and

$$\boldsymbol{S}_t^{(m)}(\operatorname{supp} \mu) \subset \boldsymbol{B}$$

for all $t \in [0, T]$ and $m=1, 2, \ldots$ Let $\varphi \in L^1(0, T)$ and $\Phi \in \mathcal{T}_0^{\text{ind}}$. Then (using the notations of Sec. 3.2)

$$F^{(m)}(\varphi \otimes \Phi) = \int_0^T \varphi(t) \left[\int_{\boldsymbol{N}} \Phi(u) d\mu_t^{(m)}(u) \right] dt =$$

$$= \int_0^T \varphi(t) \left[\int_{\boldsymbol{N}} \Phi(S_t^{(m)} u) d\mu^{(m)}(u) \right] dt =$$

$$= \int_0^T \varphi(t) \left[\int_{\boldsymbol{N}} \Phi(S_t^{(m)} P_m u) d\mu(u) \right] dt,$$

for all $m=1, 2, \ldots$ Let $\{\varphi_j\}_{j=1}^{\infty}$ be dense in $L^1(0, T)$ and $\{\Phi_k\}_{k=1}^{\infty} \subset \mathcal{T}_0^{\text{ind}}$ be such that $\{\Phi_k \,|\, \boldsymbol{B}\}_{k=1}^{\infty}$ is dense in $\mathcal{C}(\boldsymbol{B})$, where $\boldsymbol{B}$ is endowed with the $\boldsymbol{N}$-weak topology (see Lemma 7, Sec. 3.3). By Cantor's diagonal process we can obtain a subsequence $\{m_i\}_{i=1}^{\infty}$ such that $\{F^{(m_i)}(\varphi_j \otimes \Phi_k)\}_{i=1}^{\infty}$ is convergent for all $j, k=1, 2, \ldots$ Let our w^*-cluster point F (see the proof in Sec. 3.2.*c*)) be a w^*-cluster point in $\mathfrak{L}_{1,1}^*$ not only for $\{F^{(m)}\}_{m=1}^{\infty}$

but also for its subsequence $\{F^{(m_i)}\}_{i=1}^{\infty}$. Then we obtain

$$(5.29)\quad \int_0^T \varphi_j(t)\left[\int_N \Phi_k(u)d\mu_t(u)\right]dt = F(\varphi_j \otimes \Phi_k) = \lim_{i\to\infty} F^{(m_i)}(\varphi_j \otimes \Phi_k) =$$

$$= \lim_{i\to\infty} \int_0^T \varphi_j(t)\left[\int_N \Phi_k(S_t^{(m_i)} P_{m_i} u)d\mu(u)\right]dt,$$

for all j, $k=1, 2, \ldots$ Now the integrals on $\boldsymbol{N}$ are actually only on $\boldsymbol{B}$, so that, using the special choice of the sequence $\{\Phi_k\}$ we deduce that (5.29) holds even if Φ_k is replaced by any $\Phi \in C(\boldsymbol{B})$. Similarly, it holds also if φ_j is replaced by any $\varphi \in L^1(0, T)$. Thus

$$(5.29')\quad \int_0^T \varphi(t)\left[\int_N \Phi(u)d\mu_t(u)\right]dt =$$

$$= \lim \int_0^T \varphi(t)\left[\int_N \Phi(S_t^{(m_i)} P_{m_i} u)d\mu(u)\right]dt,$$

for all $\varphi(\cdot) \in L^1(0, T)$ and all real functionals Φ defined on $\boldsymbol{N}$ and $\boldsymbol{N}$-weakly contimous on $\boldsymbol{B}$. Let $t_0 \in (0, T)$ be fixed and $\boldsymbol{w}_0$ denote the of those $u_0 \in \boldsymbol{N}^1$ for which $T(t_0)u_0$ is defined. Using the same technique as in the discussion preceding Theorem 2′ in Sec. 5.1.*c*) one can verify that $\boldsymbol{w}_0$ is open in $\boldsymbol{N}^1$, thus $\boldsymbol{w}_0$ is a Borel set in $\boldsymbol{N}$. Suppose now in (5.29′) that $\Phi \geq 0$, $\varphi \geq 0$ and $\varphi(t)=0$ for $t \in (t_0, \mathrm{T})$. It results

$$(5.29'')\quad \int_0^{t_0} \varphi(t)\left[\int_N \Phi(u)d\mu_t(u)\right]dt \geq$$

$$\geq \lim_{i\to\infty} \int_0^{t_0} \varphi(t)\left[\int_{w_0} \Phi(S_t^{(m_i)} P_{m_i} u)d\mu(u)\right]dt =$$

$$= \lim_{i\to\infty} \int_0^{t_0} \varphi(t)\left[\int_{w_0 \cap \operatorname{supp}\mu} \Phi(S_t^{(m_i)} P_{m_i} u)d\mu(u)\right]dt.$$

Let us prove now that for any $u \in \mathfrak{w}_0$ and $t \in [0, t_0]$, $\{S_t^{(m_i)} P_{m_i}\}$ $\boldsymbol{N}$-weakly converges to $T(t)u$ for $i \to \infty$. We start the proof by noting that in the contrary case we find an $t_1 \in (0, t_0]$ and a subsequence $\{m'_i\} \subset \{m_i\}$ such that $\{S_{t_1}^{(m'_i)} P_{m'_i} u\}$ $\boldsymbol{N}$-weakly convergent to an element $u_1 \in \boldsymbol{N}$, $u_1 \neq T(t_1)u$. From this subsequence we can extract another subsequence $\{m''_i\} \subset \{m'_i\}$ such that $S_t^{(m''_i)} P_{m''_i} u$ converge (in $\boldsymbol{N}$) a.e. on $[0, t_0]$ to an individual solution with initial data u (see Foiaş [2], § 3), thus necessarily to $T(\cdot)u$. On the other hand *the function* $S_t^{(m)} u$ *in* $t \in [0, T]$ *is* $\boldsymbol{N}$*-weakly continuous, uniformly in* $m = 1, 2, \ldots$ Indeed for $v \in \boldsymbol{N}^1$ we easily deduce from (3.44_m-$44'''_m$) that for $0 \leq t_1 \leq t_2 \leq T$, we have

$$(5.30) \qquad |(S_{t_2}^{(m)} u, v) - (S_{t_1}^{(m)} u, v)| \leq$$

$$\leq c' \cdot \| v \| \cdot \int_{t_1}^{t_2} (\| S_\tau^{(m)} u \| + | S_\tau^{(m)} u |^{\frac{1}{2}} \, \| S_\tau^{(m)} u \|^{\frac{3}{2}}) d\tau +$$

$$+ | f | \cdot | v | \cdot (t_2 - t_1) \leq$$

$$\leq c'' \, \| v \| \left| \int_{t_1}^{t_2} \| S_\tau^{(m)} u \|^{\frac{3}{2}} d\tau + t_2 - t_1 \right| \leq$$

$$\leq c''' \left[\left(\int_{t_1}^{t_2} \| S_\tau^{(m)} u \|^2 \, d\tau \right)^{\frac{3}{4}} (t_2 - t_1)^{\frac{1}{4}} + t_2 - t_1 \right| \leq$$

$$\leq c^{\mathrm{IV}} \, \| v \| \cdot (t_2 - t_1)^{\frac{1}{4}}$$

where c^I - c^{IV} are some convenient constants (independent of t_1, t_2, m and v). Since $\{S_t^{(m)} u : t \in [0, T], \ m = 1, 2, \ldots\}$ is bounded in $\boldsymbol{N}$, the above inequalities are sufficient in order that the desired uniform $\boldsymbol{N}$-weak continuity be valid. Using this continuity, we can in virtue of Arzela-Ascoli theorem (see Dunford-Schwartz [1], Ch. IV, 6.7) extract a subsequence $\{m'''_i\} \subset \{m''_i\}$ such that $\{S_\cdot^{(m'''_i)} P_{m'''_i} u\}_{i=1}^{\infty}$ pointwise $\boldsymbol{N}$-weakly converges, uniformly on $[0, T]$; in particular on $[0, t_0]$, the limit must be $T(\cdot)u$. But this limit has in t_1 the value $u_1 \neq T(t_1)u$: Contradiction. In this manner we have verified that for any $u \in \mathfrak{w}_0$ and $t \in [0, t_0]$ $\{S_t^{(m_i)} P_{m_i} u\}$ $\boldsymbol{N}$-weakly converges to $T(t)u$, for $i \to \infty$.

We can therofore intertwine che limit and the integrals in the last term of (5.29″), obtaining

$$\int_0^{t_0} \varphi(t) \left[\int_{N} \Phi(u) d\mu_t(u) \right] dt \geq$$

$$\geq \int_0^{t_0} \varphi(t) \left[\int_{\boldsymbol{w}_0 \cap \operatorname{supp} \mu} \Phi(T(t)u) d\mu(u) \right] dt =$$

$$= \int_0^{t_0} \varphi(t) \left[\int_{\boldsymbol{w}_0} \Phi(T(t)u) d\mu(u) \right] dt.$$

This being valid for all $\varphi \in L^1(0, t_0)$, $\varphi \geq 0$, we deduce

$$\int_{N} \Phi(u) d\mu_t(u) \geq \int_{\boldsymbol{w}_0} \Phi(T(t)u) d\mu(u)$$

a.e. on $[0, t_0]$. But in virtue of the Theorem in Sec. 3.3, the left term is continuous in t, while the second is obviously also. We can conclude that

$$\int_{N} \Phi(u) d\mu_{t_0}(u) \geq \int_{\boldsymbol{w}_0} \Phi(T(t_0)u) d\mu(u). \tag{5.31}$$

In (5.31), the first integral is actually only over $\boldsymbol{B}$ while the second only over $\boldsymbol{w}_0 \cap \operatorname{supp} \mu$ and $T(t_0)(\operatorname{supp} \mu) \subset \boldsymbol{B}$. Moreover Φ is on $\boldsymbol{B}$ an arbitrary function in $\mathcal{C}(\boldsymbol{B})$. Define ν_{t_0} on the Borel subsets of $\boldsymbol{B}$ (which are also Borel sets in $\boldsymbol{N}$) by

$$\begin{aligned} \nu_{t_0}(\omega) &= \mu(\boldsymbol{w}_0 \cap T(t_0)^{-1}\omega) = \\ &= \mu(T(t_0)^{-1}\omega), \qquad \omega \text{ Borel set} \subset \boldsymbol{B}. \end{aligned} \tag{5.31′}$$

Then (5.31) can be written under the form

$$\int_{B} \varphi(u) d\mu_{t_0}(u) \geq \int_{B} \varphi(u) d\nu_{t_0}(u) \tag{5.31″}$$

for all $\varphi \in \mathcal{C}(\boldsymbol{B})$. This implies that $\mu_{t_0}(\omega) \geq \nu_{t_0}(\omega)$ for all Borel subsets $\omega \subset \boldsymbol{B}$, and consequently for any Borel set ω in $\boldsymbol{N}$ we shall have

$$\begin{aligned}\mu_{t_0}(\omega) &= \mu_{t_0}(\omega \cap \boldsymbol{B}) \geq \nu_{t_0}(\omega \cap \boldsymbol{B}) = \\ &= \mu(T(t_0)^{-1}(\omega \cap \boldsymbol{B})) = \mu((T(t_0)^{-1}\omega) \cap (T(t_0)^{-1}\boldsymbol{B})) = \\ &= \mu(T(t_0)^{-1}\omega) - \mu((T(t_0)^{-1}\omega) \cap (T(t_0)^{-1}(\boldsymbol{N} \setminus \boldsymbol{B}))) = \\ &= \mu(T(t_0)^{-1}\omega),\end{aligned}$$

since $T(t_0)^{-1}(\boldsymbol{N} \setminus \boldsymbol{B})$ does not intersect $\operatorname{supp} \mu$.

In this manner we obtained the following theorem which improves in case $n=3$ the theorems in Sec. 3.2-3.

THEOREM 5. *Let the space-dimension n be $=3$, and let μ have a bounded support in $\boldsymbol{N}$. Then among the statistical solutions (with initial data μ) constructed in Sec. 3.2-3, there exists one such that*

$$\mu_t(\omega) \geq \mu(T(t)^{-1}\omega) \tag{5.32}$$

for all $t \in [0, T]$ and all Borel subsets ω in $\boldsymbol{N}$.

In the sequel such solutions will be called *accretive*.

REMARKS. 1) The condition that the support of μ be bounded in $\boldsymbol{N}$ is inessential. Indeed the following construction works for any (Borel) probability μ on $\boldsymbol{N}$, satisfying (3.5), and leads to an accretive statistical solution $\{\mu_t\}_{0<t<T}$ with initial data μ and satisfying the strengthened energy inequality:

Split $\mu = \sum_{n=1}^{\infty} \varepsilon_n \mu^{(n)}$ such that $\varepsilon_n \geq 0$ $(n=1, 2, \ldots)$, $\sum_{n=1}^{\infty} \varepsilon_n = 1$ and $\mu^{(n)}$ $(n=1, 2, \ldots)$ have bounded supports in $\boldsymbol{N}$. Let $\{\mu_t^{(n)}\}_{0<t<T}$ be the solution with initial data $\mu^{(n)}$ provided by Theorem 5. Put $\mu_t = \sum_{n=1}^{\infty} \varepsilon_n \mu_t^{(n)}$ for all $t \in [0, T]$. It is easy to verify that $\{\mu_t\}_{0<t<T}$ has the desired properties.

2) Since in whole the § 5 we have assumed that the right term of the Navier-Stokes equations is time-independent, it is easy to show (see for instance the beginning of § 6) that in Theorem 5 and Remark 2) above we are allowed to consider $[0, \infty)$ instead $[0, T]$ with $T < \infty$. We shall use below this permissibility.

Taking into account the preceding Remark 1) and 2) and lightly improving the proof of Theorem 5 we can also obtain the following more involved

THEOREM 5′ *Let $n=3$ and let μ be any (Borel) probability in* $\boldsymbol{N}$ *satisfying the condition* (3.5). *Then there exists a statistical solution* $\{\mu_t\}_{0<t<T}$ *with initial data* μ, *satisfying the strengthened energy inequality and*

$$(5.32') \qquad \mu_{t_2}(\omega) \geq \mu_{t_1}(T(t_2-t_1)^{-1}\omega)$$

for all $T \geq t_2 \geq t_1 \geq 0$ *and Borel sets* ω *in* $\boldsymbol{N}$.

This theorem will be used in Sec. 6.2.*c*).

REMARK. 3) *Individual solutions, as statistical solutions are accretive*; actually they satisfy also the stronger condition (5.32′) for all $t_2 \geq t_1 \geq 0$ and Borel sets ω in $\boldsymbol{N}$. Indeed if t_2, t_1 and ω as before are fixed, and $\delta_2 = \delta_{u(t_2)}(\omega)$, $\delta_1 = \delta_{u(t_1)}(T(t_2-t_1)^{-1}\omega)$, then $\delta_1 = 1$ if $u(t_1)$ belongs to the domains of $T(t_2-t_1)$ and $T(t_2-t_1)u(t_1) \in \omega$; by the uniqueness of the regular individual solutions we have $u(t_2) = T(t_2-t_1)u(t_1) \in \omega$, thus $\delta_2 = 1$, hence $\delta_2 \geq \delta_1$, i.e. (5.32′) is fulfilled.

b) It seems very reasonable to expect that for the solutions yielded by Theorems 5 or 5′, a uniqueness theorem, less restrictive than Theorem 2 in Sec. 5.1.*b*), is valid. We shall give a sample of such loosened condition.

THEOREM 6. *Let $n=3$ and be a (Borel) probability on* $\boldsymbol{N}$ *satisfying* (3.5). *Let moreover* $\{\mu_t\}_{0<t<T}$ *be any accretive statistical solution with initial data* μ. *Then the condition*

$$(5.34) \qquad \int_0^T \left| \int_{\boldsymbol{N}} \| u \|^4 \, d\mu_t(u) \right| dt < \infty$$

implies

$$(5.35) \qquad \mu_t(\omega) = \mu(T(t)^{-1}\omega)$$

for all $t \in [0, T]$ *and Borel sets* ω *in* $\boldsymbol{N}$, *thus, in particular, the uniqueness of* $\{\mu_t\}_{0<t<T}$.

PROOF. For $u \in \boldsymbol{N}^1$ let $t(u)$ denote the supremum of those t for which $T(t)u$ is defined. Put $\pi(u)=0$ if $t(u)=\infty$ and $\pi(u)=t(u)^{-1}$ if $t(u)<\infty$; obviously $t(u)>0$, so that $\pi(u)$ is defined on whole $\boldsymbol{N}^1$. Since for any real number ξ, the set $\{u : u \in \boldsymbol{N}^1,\ t(u)>\xi\}$ is open in $\boldsymbol{N}^1$, (as union of open sets), $\pi(\cdot)$ is a Borel function on $\boldsymbol{N}^1$, thus

$$\int_{\boldsymbol{N}} \pi(u)d\mu(u)=\int_{\boldsymbol{N}^1} \pi(u)d\mu(u)$$

makes sens. Since $\{\mu_t\}_{0<t<T}$ is accretive we will have for any $t_0 \in [0, T]$

$$\int_{\boldsymbol{w}_0} \pi(T(t_0)u)d\mu(u) \leq \int_{\boldsymbol{N}} \pi(u)d\mu_{t_0}(u), \tag{5.36}$$

where again $\boldsymbol{w}_0$ denotes the domain of $T(t_0)$, i.e.

$$\omega_0=\{u : u \in \boldsymbol{N}^1,\ t(u)>t_0\}=\{u : u \in \boldsymbol{N}^1,\ \pi(u)^{-1}>t_0\}.$$

In virtue of (2.16″) we have

$$\int_{\boldsymbol{N}} \pi(u)d\mu_t(u) \leq c_7^{-1} \int_{\boldsymbol{N}} (\| u \|^2+1)^2 d\mu_t(u), \qquad t \in [0, T], \tag{5.37}$$

whence

$$\int_0^T \left[\int_{\boldsymbol{N}} \pi(u)d\mu_t(u) \right] dt \leq c_7^{-1} \int_0^T \left[\int_{\boldsymbol{N}} (\| u \|^2+1)^2 d\mu_t(u) \right] dt = c'_7 < \infty, \tag{5.37′}$$

by (5.34). Integrating (5.36) with respect to $t_0 \in [0, T]$ we obtain

$$\int_0^T \left[\int_{\boldsymbol{w}_0} \pi(T(t_0)u)d\mu(u) \right] dt_0 \leq c'_7,$$

thus by Fubini's theorem

$$\int_{\{u : u \in \boldsymbol{N}^1,\ \pi(u)>0\}} \pi(u) \left[\int_0^{\inf\{\pi(u)^{-1}, T\}} \frac{dt_0}{1-t_0\pi(u)} \right] d\mu(u) \leq c'_7 < \infty.$$

where the integrant is $=\infty$ whenever $\pi(u)^{-1}\leq T$. Therefore necessarily we have

$$\mu(\{u: u\in \boldsymbol{N}^1,\ t(u)>T\})=1. \tag{5.38}$$

Since $t_0\in[0, T]$, the last relation implies

$$\mu(\boldsymbol{w}_0)=1. \tag{5.38$'$}$$

The relation (5.35) is an easy consequence of this last fact. Indeed if $t_0>0$ is again fixed and $\boldsymbol{w}_0$ has the same meaning as above, and if $\mu_{t_0}(\omega)>\mu(T(t_0)^{-1}\omega)$ for a certain Borel set ω in $\boldsymbol{N}$, then

$$1=\mu_{t_0}(\boldsymbol{N})=\mu_{t_0}(\omega)+\mu_{t_0}(\boldsymbol{N}\backslash\omega)>\mu(T(t_0)^{-1}\omega)+\mu(T(t_0)^{-1}(\boldsymbol{N}\backslash\omega))=$$
$$\mu(T(t_0)^{-1}\boldsymbol{N})=\mu(T(t_0)^{-1}\boldsymbol{N}^1)=\mu(\boldsymbol{w}_0)=1,$$

that is: A contradiction! Therefore (5.35) is valid for all $t\in[0, T]$ and Borel sets ω in $\boldsymbol{N}$. The proof is complete.

REMARKS. 1) It is clear that if $\pi(u)$ and $t(u)$ conserves their meaning as in the proof above, then the Theorem 6 holds true also in the case when (5.34) is replaced by the weaker condition

$$\int_0^T\left[\int_{\boldsymbol{N}}\pi(u)d\mu_t(u)\right|dt<\infty. \tag{5.34$'$}$$

2) During the proof of Theorem 6 we established also the fact that (5.34′) (hence (5.34) also) implies that except a set of μ-measure 0, for any u_0 there exists a regular (and unique) individual solution on whole $[0, T]$ with initial value u_0 (see (5.38)).

3) In virtue of Remark 3) in the preceding Sec. 5.4.a), Theorem 6 yields the following uniqueness theorem for individual solutions: Let $n=3$; then for any $u_0\in\boldsymbol{N}$ there exists at most one individual solution $u(\cdot)$ on $(0, T)$ with initial data u_0, such that

$$\int_0^T\| u(t)\|^4\,dt<\infty.$$

However in this case a much better result is known, namely (Prodi [1]): Let $n=3$ and $3<p\leq\infty$; if there exists an individual solution $u(\cdot)$ on $(0, T)$ with initial value $u_0\in \boldsymbol{N}$ such that

$$\int_0^T (| u(t) |_p^{\frac{2p}{p-3}} dt < \infty$$

then $u(\cdot)$ is the unique solution on $(0, T)$ with initial value u_0. It is very probable that there exists a « statistical analogue » of this theorem, which would present an obvious interest.

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