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# A Characterization of Algebraic Measures.

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1. Let $\mathcal{G}$ be a locally compact Hausdorff abelian topological group. Let $M(\mathcal{G})$ be the set of complex measures $\mu$ on the Borel sets of $\mathcal{G}$ such that $\|\mu\|$ (defined in the usual way, *vide* [2]) is finite. $M(\mathcal{G})$ is an algebra over the complex numbers $C$ with the convolution operation $*$ (*vide* [4]) as multiplication on $M(\mathcal{G})$.

Cohen [1, 4] completely determined the measures for which

$$\mu * \mu = \mu .$$

Such measures are called *idempotent*. The problem considered and solved in the paper is the characterization of all $\mu$ that satisfy an algebraic equation.

More presisely, define

$$\mu^0 = \delta$$
$$\mu^n = \mu * \mu^{n-1} , \qquad n \geqslant 1$$

where $\delta$ is the unit element of $M(\mathcal{G})$, picturesquely described as « unit mass concentrated at the origin ». A complete characterization is given of those measures $\mu$ for which there exists a set (in general dependent

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on $\mu$ and not unique) of complex numbers $z_i$, $0 \leqslant i \leqslant n$, $z_n \neq 0$, such that

$$\sum_{i=0}^{n} z_i \mu^i = 0 .$$

Such measures are called *algebraic.* They were first considered Istratescu, who proved that the carrier-group of an algebraic measure is compact [3].

The main result of this paper is the theorem in Section 4 that characterizes an algebraic measure as one such that the partition induced on the dual group by the Fourier-Stieltjes transform of the measure is generated by cosets of the dual group.

**2.** Let $\Gamma$ be the dual group of $\mathcal{G}$. Let

$$\hat{\mu}: \Gamma \to C$$

be the Fourier-Stieltjes transform of $\mu$ [4]. Let $P$ be the formal polynomial

$$P(X) = \sum_{i=0}^{n} c_i X^i$$

where the $c_i$'s are complex numbers.

$$P(\hat{\mu}(\gamma)) = (P(\hat{\mu}))(\gamma) = \\ = (P(\mu))\hat{}\,(\gamma) = 0 .$$

Thus $\hat{\mu}(\gamma)$ must be a root of $P$ in $C$.

Conversely if $\hat{\mu}(\gamma)$ is always one of the complex roots of $P$, for all $\gamma \in \Gamma$, then $P(\hat{\mu})$ vanishes identically on $\Gamma$ and the uniqueness theorem for Fourier-Stieltjes transforms [4] shows that $P(\mu) = 0$. Since the functions $C \to C$ with finite images are exactly the functions that can be written as polynomials it follows that the algebraic measures are exactly the measures $\mu$ such that the image of $\hat{\mu}$ is finite. It is now clear that the sum of algebraic measures is algebraic and so is the product.

**3.** For our purposes a partition of a set $A$ is any set of pairwise disjoint subsets of $A$ whose union is $A$. In particular partitions are allowed to have empty members. Let $C$ be given some linear ordering which will be kept fixed. Then an injective mapping $f$ from algebraic measures to ordered partitions is given by

$$f(\mu) = \langle \{\gamma | \hat{\mu}(\gamma) = z\} | z \in C \rangle \, .$$

The main result of this paper is the explicit description of the image of $f$. Clearly any partition may be replaced by an equivalent one by throwing away some or all of the empty members. If the image of $\hat{\mu}$ is a subset of a set $K$ we write

$$f_k(\mu) = \langle \{\gamma | \hat{\mu}(\gamma) = z\} | z \in K \rangle \, .$$

Recall that a ring of sets is a set of sets stable under the formation of complements and finite unions (and hence under the formation of finite intersections).

In the case of idempotent measures the polynomial involved, *viz.* $P(X) = X^2 - X$ has only two distinct roots in $C$, 0 and 1. Thus the idempotent measures can be described by considering one member of the partition induced by the polynomial. Let

$$S(\mu) = \{\gamma \in \Gamma | \hat{\mu}(\gamma) = 1\}$$

for idempotent $\mu$. Cohen showed that a subset $\Lambda$ of $\Gamma$ has the form $S(\mu)$ for some idempotent $\mu$ if and only if $\Lambda$ lies in the ring of sets generated by the cosets of open subgroups of $\Gamma$.

Let $A$ be a set. An ordered $m$-partition of $A$ is an ordered $m$-tuple of pairwise disjoint sets whose union is $A$. Let $\mathcal{S} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$ be a finite set of $m$-partitions of $A$. Let $\mathcal{T}_{ij}$ be the $j$-th member of $\mathcal{T}_i$: The ordered $m$-partition $\mathcal{T}$ is *primitively generated* by $\mathcal{S}$ if every finite intersection of form $\bigcap_{i=1}^{n} \mathcal{T}_{ij_i}$ is a subset of some member of $\mathcal{T}$. Hence every member of $\mathcal{T}$ must be the union of intersections of this form. Let $\mathcal{C}$ be a set of ordered $m$-partitions of $A$. $\mathcal{C}$ is an $m$-partition algebra if $\mathcal{C}$ contains every ordered $m$-partition primitively generated by a finite subset of $\mathcal{C}$. It is clear that the intersection of a collection of $m$-partition algebras is itself an $m$-partition algebra. Thus there is a smallest $m$-partition algebra containing any set of ordered $m$-partitions. This algebra is called the $m$-partition algebra *generated* by the given set of partitions.

A *Type I* partition of $\Gamma$ is a finite ordered partition of $\Gamma$ in which at most two members are non-empty and in which one member is a coset of an open subgroup of $\Gamma$.

**4.** We shall now state the theorem that characterizes algebraic measures.

THEOREM. *Let $P$ be a polynomial over $C$ with distinct roots* $\{c_1, c_2, \ldots, c_m\} = K$ in $C$. *Let $\mu \in M(\mathcal{G})$. $P(\mu) = 0$ if and only if*: (*a*) *for all $\gamma \in \Gamma$, $\hat{\mu}(\gamma) \in K$ and* (*b*) *$f_k(\mu)$ belongs to the m-partition algebra generated by the ordered m-partitions of Type I.*

First let us look at Cohen's result in our terminology. The ring of sets generated by a set of sets $\mathcal{V}$ is precisely the set of sets $B$ that can be written as finite unions of finite intersections

$$B = \cup \cap B_{ij}$$

where every $B_{ij}$ or its complement $B'_{ij}$ belongs to $\mathcal{V}$. Thus the ordered partition $\langle B, B' \rangle$ is primitively generated by the set of ordered partitions of form $\langle B_{ij}, B'_{ij} \rangle$. The ring of sets generated by the cosets of open subgroups of $\Gamma$ thus corresponds to the 2-partition algebra generated by Type I ordered 2-partitions. Cohen's Theorem is seen to be a special case of our theorem.

The remainder of this paper will be devoted to a proof of our theorem above. We shall prove six lemmas and then the theorem.

**5.** LEMMA 1. Let $P(\mu) = 0$, $\mu \in M(\mathcal{G})$. *Let $c_1, \ldots, c_m$ be the distinct complex roots of $P$. Let $k_1, \ldots, k_m$ be complex numbers, not necessarily distinct. There exists $\nu \in M(\mathcal{G})$ such that $\hat{\mu}(\gamma) = c_i$ if and only if $\hat{\nu}(\gamma) = k_i$.*

PROOF. There exists a polynomial $P_1(X)$ over $C$ such that $P_1(c_i) = k_i$ for all $i$. $P_1(\mu) \in M(\mathcal{G})$. $P_1(\mu)\hat{}(\gamma) = P_1(\hat{\mu}(\gamma))$. Thus $\nu$ may be chosen as $P_1(\mu)$.

Two particular cases of this lemma are of special interest. The $k_i$ may be chosen so that each is equal to some $c_j$. Thus if an algebraic measure $\mu$ is given, which induces an ordered partition $\mathcal{P}$ on $C$ and if an ordered partition $\mathcal{P}'$ is given such that every member of $\mathcal{P}'$ is equal to the union of members of $\mathcal{P}$, there exists a measure $\nu$ that induces $\mathcal{P}'$ on $C$. Another special case of the lemma is obtained when the $c_i$ are all non-zero. Then a measure $\nu$ exists such that $\hat{\nu}(\gamma) = (\hat{\mu}(\gamma))^{-1}$ for all $\gamma \in \Gamma$. Then $\nu$ is the convolution inverse of $\mu$. An algebraic measure $\mu$ is therefore invertible if and only if $\hat{\mu}(\gamma)$ is never 0.

LEMMA 2. *Let $\Lambda$ be an open subgroup of $\Gamma$. Let $c_1$ and $c_2$ be two complex roots of $P$. There exists $\mu \in M(\mathcal{G})$ such that*

$$P(\mu) = 0 ; \quad \hat{\mu}(\gamma) = c_1 , \quad \gamma \in \Lambda ; \quad \hat{\mu}(\gamma) = c_2 , \quad \gamma \notin \Lambda .$$

PROOF. Let $H$ be the annihilator of $\Lambda$. $H$ is compact and isomorphic to the dual of $\Gamma/\Lambda$. Let $m$ be Haar measure on $H$ with $m(H) = 1$. $m$ defines a measure $m_1$ in $M(\mathcal{G})$ by

$$m_1(B) = m(B \cap H)$$

for all Borel sets $B$ in $\mathcal{G}$. $\hat{m}_1(\mu)$ is 1 if $\gamma \in \Lambda$ and 0 otherwise. The previous lemma shows the existence of a measure with the desired properties.

LEMMA 3. *Let $\Lambda$ be an open subgroup of $\Gamma$. Let $\gamma_0 \in \Gamma$. Let $c_1$ and $c_2$ be two roots of $P$. There exists $\mu \in M(\mathcal{G})$ with $P(\mu) = 0$, $\hat{\mu}(\gamma) = c_1$ if $\gamma \in \gamma_0 + \Lambda$, $\hat{\mu}(\gamma) = c_2$ otherwise.*

PROOF. By the previous lemma there exists $\mu_1$ such that:

$$\hat{\mu}_1(\gamma) = c_1 \text{ if } \gamma \in \Lambda$$

$$\hat{\mu}_1(\gamma) = c_2 \text{ otherwise} .$$

Let

$$d\mu = \gamma_0 d\mu_1$$

then

$$\hat{\mu}(\gamma) = \hat{\mu}_1(\gamma - \gamma_0)$$

$$\hat{\mu}(\gamma) = c_1 \quad \text{if } \gamma \in \gamma_0 + \Lambda$$

$$\hat{\mu}(\gamma) = c_2 \quad \text{if } \gamma \notin \gamma_0 + \Lambda .$$

LEMMA 4. *Let $P(\mu_i) = 0$, $\mu_i \in M(\mathcal{G})$, for $1 \leqslant i \leqslant n$. Let $\mathcal{T}_i$ be the ordered partition of $\Gamma$ induced by $\mu_i$. Let $\mathfrak{R}$ be an ordered partition of $\Gamma$ primitively generated by the $\mathcal{T}_i$. There exists a measure $\mu_i \in M(\mathcal{G})$ whose induced partition is $\mathfrak{R}$.*

PROOF. Let $\mathcal{T}_{ij} = \{\gamma \in \Gamma | \hat{\mu}_i(\gamma) = c_j\}$ .

To every intersection of form $\bigcap_{i=1}^{n} \mathcal{T}_{ij_i}$ we associate the point $(c_{j_1}, \ldots, c_{j_n})$. Thus partitions of $\Gamma$ primitively generated by the $\mathcal{T}_i$

correspond to partitions of the finite set

$$\{(\hat{\mu}_1(\gamma), \dots, \mu_n(\gamma)) | \gamma \in \Gamma\} = S .$$

Let $S_i$ be the subset of $S$ corresponding to $\mathfrak{R}_i$. Let $Q$ be a polynomial in $X_1, \dots, X_n, \bar{X}_n, \dots, \bar{X}_1$ over $C$ such that $Q(S_i) = c_i$ for all $i$. Then $Q(\mu_1, \dots, \mu_n, \tilde{\mu}_1, \dots, \tilde{\mu}_n)(\gamma) = Q(\mu_1(\gamma), \dots, \mu_n(\gamma), \bar{\mu}_1(\gamma), \dots, \bar{\mu}_n(\gamma))$. Thus $\mu = Q(\mu_1, \dots, \tilde{\mu}_1, \dots, \tilde{\mu}_n)$ is the desired measure.

LEMMA 5. *Let $\mathfrak{T}$ be a member of the algebra of ordered m-partitions generated by the set of all m-partitions of Type I. Let $\mathfrak{T}$ be a polynomial with distinct complex roots $c_1, \dots, c_m$. Then there exists $\mu \in M(\mathfrak{G})$ such that $P(\mu) = 0$ and $\mathfrak{T}_i = \{\gamma \in \Gamma | \hat{\mu}(\gamma) = c_i\}$.*

PROOF. By Lemma 4 the set $\mathfrak{C}$ of all ordered $m$-partitions which arise from some $\mu$ such that $P(\mu) = 0$ is an algebra. By Lemma 3, $\mathfrak{C}$ contains all the partitions of Type I. Thus $\mathfrak{C}$ contains the algebra generated by the partitions of Type I.

Our next test is to show that $\mathfrak{C}$ is the algebra generated by the partitions of Type *I*.

LEMMA 6. *If $P(\mu) = 0$ there exist $c_i \in C$, and idempotent $\mu_i \in M(\mathfrak{G})$, $1 \leqslant i \leqslant m$ such that*

$$\mu = \sum_{i=1}^{m} c_i \mu_i .$$

PROOF. Let $c_i$ be the distinct roots of $P$ in $C$. There exist polynomials $P_i$ over $C$ such that $P_i(c_j) = \delta_{ij}$. Let $\mu_i = P_i(\mu)$. Clearly $\hat{\mu}_i(\gamma) = 1$ if $\hat{\mu}(\gamma) = c_i$ and $\hat{\mu}_i(\gamma) = 0$ otherwise. Thus every $\mu_i$ is idempotent. It is also clear that $\mu = \sum_{i=1}^{m} c_i \mu_i$.

We shall now prove the theorem stated in Section 4.

PROOF. Let $P(\mu) = 0$. By Lemma 6 there exist $c_i \in C$ and $\mu_i$ idempotent, $1 \leqslant i \leqslant m$, such that $\mu = \sum_{i=1}^{m} c_i \mu_i$.

Each idempotent $\mu_i$ induces an $m$-partition $\mathfrak{T}_i$ of $\Gamma$ with

$$\mathfrak{T}_{i1} = \hat{\mu}_i^{-1}(0)$$

$$\mathfrak{T}_{i2} = \hat{\mu}_i^{-1}(1)$$

$$\mathfrak{T}_{ij} = \emptyset \qquad \text{otherwise} .$$

Then in the partition $\mathcal{T}^*$ of $\Gamma$ induced by $\mu$, $\mathcal{T}_j^*$ is exactly the union of all intersections of form $\mathcal{T}_{1i_1} \cap \mathcal{T}_{2i_2} \cap \ldots \cap \mathcal{T}_{mi_m}$ ($i_k = 1$ or $2$) such that $\sum_1^m {}_k c_k \delta_{1,i_k-1} = c_j$.

By Cohen's Theorem each $P_{ki_k}$ lies in the ring of sets generated by the open cosets in $\Gamma$. In our terminology each partition $\mathcal{T}_i$ is primitively generated by Type I partitions. But the partition $\mathcal{T}^*$ is primitively generated by the partitions $\mathcal{T}_i$. Thus $\mathcal{T}^*$ lies in the algebra generated by the Type I partitions.

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