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ANTONIO PASINI

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Some Fixed Point Theorems of the Mappings of Partially Ordered Sets.

ANTONIO PASINI (Firenze) (*)

1. Introduction.

In this paper we give new simple proofs of some fixed point theorems, and strengthen others. The methods we shall use base themselves on two « strong » induction principles, we stated and utilized in [5]. We shall show, moreover, that one of them is equivalent to Axiom of Choice.

Let's now recall some results on the fixed points of a function defined from a partially ordered set $\langle P; \leqslant\rangle$ into itself.

PROPOSITION A. *Let* $\langle P; \leqslant\rangle$ *be a nonempty partially ordered set every well ordered subset of which has an upper bound. And let* f *be a function from* P *into* P *such that* $x \leqslant f(x)$ *for every* x; *then* f *has a fixed point.*

The preceding result is proved in [2] by using Axiom of Choice. As a corollary we get:

PROPOSITION B. *Let* $\langle P; \leqslant\rangle$ *be a nonempty partially ordered set every well ordered subset of which has a least upper bound, and let* f *be a function just like propositon A's one*; *then* f *has a fixed point.*

The preceding proposition, however, may be proved indipendently, and without using Axiom of Choice. (see [1] and [2]).

(*) Indirizzo dell'A.: Istituto Matematico « U. Dini » dell'Università di Firenze.

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PROPOSITION C (see [3]). *Let* $\langle P; \leqslant \rangle$ *be a nonempty partially ordered set every well ordered subset* B *of which has an upper bound which is minimal in the set of upper bounds of* B. *Let* f *be a mapping from* P *into* P *such that* $a \leqslant f(a)$ *for some* a *in* P, *and such that* $x \leqslant f(x) \leqslant y$ *implies* $f(x) \leqslant f(y)$, *for every* x *and* y *in* P; *moreover* $\{x, f(x)\}$ *has a greatest lower bound in* $\langle P; \leqslant \rangle$; *then* f *has a fixed point.*

PROPOSITION D (see [1]; see also [2]). *Let* $\langle P; \leqslant \rangle$ *be a nonempty partially ordered set every well ordered subset of which has a least upper bound. And let* f *be a mapping from* P *into* P *such that* $a \leqslant f(a)$ *for some* a *in* P, *and such that* $x \leqslant y$ *implies* $f(x) \leqslant f(y)$, *for every* x *and* y *in* P. *Then* f *has a fixed point.*

Proposition D is proved in [1] and in [2] without using Axiom of Choice.

PROPOSITION E (see [4]). *Let* $\langle P; \leqslant \rangle$ *be a nonempty well ordered set every subset of which has a least upper bound. Let* f *be just like in proposition* D; *then the set* $\{p | f(p) = p\}$ *is nonempty and has a maximal element.*

Proposition E is proved in [4] without any use of Axiom of Choice. But this is possible because $\langle P; \leqslant \rangle$ is well ordered.

We shall prove (but postpone the proofs):

THEOREM A. *Let* $\langle P; \leqslant \rangle$ *be a nonempty partially ordered set every well ordered subset* B *of which has an upper bound which is minimal in the set of upper bounds of* B. *And let* f *be a function from* P *into* P *such that:*

(1) $x \leqslant f(x) \leqslant y$ *implies* $f(x) \leqslant f(y)$ *for every* x *and* y *in* P.

(2) $\{x, f(x)\}$ *has a greatest lower bound in* P, *for every* x *in* P.

(3) *There is an element* a *of* P *such that* $a \leqslant f(a)$.

Then the set $\{p | f(p) = p\}$ *is nonempty and has a maximal element.*

Theorem A strengthens proposition C. Theorem A's proof makes use of Axiom of Choice.

THEOREM B. *Let* $\langle P; \leqslant \rangle$ *be a nonempty partially ordered set every well ordered subset of which has a least upper bound. And let* f *be a mapping from* P *into* P *such that* $x \leqslant f(x) \leqslant y$ *implies* $f(x) \leqslant f(y)$ *for every* x *and* y *in* P. *And let be* $a \leqslant f(a)$ *for some* a *in* P. *Then the set* $\{p | f(p) = p\}$ (*is nonempty and*) *has a maximal element.*

REMARK. Theorem B generalizes propositions E and D: in the fact (we require on P more general conditions than in E and) what we require on f is less strong than isotonicity: clearly if f is isotone $x \leqslant f(x) \leqslant y$ implies $f(x) \leqslant f(y)$, but there are non isotone functions f such that $x \leqslant f(x) \leqslant y$ implies $f(x) \leqslant f(y)$, and is $a \leqslant f(a)$ for a convenient a in P. Let consider, for instance, the ordered set $\langle\{1, 2, 3\}; \leqslant\rangle$, where $1 < 2 < 3$, and let's pose $f(1) = 3$, $f(3) = 3$, $f(2) = 1$.

In theorem B we doesn't need Axiom of Choice in proving $\{p \mid f(p) = p\} \neq \emptyset$. So we get a proof of proposition D which doesn't use Axiom of Choice. Moreover, if we assume $\langle P; \leqslant\rangle$ is a well ordered set (as in proposition E), our methods give a proof of the existence of a maximal element in $\{p \mid f(p) = p\}$, which doesn't use Axiom of Choice.

THEOREM C. *Let $\langle P; \leqslant\rangle$ and f be just like in proposition A. Then $\{p \mid f(p) = p\}$ (is nonempty and) has a maximal element.*

2. « Strong » transfinite induction principles.

Let be given a class A and a limit ordinal number α_0. Let α be a nonzero ordinal number less than α_0, and Φ_α be a function from the set $[0, \alpha)$ (*i.e.*: the set of all ordinal numbers less than α) into A. Let $P(x, y)$ be a first order quantification scheme free on x, y, belonging to a language which formalizes set theory. And let $P(\alpha, \Phi_\alpha)$ be the sentence we give from $P(x, y)$ by substituting x with α and y with Φ_α. Moreover let $P(\alpha, \Phi_\alpha)$ be of the form:

$$(\forall\gamma_1) \ldots (\forall\gamma_n)\Big(\big((\gamma_1 < \alpha) \,\&\, \ldots \,\&\, (\gamma_n < \alpha)\big) \rightarrow Q(\Phi_\alpha, \gamma_1, \ldots, \gamma_n)\Big)$$

where $\gamma_1, \ldots, \gamma_n$ vary on $[0, \alpha)$ and $Q(\Phi_\alpha, \gamma_1, \ldots, \gamma_n)$ satisfies the following condition:

CONDITION A. *For any choice of $\gamma_1, \ldots, \gamma_n$ less than α, $Q(\Phi_\alpha, \gamma_1, \ldots, \gamma_n)$ is true only if for every β greater than zero, less than α, and such that is an upper bound for $\{\gamma_1, \ldots, \gamma_n\}$, $Q(\Phi_{\alpha|[0,\beta)}, \gamma_1, \ldots, \gamma_n)$ is true (where $\Phi_{\alpha|[0,\beta)}$ is the restriction of Φ_α to $[0, \beta)$).*

We proved in [5] (section 2, corollaries 2.1 and 2.2) the following results:

Lemma A. *If for every α less than α_0 and greater than 0, for every function Φ_α such that $P(\alpha, \Phi_\alpha)$ is true, there is only one function $\Phi_{\alpha+1}$ such that $P(\alpha+1, \Phi_{\alpha+1})$ is true and $\Phi_{\alpha+1|[0,\alpha)} = \Phi_\alpha$, and if there is a function Φ_1 such that $P(1, \Phi_1)$ is true, then for every β (less than α_0 and greater than 0) there exists a function Φ_β such that $P(\beta, \Phi_\beta)$ is true.*

Lemma A's proof doesn't need Axiom of Choice. *Let's now suppose A is a set.* Then we have:

Lemma B. *If for every α less than α_0 and greater than 0, for every function Φ_α such that $P(\alpha, \Phi_\alpha)$ is true, there is a function $\Phi_{\alpha+1}$ such that $P(\alpha+1, \Phi_{\alpha+1})$ is true and $\Phi_{\alpha+1|[0,\alpha)} = \Phi_\alpha$, and if there is a function Φ_1 such that $P(1, \Phi_1)$ is true, then for every β (less than α_0 and greater than 0) there exists a function Φ_β such that $P(\beta, \Phi_\beta)$ is true.*

In lemma B's proof we made use of Axiom of Choice (formulated as well ordering principle); conversely we shall see now that lemma B implies the well ordering principle, and then the Axiom of Choice. Let A be a nonempty set, and a an element of A; let's consider the set $A^* = A \cup \{A\}$, and let α_0 be a limit ordinal of power greater than $2^{|A^*|}$ (where $|A^*|$ is the cardinality of A^*) (1). Let $P(\alpha, \Phi_\alpha)$ be the following statement:

For every $\gamma < \alpha$, if $\{\Phi_\alpha(\delta) | \delta < \gamma\}$ is properly contained in A, then $\Phi_\alpha(\gamma) \in A$, and $\Phi_\alpha(\delta) \neq \Phi_\alpha(\gamma)$ for every $\delta < \gamma$;

It's easily seen that this statement satisfies condition A. Let now Φ_α be a function from A^* into A^* such that $P(\alpha, \Phi_\alpha)$ is true. If $\Phi_\alpha(\delta) = \{A\}$ for a $\delta < \alpha$, then let δ_0 be the least δ such that $\Phi_\alpha(\delta) = \{A\}$. $\{\Phi_\alpha(\gamma) | \gamma < \delta_0\} \subseteq A$; let's suppose $\{\Phi_\alpha(\gamma) | \gamma < \delta_0\} \neq A$. Then $\Phi_\alpha(\delta_0) \in A$, by proposition $P(\alpha, \Phi_\alpha)$; so we get a contradiction, and have to admit $\{\Phi_\alpha(\gamma) | \gamma < \delta_0\} = A$. So we can define $\Phi_{\alpha+1}$ by: ($\Phi_{\alpha+1|[0,\alpha)} = \Phi_\alpha$, and) $\Phi_{\alpha+1}(\alpha) = \{A\}$. And it's easily seen that $P(\alpha+1, \Phi_{\alpha+1})$ is true. Now we have to consider the second case, that's when $\{A\} \notin \{\Phi_\alpha(\delta) | \delta < \alpha\}$. We have two subcases: $\{\Phi_\alpha(\delta) | \delta < \alpha\} = A$ and $\{\Phi_\alpha(\delta) | \delta < \alpha\} \neq A$. In the first subcase we pose $\Phi_{\alpha+1}(\alpha) = \{A\}$; in the second subcase we equate $\Phi_{\alpha+1}(\alpha)$ to an arbitrary element of $A - \{\Phi_\alpha(\delta) | \delta < \alpha\}$. And it's trivial to see that $P(\alpha+1, \Phi_{\alpha+1})$ is true (in the first as well as in the second subcase). So, by lemma B, if β

(1) It's known that this can be done without any use of Axiom of Choice.

is an ordinal of power greater tha $2^{|A^*|}$ and less than α_0 (Cobviously we can always choose α_0 such that such a β exists), there is a Φ_β such that $P(\beta, \Phi_\beta)$ is true. Let's suppose, by absurde, $\{\Phi_\beta(\gamma)|\gamma<\beta\}$ properly contained in A. Then Φ_β is injective, and so has as many values as the power of β; this gets a contradiction. So we must have $\{\Phi_\beta(\gamma)|\gamma<\beta\}\supseteq A$. And it's trivial to check that Φ_β defines a well ordering on A; as we wished to prove. Then:

Lemma B is equivalent to Axiom of Choice.

3. Proofs.

PROOF OF THEOREM A. Let a be an element of P such that $f(a)\geqslant a$. Let α_0 be a limit ordinal whose power is greater than $2^{|P|}$ ($|P|$ is P's cardinality). Let Φ_α indicate a function from $[0, \alpha)(0<\alpha<<\alpha_0)$ into P, and let $P(\alpha, \Phi_\alpha)$ be the following statement:

$\Phi_\alpha(0)=a$. And for every ordinal number γ, if $\gamma<\alpha$, then is $\gamma+1<\alpha$,

$\Phi_\alpha(\gamma+1)=f(\Phi_\alpha(\gamma))$, and if γ is a limit ordinal, then the set B_γ of upper bounds of $\{\Phi_\alpha(\delta)|\delta<\gamma\}$ is nonempty and has a minimal element, and $\Phi_\alpha(\gamma)$ is such an element.

It's easily seen that $P(\alpha, \Phi_\alpha)$ satisfies condition A. Let's now suppose Φ_α is a function verifying $P(\alpha, \Phi_\alpha)$. Now we prove that there is a function $\Phi_{\alpha+1}$ such that $P(\alpha+1, \Phi_{\alpha+1})$ is true and $\Phi_{\alpha+1|[0,\alpha)}=\Phi_\alpha$. We have to distinguish two cases: $\alpha=\delta+1$ for some ordinal number δ, or α is a limit ordinal. First we verify $\{\Phi_\alpha(\gamma)|\gamma<\alpha\}$ is a well ordered subset of $\langle P;\leqslant\rangle$. We start by proving (by transfinite induction on $\gamma<\alpha$) the following statement $S(\gamma)$:

$\Phi_\alpha(\delta)\leqslant\Phi_\alpha(\gamma)$ for every δ, with $\delta\leqslant\gamma$.

Let's now suppose $S(\gamma)$ true for any $\gamma<\mu$ (μ is a given ordinal number less than α). We shall prove $S(\mu)$ is true. We must distinguish two cases: there is an ordinal number ν such that $\mu=\nu+1$, or μ is a limit ordinal. If $\mu=\nu+1$, then $\Phi_\alpha(\mu)=f(\Phi_\alpha(\nu))$. We have two subcases: $\nu=\lambda+1$ or ν is a limit ordinal; if $\nu=\lambda+1$, $\Phi_\alpha(\nu)=f(\Phi_\alpha(\lambda))$, and then $\Phi_\alpha(\mu)=f\big(f(\Phi_\alpha(\lambda))\big)$. But $f(\Phi_\alpha(\lambda))=\Phi_\alpha(\nu)\geqslant\Phi_\alpha(\lambda)$ (by inductive hypothesis). So $\Phi_\alpha(\lambda)\leqslant f(\Phi_\alpha(\lambda))=\Phi_\alpha(\nu)$; then we get $f(\Phi_\alpha(\lambda))\leqslant f(\Phi_\alpha(\nu))$, that's $\Phi_\alpha(\mu)\geqslant\Phi_\alpha(\nu)$. And this suffices to prove $S(\mu)$.

Let's now consider the second subcase: that's when ν is a limit ordinal. We have $\Phi_\alpha(\nu) \geqslant \Phi_\alpha(\lambda)$ for every $\lambda < \nu$, and $f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\lambda + 1) \leqslant \Phi_\alpha(\nu)$ (by inductive hypothesis). And $\Phi_\alpha(\lambda) \leqslant f(\Phi_\alpha(\lambda))$ (by inductive hypothesis). So $\Phi_\alpha(\lambda) \leqslant f(\Phi_\alpha(\lambda)) \leqslant \Phi_\alpha(\nu)$: then we get $f(\Phi_\alpha(\lambda)) \leqslant f(\Phi_\alpha(\nu))$; then $\Phi_\alpha(\lambda) \leqslant \Phi_\alpha(\mu)$. So: $\Phi_\alpha(\mu) \geqslant \Phi_\alpha(\lambda)$ for every $\lambda < \nu$; hence $\Phi_\alpha(\mu)$ is an upper bound of $\{\Phi_\alpha(\lambda) | \lambda < \nu\}$. But $\Phi_\alpha(\mu) = f(\Phi_\alpha(\nu))$; then does exists the greatest lower bound of $\{\Phi_\alpha(\nu), \Phi_\alpha(\mu)\}$, and it's clearly an upper bound of $\{\Phi_\alpha(\lambda) | \lambda < \nu\}$; but $\Phi_\alpha(\nu)$ is a minimal upper bound of $\{\Phi_\alpha(\lambda); \lambda < \nu\}$. Then $\Phi_\alpha(\nu) = \text{g.l.b.}\{\Phi_\alpha(\nu), \Phi_\alpha(\mu)\}$, that's $\Phi_\alpha(\nu) \leqslant \Phi_\alpha(\mu)$, as we wished to prove; finally we have $S(\mu)$.

Let's now consider the second case, that's when μ is a limit ordinal; then, by $P(\alpha, \Phi_\alpha)$, $\Phi_\alpha(\mu)$ is an upper bound for $\{\Phi_\alpha(\lambda) | \lambda < \mu\}$, and $S(\mu)$ is trivially verified; then $S(\gamma)$ is true for every $\gamma < \alpha$. This implies Φ_α is a monotone function from $[0, \alpha)$ into P; and therefore $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$ is a well ordered set; then, by hypothesis, does exists a minimal upper bound b of $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$, and we can pose $\Phi_{\alpha+1}(\alpha) = b$.

And so, by lemma B, for every $\beta < \alpha_0$ is definable a Φ_β such that $P(\beta, \Phi_\beta)$ is true. We have already seen that Φ_β is a monotone funtion from $[0, \beta)$ into $\langle P; \leqslant \rangle$. Let's now suppose $x \neq f(x)$ for every x; in P; then Φ_β has as many values as the power of β. And we get an absurde because we can choose β with power greater than $|P|$. So we must admit $\bar{p} = f(\bar{p})$ for a convenient $\bar{p} \in P$.

We remark that we can choose $\bar{p}$ greater than (or equal to) a.

We shall now prove that the set of fixed points of f has a maximal element. Let's consider another proposition $P'(\alpha, \Phi'_\alpha)$ as follows:

> for every γ_1 and γ_2 less than α, if $\gamma_1 < \gamma_2$, then, if $\Phi'_\alpha(\gamma_1)$ is maximal in $\{p | f(p) = p\}$, is $\Phi'_\alpha(\gamma_1) = \Phi'_\alpha(\gamma_2)$. And, if it isn't maximal in $\{p | f(p) = p\}$, is $\Phi'_\alpha(\gamma_1) < \Phi'_\alpha(\gamma_2)$. Moreover, for every γ less than α, $\Phi'_\alpha(\gamma)$ is a fixed point for f. And is $\Phi'_\alpha(0) = \bar{p}$ (where $\bar{p}$ is the before found fixed point of f).

It's a trivial question to check that condition A is satisfied by $P'(\alpha, \Phi'_\alpha)$. Now we must see that, if there's a function Φ'_α such that $P'(\alpha, \Phi'_\alpha)$ is true, then we can construct a function $\Phi'_{\alpha+1}$ such that $P'(\alpha + 1, \Phi'_{\alpha+1})$ is true and $\Phi'_{\alpha+1|[0,1)} = \Phi'_\alpha$: We distinguish two cases: when is $\alpha = \delta + 1$ for a convenient δ, and when α is a limit ordinal.

Let's suppose is $\alpha = \delta + 1$; we have two subcases: $\Phi'_\alpha(\delta)$ is maximal in $\{p | f(p) = p\}$, or not. If $\Phi'_\alpha(\delta)$ is such a maximal element, then we set $\Phi'_{\alpha+1}(\alpha) = \Phi'_\alpha(\delta)$ (and $\Phi'_{\alpha+1|[0,\alpha)} = \Phi'_\alpha$). It's easily seen that $P'(\alpha + 1, \Phi'_{\alpha+1})$ is true. Let's now consider the subcase when $\Phi'_\alpha(\delta)$

isn't maximal in $\{p|f(p)=p\}$: then there is a fixed point q of f such that $q > \Phi'_\alpha(\delta)$. If we set $\Phi'_{\alpha+1}(\alpha)=q$ (and $\Phi'_{\alpha+1|[0,\alpha)}=\Phi'_\alpha$) we get a function satisfying $P'(\alpha+1, \Phi'_{\alpha+1})$. Let's now consider the case when α is a limit ordinal. We have two subcases: there is an ordinal γ less than α such that $\Phi'_\alpha(\gamma)$ is maximal in $\{p|f(p)=p\}$, or there's no such ordinal. In the first subcase is $\Phi'_\alpha(\lambda)=\Phi'_\alpha(\gamma)$ for every λ (less than α and) greater than k. So we can set $(\Phi'_{\alpha+1|[0,\alpha)}=\Phi'_\alpha$ and) $\Phi'_{\alpha+1}(\alpha)=\Phi'_\alpha(\gamma)$. $\Phi'_{\alpha+1}$, so defined, verifies $P'(\alpha+1, \Phi'_{\alpha+1})$. In the second subcase $\{\Phi'_\alpha(\gamma)|\gamma<\alpha\}$ is a well ordered set. Then by hypothesis, it has a minimal upper bound b. We shall see now that there is a fixed point q greater than (or equal to) b. We have $\Phi'_\alpha(\gamma)=f(\Phi'_\alpha(\gamma))$, and $b > \Phi'_\alpha(\gamma)$, for every γ less than α. That's: $\Phi'_\alpha(\gamma)\leqslant f(\Phi'_\alpha(\gamma))\leqslant b$. We get: $f(\Phi'_\alpha(\gamma))\leqslant f(b)$; that's $\Phi'_\alpha(\gamma)\leqslant f(b)$, for every γ less than α. So $f(b)$ in an upper bound for the set $\{\Phi'_\alpha(\gamma)|\gamma<\alpha\}$; but the set $\{b, f(b)\}$ has a greatest lower bound. This, and the minimality of b in the set of upper bounds of $\{\Phi'_\alpha(\gamma)|\gamma<\alpha\}$, give $b\leqslant f(b)$. By the first part of the proof of this theorem, we can find a fixed point (of f) q, such that $q\geqslant b$. If we set $(\Phi'_{\alpha+1|[0,\alpha)}=\Phi'_\alpha$ and) $\Phi'_{\alpha+1}(\alpha)=q$, we have a function $\Phi'_{\alpha+1}$ satisfying $P(\alpha+1, \Phi'_{\alpha+1})$. Then, by lemma B, for every nonzero ordinal β less than α_0, we have a function Φ'_β such that $P'(\beta, \Phi'_\beta)$ is true. Let's choose β of power greater than $|P|$. If no $\Phi'_\beta(\gamma)$ is maximal in the set of fixed points of f, Φ'_β is an injection, and therefore has as many values as the power of β; therefore we get an absurde. And we have to admit that there is an ordinal γ such that $\Phi'_\beta(\gamma)$ is maximal in $\{p|f(p)=p\}$. As we wished to prove.

Proof of Theorem B. The proof is quite similar to theorem A's one; then various details will be omitted. α_0 is an ordinal number of power greater than $2^{|P|}$, Φ^a_α a function from $[0,\alpha)$ into P, a an element of P such that $a\leqslant f(a)$, and $P(\alpha, \Phi^a_\alpha)$ is the following statement:

$\Phi^a_\alpha(0)=a$. And for every ordinal number γ, if $\gamma<\alpha$, then if $\gamma+1<\alpha$, $\Phi^a_\alpha(\gamma+1)=f(\Phi^a_\alpha(\gamma))$, and if γ is a limit ordinal, then does exists in P the least upper bound of $\{\Phi^a_\alpha(\lambda)|\lambda<\gamma\}$, say it l.u.b. $(\Phi^a_\alpha(\lambda)|\lambda<\gamma)$, and is l.u.b. $(\Phi^a_\alpha(\lambda)|\lambda<\gamma)=\Phi^a_\alpha(\gamma)$.

It's easily seen that $P(\alpha, \Phi^a_\alpha)$ satisfies condition A. Let now Φ^a_α be a function verifying $P(\alpha, \Phi^a_\alpha)$. We shall show that there is (only) one function $\Phi^a_{\alpha+1}$ such that $P(\alpha+1, \Phi^a_{\alpha+1})$ is true and is $\Phi^a_{\alpha+1|[0,\alpha)}=\Phi^a_\alpha$; We have to distinguish two cases: $\alpha=\delta+1$ for some ordinal number δ, or α is a limit ordinal. If $\alpha=\delta+1$ we define $\Phi^a_{\alpha+1}$ by posing $(\Phi^a_{\alpha+1|[0,\alpha)} = \Phi^a_\alpha$ and$)$ $\Phi^a_{\alpha+1}(\alpha) = f(\Phi^a_\alpha(\delta))$. And it's easily seen that $P(\alpha+1, \Phi^a_{\alpha+1})$ is true. Let's now suppose α is a limit ordinal. We have to see $\{\Phi^a_\alpha(\lambda) | \lambda < \alpha\}$ is a well ordered subset of $\{P; \leqslant\}$. As we did do in theorem A, let's consider the following statement $S(\gamma)$:

$$\Phi^a_\alpha(\delta) \leqslant \Phi^a_\alpha(\gamma) \text{ for every } \delta(\delta < \gamma < \alpha).$$

Now, given $\mu < \alpha$, let's suppose that $S(\gamma)$ is true for every $\gamma < \mu$. We shall prove that $S(\mu)$ is true. We must distinguish two cases: $\mu = \nu + 1$ or μ is a limit ordinal. Let's now suppose $\mu = \nu + 1$. If $\nu = \lambda + 1$, $S(\mu)$ is proved in quite a similar manner as in theorem A's proof. Let's now suppose ν is a limit ordinal. As in Theorem A's proof, we get $\Phi^a_\alpha(\mu) \geqslant \Phi^a_\alpha(\lambda)$ for every $\lambda < \nu$; then $\Phi^a_\alpha(\mu) \geqslant$ l.u.b.$\cdot$ $\cdot(\Phi^a_\alpha(\lambda) | \lambda < \nu)$, that's $\Phi^a_\alpha(\mu) \geqslant \Phi^a_\alpha(\nu)$. Hence $S(\mu)$ is true. Let's now consider the second case, that's when μ is a limit ordinal: now $S(\mu)$ is trivial, by $P(\alpha, \Phi^a_\alpha)$. So $S(\mu)$ is true in any case. Hence $S(\gamma)$ is true for every $\gamma < \alpha$; this implies that Φ^a_α is a monotone function from the well ordered set $[0, \alpha)$ into $\langle P; \leqslant\rangle$; then $\{\Phi^a_\alpha(\gamma) | \gamma < \alpha\}$ is a well ordered subset of $\langle P; \leqslant\rangle$, and l.u.b. $(\Phi^a_\alpha(\gamma) | \gamma < \alpha)$ does exists. So we can define $\Phi^a_{\alpha+1}(\alpha) =$ l.u.b. $(\Phi^a_\alpha(\gamma) | \gamma < \alpha)$ (and $\Phi^a_{\alpha+1|[0,\alpha)} = \Phi^a_\alpha$). $\Phi^a_{\alpha+1}$ verifies $P(\alpha+1, \Phi^a_{\alpha+1})$. Obviously there is only one such $\Phi^a_{\alpha+1}$. By lemma A it follows that for every $\beta < \alpha_0$ is definable a function Φ^a_β such that $P(\beta, \Phi^a_\beta)$ is true. Now the fact that f has a fixed point p follows as in theorem A's proof. (condition $a \leqslant p$ may be required on p). In this first part of the proof no use of Axiom of Choice is done.

The existence of a maximal element in $\{p | f(p) = p\}$ is proved similarly as in theorem A; therefore we omit this proof; it needs Axiom of Choice.

REMARK. If we suppose $\langle P; \leqslant\rangle$ is well ordered, we don't need Axiom of Choice. First we note that in the preceding proof we get a unique Φ^a_β, for every β less than α_0. Let β_0 be the first ordinal number β such that $f(\Phi^a_\beta(\gamma)) = \Phi^a_\beta(\gamma)$ for a convenient k. And let γ_0 be the first ordinal γ such that $f(\Phi^a_{\beta_0}(\gamma)) = \Phi^a_{\beta_0}(\gamma)$; let's pose $p(a) = \Phi^a_{\beta_0}(\gamma_0)$ So we define a mapping p from the set $\{x | x \leqslant f(x)\}$ into the set $\{x | x = f(x)\}$ Clearly is $x \leqslant p(x)$ for every x. Let's now consider a proposition $P(\alpha, \psi_\alpha)(\psi_\alpha : [0, \alpha) \to P)$ as follows:

For every γ_1 and γ_2 less than α, if $\gamma_1 < \gamma_2$, then, if $\psi_\alpha(\gamma_1)$ is maximal in $\{x | f(x) = x\}$, is $\psi_\alpha(\gamma_1) = \psi_\alpha(\gamma_2)$, and, if it isn't such a maximal element, is $\psi_\alpha(\gamma_1) < \psi_\alpha(\gamma_2)$. Moreover, for every γ less than α, $\psi_\alpha(\gamma)$ is a fixed point of f. And is $\psi_\alpha(0) = p(a)$, And if $\gamma + 1 < \alpha$, if $\psi_\alpha(\gamma)$

is not maximal in $\{x|f(x)=x\}$, is $\psi_\alpha(\gamma+1)=$ g.l.b. $(x|f(x)=x$ and $x>\psi_\alpha(\gamma))$. And if γ is a limit ordinal, then $\psi_\alpha(\gamma)=p\big(\text{l.u.b.}\cdot$ $\cdot(\psi_\alpha(\delta)|\delta<\gamma)\big)$.

It may be seen that $P(\alpha, \psi_\alpha)$ so defined verifies condition A, and every ψ_α such that $P(\alpha, \psi_\alpha)$ is true may be extended in a unique manner to a $\psi_{\alpha+1}$ such that $P(\alpha+1, \psi_{\alpha+1})$ is true. So we can find, by lemma A (and, therefore, without using Axiom of Choice) a ψ_β for every $\beta<\alpha_0$. If $|P|<|\beta|$, is $\psi_\beta(\gamma)$ maximal in $\{x|f(x)=x\}$ for an ordinal γ ($|\beta|$ is the power of β).

Proof of Theorem C. Let α_0 be a limit ordinal whose power is greater than $2^{|P|}$. Let Φ_α indicate a function from $[0, \alpha)(0<\alpha<\alpha_0)$ into P. Let $P(\alpha, \Phi)_\alpha$ be the following statement:

for every ordinal number γ in $[0, \alpha)$, if $\gamma+1<\alpha$, $\Phi_\alpha(\gamma+1)=$ $=f(\Phi_\alpha(\gamma))$, and, if γ is a limit ordinal, then does exist an upper bound of $\{\Phi_\alpha(\lambda)|\lambda<\gamma\}$ in $\langle P; \leqslant\rangle$, and $\Phi_\alpha(\gamma)$ is such an upper bound.

It's easily seen that $P(\alpha, \Phi_\alpha)$ verifies condition A. Moreover, given Φ_α verifying $P(\alpha, \Phi_\alpha)$, there is a $\Phi_{\alpha+1}$ verifying $P(\alpha+1, \Phi_{\alpha+1})$ and such that $\Phi_{\alpha+1|[0,\alpha)}=\Phi_\alpha$. We must distinguish two cases: $\alpha=\delta+1$ for a convenient ordinal δ, or α is a limit ordinal. If $\alpha=\delta+1$, we pose $\Phi_{\alpha+1}(\alpha)=f(\Phi_\alpha(\delta))$; if α is a limit ordinal, then we set $\Phi_{\alpha+1}(\alpha)$ equal to an upper bound of $\{\Phi_\alpha(\gamma)|\gamma<\alpha\}$; and such an upper bound does exist because is $\Phi_\alpha(\gamma)\leqslant\Phi_\alpha(\lambda)$ whenever is $\gamma\leqslant\lambda$, and so Φ_α realizes an order-homomorphism from the well-ordered set $[0, \alpha)$ into $\langle P; \leqslant\rangle$, and then $\{\Phi_\alpha(\gamma)|\gamma<\alpha\}$ is a well-ordered subset of $\langle P; \leqslant\rangle$. Then it follows from lemma B that for every $\beta<\alpha_0$ there is a function Φ_β such that $P(\beta, \Phi_\beta)$ is true. As α_0 has power greater than $2^{|P|}$, we can choose β in $[0, \alpha_0)$ of power greater than to $2^{|P|}$. Let's now suppose (by absurde) $x<f(x)$ for every x in P; then we get $\Phi_\beta(\gamma)<\Phi_\beta(\mu)$ for every choose of γ, μ such that $\gamma<\mu$. So Φ_β is injective, and there fore has more than $2^{|P|}$ values. But, as Φ_β's values are in P, Φ_β has-at most $|P|$ values; we get an absurde. This constrains us to admit there is a fixed point of f, say it p.

Now we shall prove that the nonempty set of fixed points of f has a maximal element. Let's consider the statement $P'(\alpha, \Phi'_\alpha)$ as follows:

For every γ less than α, $\Phi'_\alpha(\gamma)$ is a fixed point for f. And is $\Phi'_\alpha(0)=p$. And, for every γ_1, γ_2 less than α, if $\gamma_1<\gamma_2$. Then, if

$\Phi'_\alpha(\gamma_1)$ is maximal in $\{x|f(x)=x\}$, is $\Phi'_\alpha(\gamma_1)=\Phi'_\alpha(\gamma_2)$; and, if it isn't such a maximal element, is $\Phi'_\alpha(\gamma_1)<\Phi'_\alpha(\gamma_2)$.

$P'(\alpha, \Phi'_\alpha)$ verifies condition A. Now we shall see that, given a function Φ'_α such that $P'(\alpha, \Phi'_\alpha)$ is true, we can construct a function $\Phi'_{\alpha+1}$ which extends Φ'_α and such that $P'(\alpha+1, \Phi'_{\alpha+1})$ is also true. If α isn't a limit ordinal, proof is just like in theorem A. If α is a limit ordinal, but there's a γ less than α and such that $\Phi'_\alpha(\gamma)$ is maximal in $\{x|f(x)=x\}$, proof is just like in theorem A. The only case we must check is when (α is a limit ordinal and) $\Phi'_\alpha(\gamma)$ isn't maximal in the set of fixed points of f, for every γ less than α. In this case the set $\{\Phi'_\alpha(\gamma)|\gamma<\alpha\}$ is a well ordered set. Then there exists an upper bound b of it. In the ordered set Q of all elements of P greater or equal to b, f has a fixed point q. If we set $\Phi'_{\alpha+1}(\alpha)=q$, we get a function as we required. Now the existence of a maximal element in $\{x|f(x)=x\}$ follows as in theorem A.

As we wished to prove.

REMARK. Let's assume on P and f the hypothesis of proposition B. Then, if we substitute in the preceding proof the statement $P(\alpha, \Phi_\alpha)$ with the statement:

for every ordinal γ in $[0, \alpha)$, if $\lambda+1<\alpha$, $\Phi_\beta(\gamma+1)=f(\Phi_\alpha(\gamma))$ and, if γ is a limit ordinal, $\Phi_\alpha(\gamma)=\text{l.u.b.}\,(\Phi_\alpha(\lambda)|\lambda<\nu)$
we have a proof of proposition **A** which utilizes only lemma A, and therefore doesn't need Axiom of Choice.

BIBLIOGRAPHY

[1] S. ABIAN - A. BROWN, *A theorem on partially ordered sets, with application to fixed point theorems*, Can. J. Math., **13**, no I (1961).

[2] A. ABIAN (= S. ABIAN), *Fixed point theorems on the mappings of partially ordered sets*, Rendiconti del Circolo Matematico di Palermo, Serie II, Tomo XX (1971), pp. 139-142.

[3] R. SALINAS BALTASAR, *Fixed points in ordered sets*, In Honor of Prof. Dr. Iniguez y Almech on the Occasion of his Seventieth Birthday and

Academic Retirement, pp. 163-172, Consejo Sup. Investigacion Ci. Fac. C. Zaragoza, 1969 (see Mathematical Reviews, 1970, rep. n. 4171).

[4] A. PELCZAR, *On the invariant points of a transformation*, Ann. Pol. Math., **11** (1961), pp. 199-202.

[5] A. PASINI, *Sulla sottoalgebra di Frattini* (it has to appear on « Le Matematiche »).

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