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B. E. Rhoades

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A Common Fixed Point Theorem.

B. E. Rhoades (*)

In this note we establish the following fixed point theorem, which is a generalization oh that of Iseki [1].

Theorem. Let X be a metric space with two metrics d and δ satisfying the following conditions:

1) $d(x, y) \leqslant \delta(x, y)$ for each $x, y \in X$,

2) X is complete with respect to d,

3) $f, g\colon X \to X$, each continuous with respect to d and satisfying the following contractive condition: there exists a real number h $0 < h < 1$ such that, for each $x, y \in X$,

$$\delta(f(x), g(y)) \leqslant h \max\{\delta(x, y), \delta(x, f(x)), \delta(y, g(y)), [\delta(x, g(y)) + \delta(y, f(x))]/2\}.$$

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by

$$x_1 = f(x_0), \quad x_2 = g(x_1), \ldots, x_{2n} = g(x_{2n-1}), \quad x_{2n+1} = f(x_{2n}), \ldots .$$

As in the proof of Theorem 14 of [2], one can show that, for $m > n$, $\delta(x_m, x_n) \leqslant h^{2n} r(x_0)(1 - h)^{-1}$, where $r(x_0) = \max\{\delta(x_0, x_1), \delta(x_1, x_2)\}$.

(*) Indirizzo dell'A.: Dept. of Mathematics, Indiana University, Swain-Hall East, Bloomington, Indiana, U.S.A.

Therefore $d(xh, x_n) \to 0$ as $n \to +$ so that $\{x_n\}$ is Cauchy, hence convergent. Call the limit z. Since f and g are continuous with respect to the metric d it then follows that z is a common fixed point.

Suppose w is also a common fixed point. Using (3), $\delta(z, w) \leqslant h\delta(z, w)$, which implies $z = w$.

REFERENCES

[1] K. Iseki, *A Common Fixed Point Theorem*, Rend. Sem. Mat. Padova, **53** (1975), pp. 13-14.

[2] B. E. Rhoades, *A Comparison od Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc. 226 (1977) pp. 257-290.

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