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## A Common Fixed Point Theorem.

B. E. RHOADES (\*)

In this note we establish the following fixed point theorem, which is a generalization of that of Iseki [1].

**THEOREM.** Let  $X$  be a metric space with two metrics  $\bar{d}$  and  $\delta$  satisfying the following conditions:

1)  $\bar{d}(x, y) \leq \delta(x, y)$  for each  $x, y \in X$ ,

2)  $X$  is complete with respect to  $\bar{d}$ ,

3)  $f, g: X \rightarrow X$ , each continuous with respect to  $\bar{d}$  and satisfying the following contractive condition: there exists a real number  $h$   $0 < h < 1$  such that, for each  $x, y \in X$ ,

$$\delta(f(x), g(y)) \leq h \max \{ \delta(x, y), \delta(x, f(x)), \delta(y, g(y)), \\ [\delta(x, g(y)) + \delta(y, f(x))]/2 \}.$$

Then  $f$  and  $g$  have a unique common fixed point.

**PROOF.** Let  $x_0 \in X$  and define the sequence  $\{x_n\}$  by

$$x_1 = f(x_0), \quad x_2 = g(x_1), \dots, x_{2n} = g(x_{2n-1}), \quad x_{2n+1} = f(x_{2n}), \dots$$

As in the proof of Theorem 14 of [2], one can show that, for  $m > n$ ,  $\delta(x_m, x_n) \leq h^{2n} r(x_0)(1-h)^{-1}$ , where  $r(x_0) = \max\{\delta(x_0, x_1), \delta(x_1, x_2)\}$ .

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Therefore  $d(xh, x_n) \rightarrow 0$  as  $n \rightarrow +$  so that  $\{x_n\}$  is Cauchy, hence convergent. Call the limit  $z$ . Since  $f$  and  $g$  are continuous with respect to the metric  $d$  it then follows that  $z$  is a common fixed point.

Suppose  $w$  is also a common fixed point. Using (3),  $\delta(z, w) \leq h\delta(z, w)$ , which implies  $z = w$ .

#### REFERENCES

- [1] K. ISEKI, *A Common Fixed Point Theorem*, Rend. Sem. Mat. Padova, **53** (1975), pp. 13-14.
- [2] B. E. RHOADES, *A Comparison of Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc. 226 (1977) pp. 257-290.

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