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On prime ideals of C (X)

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Introduction.

In the paper [*DO*] De Marco and Orsatti have studied the mapping $\sigma : P \longmapsto P \cap C^* (X)$ of $\mathcal{S}(C(X))$ into $\mathcal{S}(C^*(X))$, the spectra of the prime ideals of $C(X)$ and $C^*(X)$ respectively.

If M is a maximal ideal of $C(X)$, $\sigma(M) = M \cap C^*$ is comparable with every prime ideal $P^* \subset M^*$ if M^* is the unique maximal ideal of C^* containing σM. σ subordinates a bijection preserving inclusion between the prime ideals of $C(X)$ contained in M and the prime ideals of $C^*(X)$ contained in $M \cap C^*(X)$.

σM is the minimum prime ideal of $C^*(X)$ comparable with every prime ideal contained in M^* iff M has the same property in $C(X)$; in this case M will be called ramified. We generalize the definition at every prime ideal (necessarily z-ideal) of $C(X)$: that is, a prime ideal P of $C(X)$ is ramified if it is the minimum prime ideal comparable with every prime ideal contained in P. We give several equivalent conditions for a prime z-ideal to be ramified; we produce a result, due to De Marco and independent from the remaining work, stating that every maximal fixed ideal of a space satisfying the first axiom of countability is ramified.

The main result of this paper is the theorem stating that every prime z-ideal in a metric space is ramified.

I. X denotes a T_2 completely regular topological space, $C(X)$ the ring of continuous functions, βX the Stone — Čech compactifi-

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cation (βX is a subspace of Spec $(C(X))$, the set of prime ideals of $C(X)$ with spectral topology). For every $p \in \beta X$, M^p will be the maximal ideal associated to it. For every ideal P of $C(X)$, $Z[P]$ is the z-filter of P. If $\mathfrak{F}$ is a z-filter, $Z^{\leftarrow}[\mathfrak{F}]$ is the z-ideal of $\mathfrak{F}$.

PROPOSITION. *Let Q be a prime ideal of $C(X)$. Let $R(Q)$ denote the ideal sum of all the minimal prime ideals contained in P; $R(Q)$ is smallest among the prime ideals comparable with every prime ideal contained in Q.*

PROOF. Let $P \subset Q$ be prime. P contains a minimal ideal P_α. As either P or $R(Q)$ contain P_α, they are comparable. If now P is comparable with every prime contained in Q, then it contains the minimal prime ideals contained in Q, then $P \supset R(Q)$.

DEFINITION 1. Let Q be a prime ideal of $C(X)$. We say Q is *ramified* if $Q = R(Q)$.

DEFINITION 2. Let $p \in \beta X$. We say p is *ramified* if M^p is ramified.

DEFINITION 3. Let $p \in \beta X$. We say p is *totally ramified* if every prime z-ideal Q is ramified, with $Q \subset M^p$.

Analogously X is said to be ramified (*r.* totally ramified) if every $p \in \beta X$ is ramified (r. totally ramified).

COROLLARY. *A ramified prime ideal is necessarily a* z*-ideal.*

PROOF. A minimal prime ideal is a z-ideal. [*GJ* 14.7].

LEMMA. *If I and J are* z*-ideals then $Z[I + J] = (Z[I], Z[J])$ (the* z*-filter generated by $Z[I]$ and $Z[J]$) and every element of $Z[I + J]$ is the intersection of two elements of $Z[I]$ and $Z[J]$.*

PROOF. Trivial.

THEOREM. *Let* Q *be a non-minimal prime* z*-ideal. The following conditions are equivalent:*

(*a*) *Q is ramified*;

(*b*) *for every $Z \in Z[Q]$ there exist $Z_1, Z_2 \in Z[Q]$ such that $Z_1 \cap Z_2 \subset Z$; and, if $Z' \in Z(X)$ is such that $Z' \supset Z_1 \setminus Z_2$ or $Z' \supset Z_2 \setminus Z_1$, then $Z' \in Z[Q]$*;

(*c*) *for every $Z \in Z[Q]$ there exist $Z_1, Z_2 \in Z[Q]$ such that $Z_1 \cap Z_2 \subset$ $\subset Z$ and $Q \in cl_{Spec(C(X))}(Z_1 \setminus Z_2) \cap cl_{Spec(C(X))}(Z_2 \setminus Z_1)$*;

(*d*) *every cozero set* $A = X \setminus Z$ *with* $Z \in Z[Q]$ *is containded in a* $B = X \setminus Z_0$ *with* $Z_0 \in Z\,[Q]$ *and* B *non* C^* — *embedded in* $B \cup \{Q\} \subset \subset Spec(C\,(X))$;

(*e*) Q *is generated by the functions of* Q *that change their sign in every zero set of* $O_Q = \cap \{P_\alpha : P_\alpha$ *minimal prime ideals contained in* $Q\}$.

PROOF.

$(b) \Longleftrightarrow (c)$.

A base of neighborhoods of Q in Spec $(C(X))$ is produced by the sets $V_f = \{P \in \text{Spec}\,(C(X)) : P \not\ni f\}$ with $f \notin Q$. But $V_f \cap X = \{p \in X : M_p \not\ni f\} = \{p \in X : f(p) \neq 0\} = X \setminus Z(f)$ (with $Z(f) \notin Z\,[Q]$) . Then we have $Q \in cl_{SpecC(X)}\ (Z_1 \setminus Z_2)$ iff for each $Z' \notin Z[Q]$ it is $(X \setminus Z') \cap (Z_1 \setminus Z_2) \neq \varnothing$ i.e. iff for each $Z' \notin Z[Q]$ it is $Z' \not\supset Z_1 \setminus Z_2$.

$(a) \Rightarrow (b)$.

Call P_α $(\alpha \in I)$ the minimal prime ideals. Put $Z \in Z[Q] = Z\big[\sum_{P_\alpha \subset Q} P_\alpha\big]$. Let $\alpha_1, \ldots, \alpha_n$ a set of indexes minimal for the property : $Z \in Z\,[P_{\alpha_1} + \ldots + P_{\alpha_n}]$. If $n > 1$, $Z = Z_1 \cap Z_2$ with $Z_1 \in Z[P_{\alpha_1} + \ldots + P_{\alpha_{n-1}}]$ and $Z_2 \in Z\,[P_{\alpha_n}]$ by lemma. Then Z_1 , Z_2 satisfy the hypothesis of (*b*). In fact they belong to $Z[Q]$ and if $Z' \in Z(X) \setminus Z[Q]$ is such that, for example, $Z' \supset Z_1 \setminus Z_2$ i.e. $Z' \cup Z \supset Z_1 \in Z\,[P_{\alpha_1} + \ldots + P_{\alpha_{n-1}}]$ being $P_{\alpha_1} + \ldots + P_{\alpha_{n-1}}$ a prime *z*-ideal, then $Z \in Z\,[P_{\alpha_1} + \ldots + P_{\alpha_{n-1}}]$ against the hypothesis. Analogously if $Z' \cup Z \supset Z_2 \in Z\,[P_{\alpha_n}]$ and $Z' \notin Z\,[Q]$ then $Z \in Z[P_{\alpha_n}]$. If now $n = 1$ i.e. if $Z \in Z[P_\alpha]$, $P_\alpha \subset Q$ being Q not minimal there exists $P_\beta \neq P_\alpha$, $P_\beta \subset Q$. Let $Z_0 = Z \cap Z' \cap Z''$ with $Z' \in Z[P_\alpha] \setminus Z[P_\beta]$ and $Z'' \in Z[P_\beta] \setminus Z[P_\alpha]$: then $Z_0 \in Z[P_\alpha + P_\beta] \setminus (Z[P_\alpha] \cup Z[P_\beta])$, and we can apply to Z_0 the previous argument.

$(b) \Rightarrow (a)$.

Put $Z \in Z[Q]$ and let Z_1 , $Z_2 \in Z[Q]$ satisfy the property (*b*) for Z . Let $I = Z^{\leftarrow}[(Z_1)]$ (where (Z_1) is the *z*-filter generated by Z_1) and $S = \{h \cdot k :$ with $Z(h) = Z_0$ and $k \notin Q\} \cup (C(X) \setminus Q)$ with $Z_0 = Z_1 \cap Z_2$. S is closed under multiplication and disjoint from I ; in fact $I \subset Q$ and if $h \cdot k \in I$ with $Z(h) = Z_0$ and $k \notin Q$, we have $Z\,(h \cdot k) \supset Z_1$, $Z(h) \cup Z(k) \supset Z_1$, $Z(k) \supset Z_1 \setminus Z_0$ and for (*b*) we have $Z(k) \in Z\,[Q]$. Then there is an ideal Q_1 containing I, disjoint from S and maximal with respect to this property : namely $Z[Q_1] \ni Z_1$, $Z[Q_1] \not\ni Z_0$ and $Q_1 \in Q$ (such an ideal is prime and a *z*-ideal because

$Z^{\leftarrow}[Z[Q_1]]$ has the same property). Doing the same for Z_2, we obtain a prime z-ideal Q_2 such that $Z[Q_2] \ni Z_2$, $Z[Q_2] \not\ni Z_0$ and $Q_2 \subset Q$. Then $Z_0 = Z_1 \cap Z_2 \in Z[Q_1 + Q_2] = (Z[Q_1], Z[Q_2])$ so that Q is ramified.

$(b) \Rightarrow (d)$.

Let $A = X \setminus Z$ with $Z \in Z[Q]$; then there exist Z_1, Z_2 satisfying the property (b). Put $B = X \setminus Z_0$ with $Z_0 = Z_1 \cap Z_2$. Being $Z_0 \subset Z$, it is $B \supset A$; now $Z_1 \setminus Z_0$ and $Z_2 \setminus Z_0$ are disjoint zero sets of B, but their closure in $B \cup \{Q\}$ contains the point Q.

$(d) \Rightarrow (c)$.

Let $Z \in Z[Q]$. Put $A = X \setminus Z$ and take $B = X \setminus Z_0$ with $Z_0 \in Z[Q]$ satisfying (d). Then there is a bounded function on B, not extensible to $B \cup \{Q\}$, hence as x approximates Q, f has two limit points, i.e. there are two disjoint zero-sets Z_1', Z_2' of B, containing Q in their closure. With $Z_1 = Z_1' \cup Z_0$ and $Z_2 = Z_2' \cup Z_0$ we have (c).

$(a) \Rightarrow (e)$.

It is sufficient to see that every $f \in P_\alpha \subset Q$, with P_α a minimal prime ideal and $f > 0$, is a sum of functions that change their sign on every zero-set of O_Q. If $f \notin O_Q$, there is $P_\beta \subset Q$ such that $f \notin P_\beta$; hence if $g \notin P_\alpha$, $g > 0$ is such that $fg = 0$ then $g \in P_\beta \subset Q$ and $f = (2f - g) - (f - g)$; and if $h \in O_Q$ it is $Z(f) \supset Z(h)$ and $Z(g) \supset Z(h)$. If $f \in O_Q$ let $f' \in P_\beta \setminus P_\alpha$ with $P_\alpha, P_\beta \subset Q$; put $g' \notin P_\alpha, f'g' = 0$; then $g' \in P_\beta$ and $f = (f + f' - g') - (f' - g')$, and, as f vanishes on every zero set of O_Q, we have (f).

$(e) \Rightarrow (a)$.

Let $f \in Q$ be a function that changes its sign on every zero set of O_Q. Put $f = f^+ - f^-$ (where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$), it is $f^+ f^- = 0$ hence if $P_\alpha \subset Q$ we have, for example, $f^+ \in P$ but $f^+ \notin O_Q$ and then there is a $P_\beta \subset Q$ such that $f^+ \notin P_\beta$; hence $f^- \in P_\beta$ and $f \in P_\alpha + P_\beta \subset Q$. (q.e.d.).

In the special case where $Q = M^p$, (c) and (f) take the form:

(c') for each $Z \in Z[M^p]$ there exist $Z_1, Z_2 \in Z[M^p]$ such that $Z_1 \cap Z_2 \subset Z$ and $p \in cl_{\beta X}(Z_1 \setminus Z_2) \cap cl_{\beta X}(Z_2 \setminus Z_1)$

(e') M^p is generated by the functions of M^p that change their sign on every zero-set of O^p.

One may wonder whether in this theorem, if $Z \notin Z[O_Q]$, $Z_1 \cap Z_2 = Z$ can be assumed; this is true only if $Z \notin \bigcup_{P_\alpha \subset Q} Z[P_\alpha]$, otherwise it may be not true. In fact let $\sigma \notin N$, put $X = N \cup \{\sigma\}$ with every point of N isolated, and let the neighborhoods of o the $\{\sigma\} \cup H_1 \cup H_2$, with $H_1 \in \mathcal{U}_1$ and $H_2 \in \mathcal{U}_2$ two distinct free ultrafilters on N. For σ there are tree prime z-ideals:

$$M_\sigma = Z^{\leftarrow}(\{\sigma\}), \quad P_1 = Z^{\leftarrow}\{\{\sigma\} \cup H_1 : H_1 \in \mathcal{U}_1\},$$
$$P_2 = Z^{\leftarrow}\{\{\sigma\} \cup H_2 : H_2 \in \mathcal{U}_2\}.$$

It is $P_1 + P_2 = M_\sigma$. P_1, P_2 are minimal and M_σ is ramified. If $f \in M_\sigma$ with $\operatorname{coz} f \in \mathcal{U}_1 \setminus \mathcal{U}_2$ (or $\operatorname{coz} f \in \mathcal{U}_2 \setminus \mathcal{U}_1$) then $\operatorname{coz} f$ is C^* — embedded in the whole X.

We note that in this space every finitely generated ideal is generated by two functions.

2. Proposition. (De Marco). *Let X satisfy the first axiom of countability. Then for every non-isolated point $p \in X$, there are non-maximal prime ideals P_1 and P_2 such that $P_1 + P_2 = M_p$.*

Proof. Let $(x_n)_{n \in N}$ be a sequence of distinct points, converging to p in X ($x_n \neq p$). Select on $D = \{x_n : n \in N\}$ two distinct free ultrafilters $\mathcal{U}_1, \mathcal{U}_2$. For $i = 1, 2$, put:
$P_i = \{f \in C(X) : Z(f) \supset A$ for an $A \in \mathcal{U}_i\}$; P_i is a prime z-ideal of $C(X)$. In fact P_i is clearly a z-ideal and if $Z(fg) \supset A$ and $A \in \mathcal{U}_i$ we have $\big(Z(f) \cap D) \cup (Z(g) \cap D\big) \supset A$: this implies that either $Z(f) \cap D$ or $Z(g) \cap D$ belongs to $\mathcal{U}_i$; hence, for example, $Z(f) \supset Z(f) \cap D \in \mathcal{U}_i$, then $f \in P_i$. Hence P_i is prime. For the continuity of the f, $f(p) = 0$ for every $f \in P_i$, hence P_i ($i = 1, 2$) is contained in M_p.

Suppose now that D is chosen as follows. Let $g \in C(X)$, $g \geq 0$ be such that $Z(g) = \{p\}$ (this is possible because in a $T_{3\frac{1}{2}}$ space every compact G_δ — set is a zero-set).

Let $V_1 \supset V_2 \supset \ldots$ be a base of neighborhoods of p and (a_n) a real sequence constructed inductively in the following way:

$$0 < a_1,\ a_1 \in g[V_1]\,;\ 0 < a_2 < a_1,\ a_2 \in g[V_2] \text{ etc.}$$

For every n, let $x_n \in V_n$ such that $g(x_n) = a_n$. Let D be the set of the x_n. Clearly $(x_n)_{n \in N}$ is formed of distinct points and converges to p. Take now $A_1 \in \mathcal{U}_1$, $A_2 \in \mathcal{U}_2$ such that $A_1 \cap A_2 = \varnothing$ and let $B_1 = g[A_1]$, $B_2 = g[A_2]$. Finally let $\varphi_i \in C(R)$ ($i = 1, 2$) such that $Z(\varphi_i) = B_i \cup \{0\}$ and $\varphi_1 + \varphi_2 = 1$. Put $u_i = \varphi_i g$. Then $u_i \in P_i$ because $Z(u_i) \supset A_i$; and we have that $u_1 + u_2 = g$ $\big((u_1 + u_2)(x) = (\varphi_1 + \varphi_2)\, g(x) = g(x)\big)$.

3. – *Theorem on metric spaces.*

LEMMA. *Let X be a perfectly normal space, Q a prime z-ideal of $C(X)$. If $Z[Q]$ contains no nowhere dense set, then Q is minimal.*

PROOF. Take $f \in Q$ and let $g \in C(X)$ be such that $Z(g) = cl_X(X \setminus Z(f))$; $g \notin Q$ because $\text{int}\,(Z(g) \cap Z(f)) = \varnothing$. Being $fg = 0$, f belongs to every prime ideal contained in Q. Hence Q is minimal.

THEOREM. *Every metric space is totally ramified.*

PROOF. Let Q be a non-minimal prime z-ideal of $C(X)$. For the lemma $Z[Q]$ contains a nowhere dense zero-set Z. For a lemma given by Hausdorff [W 4.39] there exists a discrete set $D \subset X \setminus Z$ with $cl_X D = D \cup Z$.

We want to find two disjoint subsets D_1 and D_2 of D such that $Z = cl\, D_1 \cap cl\, D_2$. Putting $Z_1 = Z \cup D_1$, $Z_2 = Z \cup D_2$ for (b) of theorem at n. 1 we have that Q is ramified.

Put $Y = D \cup Z$. For every $x \in Y$, define $\eta(x) = \min\limits_{\varepsilon > \delta(x)} w(B(x, \varepsilon))$ where $\delta(x)$ is the distance of x from Z; $B(x, \varepsilon)$ is the open ball in Y of center x and radius ε; w is the weight.

It is $\aleph_0 \leq \eta(x) \leq |D|$. For each $H \subset Y$ and each infinite cardinal $\alpha \leq |D|$, put $H_\alpha = \{x \in H : \eta(x) = \alpha\}$. We prove that: $Z_\alpha \setminus cl \bigcup\limits_{\beta < \alpha} Z_\beta \subseteq cl\, D_\alpha \setminus cl(D \setminus D_\alpha)$.

In fact if $z \in Z$ there exists $\varepsilon > 0$ such that $w(B(z, \varepsilon)) = \alpha$, if $d \in D$ with $d \in B(z, \varepsilon/2)$ it is $\varepsilon/2 > \delta(d)$, $B(d, \varepsilon/2) \subseteq B(z, \varepsilon)$ and hence $\eta(d) \leq \alpha$, i.e. $z \notin cl \bigcup\limits_{\beta > \alpha} D_\beta$; now if $z \in cl \bigcup\limits_{\beta < \alpha} D_\beta$ for each $\varepsilon' > 0$ there exists $d \in D$ with $\eta(d) < \alpha$ and $d \in B(z, \varepsilon'/2)$, hence there exists $\varepsilon'' \leq \varepsilon'/2$ such that $\varepsilon'' > \delta(d)$; $w(B(d, \varepsilon'')) = \eta(d) < \alpha$; then there is $z' \in Z$ with $z' \in B(d, \varepsilon'')$; we have $z' \in \bigcup\limits_{\beta \leqslant \eta(d)} z_\beta$ and $z' \in B(z, \varepsilon')$; hence $z \in cl \bigcup\limits_{\beta < \alpha} Z$.

Now, for a generalization of an exercise of [E 4c], there is a partition of Y_α with $Y_\alpha^i (i \in I_\alpha)$ clopen sets of Y_α that have a dense subset of cardinality not bigger than α.

Let us prove that for each $\alpha \leq |D|$, there exist two sets D_α^1, $D_\alpha^2 \subset D_\alpha$ such that $D_\alpha^1 \cap D_\alpha^2 = \varnothing$ and $Z_\alpha \subseteq (cl \bigcup\limits_{\beta < \alpha} Z_\beta) \cup cl\, D_\alpha^i$ for $i = 1, 2$; in fact $Z_\alpha = \bigcup\limits_i (Y_\alpha^i \cap Z_\alpha)$; $Y_\alpha^i \cap Z_\alpha$ contains a dense subset of cardinality a^i with $a^i \leq a$. Consider a^i as an ordinal (i.e. the minimum ordinal of cardinality a^i). We can write: $Y^i_\alpha \cap Z_\alpha = cl_{Z_\alpha} \{ y^i_1, y^i_2, \dots y^i_\nu \dots \}_{\nu < \alpha^i \leqslant \alpha}$.

Now for each $y^i_\nu \notin cl \bigcup_{\beta < \alpha} Z_\beta$ take a sequence of distinct points, $(d^{in}_\nu)_n \to y^i_\nu$ with $d^{in}_\nu \in D_\alpha \cap Y^i_\alpha$ such that $d^{in}_\nu \notin \{ d^{im}_{\nu'} : \nu' < \nu, m \in N \}$. This is possible because y^i_ν has a neighborhood disjoint from $D \setminus D_\alpha$, and besides, if $a > \aleph_0$, every neighborhood of it has weight $\geqq a$ and card $\{ d^{im}_{\nu'} : \nu' < \nu, m \in N \} < a$; if $a = \aleph_0$ then there exists a neighborhood of y^i disjoint from $\{ d^{im}_{\nu'} : \nu' < \nu, m \in N \}$, as ν is in this case finite.

Put then $D^1_\alpha = \{ d^{i(2n)}_\nu : n \in N, \nu < a^i, i \in I_\alpha \}$ and

$$D^2_\alpha = \{ d^{i(2n+1)}_\nu : n \in N, \nu < a^i, i \in I_\alpha \}.$$

If now $D_1 = \bigcup_{\alpha \leqslant |D|} D^1_\alpha$, $D_2 = \bigcup_{\alpha \leqslant |D|} D^2_\alpha$, we have $D_1 \cap D_2 = \varnothing$ and $Z = cl\, D_1 \cap cl\, D_2$.

4. – *Some problems.*

PROBLEM 1. Is there a relation for a space X between being totally ramified and having particular ramified subsets?

PROBLEM 2. Is it equivalent to ask that every cozero set of X be ramified and that X be totally ramified?

We have only a partial answer for these problems, namely.

PROPOSITION: *Let X be ramified. Every cozero set of X is ramified iff*: (i) *for each prime z-ideal Q, between Q and $R(Q)$ there are no(prime) z-ideal*; (ii) *if Q has an immediate successor in the z-ideal then it is ramified* $(Q = R(Q))$.

PROOF. As a zero-set we consider for (i) a set A belonging to $Z[Q]$ but not to the z-filter of a prime between Q and $R(Q)$; for (ii), a set A belonging to the z-filter of the successor of Q but not to $Z[Q]$: if ι is the immersion of $X \setminus A$ in X we consider the lattice-isomorfism $\iota^\#$ of [GJ 4, 12].

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