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# A Watsonian theorem for multiple series **

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## 1. - Introduction

Put

$$(1)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \forall\, n\in\{1,2,3,\ldots\}, \end{cases}$$

and let $F^{p:q;k}_{l:m;n}$ denote Kampé de Fériet's double hypergeometric function [1, p. 150] in the (modified) notation of Burchnall and Chaundy [2, p. 112], defined by

$$(2)\qquad F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ \\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\; x, y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!},$$

where, for convenience, $(a_p)$ abbreviates the sequence of $p$ parameters $a_1, \ldots, a_p$, with similar interpretations for $(b_q)$, $(c_k)$, *etc.*

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Recently, B. L. Sharma [4] gave a double-series analogue of the familiar Watson's summation theorem (*cf.*, *e. g.*, [6, p. 54, Eq. (2. 3. 3. 13)]; see also [8])

$$
(3) \qquad {}_3F_2\left[a, b, c; \frac{1}{2}(1+a+b), 2c; 1\right]
$$

$$
=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+c\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}b+c\right)},
$$

which holds true when $\operatorname{Re}(2c-a-b)>-1$; indeed, making use of the notation (2), we have Sharma's result (*cf.* [4, p. 95, Eq. (3)])

$$
(4) \qquad F^{2:1;1}_{2:0;0}\left[\begin{array}{r} \alpha, \beta:\ \gamma;\ \delta; \\ 2\alpha, (1+\beta+\gamma+\delta)/2: -; -; \end{array} 1, 1\right]
$$

$$
=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\gamma+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)},
$$

provided that $\operatorname{Re}(2\alpha-\beta-\gamma-\delta)>-1$.

In his long and involved proof of the summation formula (4), Sharma [*loc. cit.*] applies a number of results including, for example, the well-known Gaussian summation theorem for ${}_2F_1[a, b; c; 1]$, a formula of his own [5, p. 105, Eq. (5)], and indeed, the Watsonian theorem (3). The object of the present note is first to observe that the summation formula (4) is essentially equivalent to, and *not* a generalization of, Watson's theorem (3); we then show how readily one can derive a similar multiple-series analogue of (3) and (4).

## 2. – Equivalence of (3) and (4)

We begin by recalling the known result (*cf.* [7, p. 297, Eq. (16)])

$$
(5) \qquad \sum_{m,n=0}^{\infty} \Delta_{m+n}(\gamma)_m(\delta)_n \frac{x^{m+n}}{m!\,n!} = \sum_{n=0}^{\infty} \Delta_n (\gamma+\delta)_n \frac{x^n}{n!},
$$

where $\gamma$, $\delta$ are arbitrary parameters, real or complex, and $\{\Delta_n\}$ is a sequence of arbitrary complex numbers, it being assumed that the series involved are absolutely convergent.

Of our concern here is merely a special case of (5), involving hypergeometric functions, which was indeed given by Appell and Kampé de Fériet [1, p. 155] as long ago as 1926. Thus, by further specializing this 1926 result or by setting $x = 1$ and

$$(6) \qquad \Delta_n = \frac{(\alpha)_n\,(\beta)_n}{(2\alpha)_n\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}\delta\right)_n}, \quad \forall n \in \{0, 1, 2, \ldots\},$$

in (5), we readily have

$$(7) \qquad F^{2:1;1}_{2:0;0}\left[\begin{matrix} \alpha, \beta: \; \gamma; \; \delta; \\ 2\alpha, (1+\beta+\gamma+\delta)/2 : - ; - ; \end{matrix} \; 1, 1\right]$$

$$= {}_3F_2\,[\alpha, \beta, \gamma+\delta; 2\alpha, (1+\beta+\gamma+\delta)/2; 1],$$

which exhibits the fact that the first member of (4) is just the hypergeometric ${}_3F_2$ series occurring on the left-hand side of Watson's theorem (3) with, of course, $a = \beta$, $b = \gamma + \delta$, $c = \alpha$, and the equivalence of (3) and (4) evidently follows.

We remark in passing that, in view of the well-known formula [1, p. 23, Eq. (25)]

$$(8) \qquad F_1[a, b, b'; c; x, x] = {}_2F_1\,[a, b+b'; c; x], \; |x| < 1,$$

also contained in the general result (5), Sharma's formula [4, p. 96, Eq. (6)] for the Appell function $F_1$ is, in fact, equivalent to (and *not* a generalization of) Gauss's second summation theorem [6, p. 32, Eq. (1.7.1.9)], *viz*

$$(9) \quad {}_2F_1\left[a, b; \frac{1}{2}(1+a+b); \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)}.$$

## 3. – A multiple-series analogue

A multiple-series analogue of the summation formulas (3) and (4) would follow readily from a generalization of (5) considered recently by R. Panda [3], who also gave the hypergeometric form [*op. cit.*, p. 168, Eq. (12)]

$$(10)\qquad F^{p:1;\ldots;1}_{q:0;\ldots;0}\begin{pmatrix}\alpha_1,\ldots,\alpha_p:\gamma_1;\ldots;\gamma_n; & \\ & x,\ldots,x\\ \beta_1,\ldots,\beta_q:-;\ldots;-; & \end{pmatrix} = {}_{p+1}F_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\gamma_1+\ldots+\gamma_n; & \\ & x\\ \beta_1,\ldots,\beta_q; & \end{matrix}\right],$$

where, for convergence, $p<q$ and $|x|<\infty$, or $p=q$ and $|x|<1$, or $p=q$, $x=1$, and

$$(11)\qquad \operatorname{Re}\left(\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j-\sum_{j=1}^{n}\gamma_j\right)>0.$$

If, in the reduction formula (10), we set $p=q=2$, $\alpha_1=\alpha$, $\alpha_2=\beta$, $\beta_1=2\alpha$, $\beta_2=(1+\beta+\gamma_1+\ldots+\gamma_n)/2$, and $x=1$, and apply Watson's theorem (3), we shall at once get the following multiple-series analogue:

$$(12)\quad F^{2:1;\ldots;1}_{2:0;\ldots;0}\begin{pmatrix}\alpha,\beta:\gamma_1;\ldots;\gamma_n; & \\ & 1,\ldots,1\\ 2\alpha,(1+\beta+\gamma_1+\ldots+\gamma_n)/2:-;\ldots;-; & \end{pmatrix}=$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\gamma_1+\ldots+\frac{1}{2}\gamma_n\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\beta-\frac{1}{2}\gamma_1-\ldots-\frac{1}{2}\gamma_n\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\gamma_1+\ldots+\frac{1}{2}\gamma_n\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\gamma_1-\ldots-\frac{1}{2}\gamma_n\right)},$$

provided that $\operatorname{Re}(2\alpha-\beta-\gamma_1-\ldots-\gamma_n)>-1$.

Evidently, the reduction formula (10) can be applied to derive similar multiple-series analogues of several other known hypergeometric summation theorems including, for example, Gauss's second summation theorem (9) above.

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