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Charles Powder

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An Existence Theorem for Solutions of n-th Order Nonlinear Differential Equations in the Complex Domain.

CHARLES POWDER (*)

SUMMARY - In this paper, we consider n-th order nonlinear ordinary differential polynomials whose coefficients have asymptotic expansions as $x \to \infty$ in terms of logarithmic monomials (i.e., functions of the form $M(x) = kx^{a_0}(\log x)^{a_1} \ldots (\log_q x)^{a_q}$ where k is complex and non-zero, the a_i are real and $\log_m x$ is the m-th iterated logarithm. In earlier work Strodt, Wright and Bank proved the existence, in sectors, of solutions asymptotic lo logarithmic monomials. For the first and second order cases, Bank proved the existence of solutions of the form $\exp\int_{x_0}^{x} u$ where u is asymptotic to a logarithmic monomial. Such solutions have large rate of growth. In this result, we treat the n-th order case and prove an existence theorem for such solutions. It should be noted that the conditions of the theorem can be easily verified by simple computations and the use of certain algorithms.

1. Introduction.

We treat n-th differential polynomials where the coefficients are complex functions, defined in and analytic in a sector

$$a < \arg\left(x - \beta \exp\left(i(a+b)/2\right)\right) < b$$

(for fixed a and b in $(-\pi, \pi)$ and some $\beta \geqslant 0$), and where as $x \to \infty$ in this region, each non-zero coefficient has an asymptotic expansion

(*) Indirizzo dell'A.: Department of Mathematics, University of Dayton, Dayton, OH 45469, U.S.A.

in terms of logarithmic monomials (i.e., functions of the form

$$M(x) = Kx^{a_0}(\log x)^{a_1}(\log\log x)^{a_2} \dots (\log_q x)^{a_q}$$

for complex $K \neq 0$ and real a_j). This class contains, in particular, those differential polynomials having rational functions for coefficients. In [2, § 43] and [8, § 122], existence theorems were proved for solutions of these equations which are asymptotica.ly equivalent to logarithmic monomials as $x \to \infty$ over a filter base consisting essentially of those sectors here.

For the first and second order cases, Bank [4 and 5] proved the existence of solutions of the form $\exp \int_{x_0}^{x} u$ where u is $\sim$ a logarithmic monomial. Such solutions were shown to have a larger rate of growth as $x \to \infty$ than all logarithmic monomials. Such concepts and notation are reviewed in Section 2. In this paper, we prove an existence theorem for the n-th order case for solutions of the form $\exp \int_{x_0}^{x} u$.

For a given n-th order differential polynomial Ω, we consider a corresponding $(n-1)$-st order differential polynomial G (see § 3.1). Then critical monomials of G (those logarithmic monomials for which there is a function $h \sim N$ such that $G(h)$ is not $\sim G(N)$) are of particular interest. In [2, §§ 21 and 26], an algorithm was introduced for finding the set of all critical monomials of a given differential polynomial. We look for critical monomials N of G such that $N \succ x^{-1}$. (Thus $N = cx^{-1+a_0}(\log x)^{a_1} \dots (\log_t x)^{a_t}$, where $(a_0, a_1, \dots, a_t)$ is lexicographically greater than $(0, 0, \dots, 0)$.) If (a, b) is an interval on which $\cos(a_0\varphi + \arg c)$ is positive, then Theorem 4 asserts the existence, in sectorial subregions of the original sector, of at least a one-parameter family of solutions of $\Omega(y) = 0$, each having the form $\exp\int W$ for some $W \sim N$, provided certain subsidiary conditions are fulfulled. (Such solutions are automatically of larger rate of growth than all logarithmic monomials. The conditions are of two main types. One type requires that N not be a critical monomial of certain other $(n-1)$-st order differential polynomials. (This type of condition is fulfilled in general.) A second type of condition is also generally fulfilled since it requires that certain other logarithmic monomials which arise do not have certain special forms. These conditions are similar to those imposed in existence theorems in [2, § 43] and [8, § 122] for solutions of n-th order equations which are $\sim$ logarithmic monomials.

It should be noted that it is easy to test the conditions in any given example by using the algorithm in [2, §§ 21 and 26], by inspection or by simple computation. These conditions guarantee (using [2] at the outset) that $G(z) = 0$ has a solution $z \sim N$. Furthermore, the conditions enable us to use that solution to transform $\Omega(y) = 0$ into quasi-linear form. The conditions play an essential role in effecting the transformation, since they permit us, at a crucial stage, to assert the existence of a particular type of solution of a certain n-th order non-homogeneous linear differential equation (see Lemmas 5.8 and 5.9). Conditions (vi) and (vii) in Theorem 4 are not the best possible ones. They are however the most natural ones to simplify computations in Lemma 5.2 and in the main proof.

One can obtain information on the existence of solutions of $\Omega(y) = 0$ which are of smaller rate of growth than all logarithmic monomials, by making the change of variable $y = w^{-1}$ and then applying Theorem 4.

In section 6, we apply our result to an example.

2. Notation and preliminaries.

2.1. Notation. Let $-\pi \leqslant a < b \leqslant \pi$. For each non-negative real-valued function g on $(0, (b-a)/2)$, let $E(g)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors, $a + \delta < \arg(x - h(\delta)) < b - \delta$ where $h(\delta) = g(\delta) \exp(i(a+b)/2)$. The set of all $E(g)$ (for all choices of g) is denoted $F(a, b)$ and is a filter base which converges to ∞ (see [7, § 94]). Let S_1 be the subset of $F(a, b)$ consisting of those members E of $F(a, b)$ such that $|z| > 1$ for all $z \in E$. By $\log F(a, b)$ is meant the set $\{\log R: R \in S_1\}$. Each $E(g)$ is simply-connected by [7, § 93]. If W is analytic in $E(g)$ then the symbol $\int W$ stands for any primitive of W in $E(g)$. If x and x_0 are in $E(g)$, then the contour of integration for $\int_{x_0}^{x} W$ will be any rectifiable path in $E(g)$ from x_0 to x. A statement is said to hold except in finitely many directions (briefly e.f.d.) in $F(a, b)$ if there are finitely many points $r_1 < \dots < r_q$ in (a, b) such that the statement holds in each of $F(a, r_1), F(r_1, r_2), \dots, F(r_q, b)$ separately (see [1, § 6]).

2.2. Notation. If f is analytic in some $E(g)$, then $f \to 0$ in $F(a, b)$ means that for any $\varepsilon > 0$, there is a g_1 such that $|f(x)| < \varepsilon$ for all x

in $E(g_1)$. The relation of $f < 1$ in $F(a, b)$ means that in addition to $f \to 0$, all functions $\theta_j^k f \to 0$ where $\theta_j f = (x \log x \dots \log_{j-1} x) f'$ and $\theta_j^k f = \theta_j(\theta_j^{k-1} f)$. Then $f_1 < f_2$, $f_1 \sim f_2$, $f_1 \approx f_2$ and $f_1 \underset{\approx}{<} f_2$ mean respectively, $f_1/f_2 < 1$, $f_1 - f_2 < f_2$, $f_1 \sim cf_2$ for some constant $c \neq 0$, and finally either $f_1 < f_2$ or $f_1 \approx f_2$ (see [7, § 13]). If $f < 1$, then by [7, § 28], $(x \log x \dots \log_q x) f' < 1$ for all $q \geqslant 0$. If $M = Kx^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_q x)^{\alpha_q}$, then $M'/M \underset{\approx}{<} x^{-1}$ and if $V = M(1 + E)$ with $E < 1$, then by simple calculation, $V' \underset{\approx}{<} x^{-1}V$ and $V' < x^{-1}V$ if and only if $\alpha_0 = 0$. By $\delta_j(V)$ is meant α_j. If M is not constant, then it follows from [7, § 28] that $N < N$ implies $N' < M'$. If for every real α, $f < x^{\alpha}$, we say f is trivial in $F(a, b)$ and set $\delta_0(f) = -\infty$.

2.3. Definition. A logarithmic domain of rank p (briefly LD_p) over $F(a, b)$ is a complex vector space L of functions (each analytic in some $E(g)$), which contains the constants, and such that any finite linear combination of elements in L with coefficients which for some $q \geqslant p$ are logarithmic monomials of rank q (i.e., those of the form $kx^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_q x)^{\alpha_q}$), is either $\sim$ to a function of the latter form or is trivial (see [7, § 49]).

2.4. Definition. A logarithmic differential field (briefly an LDF) over $F(a, b)$, is a differential field D of functions (each analytic in come $E(g)$), for which there is an integer $q \geqslant 0$ such that D contains all logarithmic monomials of rank $\leqslant q$, and such that every non-zero element of D is $\sim$ to a logarithmic monomial of rank $\leqslant q$. (For a fixed q, the set of rational combinations of logarithmic monomials of rank $\leqslant q$, is the simplest example of an LDF) (see [9, p. 247]).

2.5. Definition. If in $F(a, b)$, W is $\sim$ to a monomial of the form,

$$Kx^{-1}(\log x)^{-1} \dots (\log_{k-1} x)^{-1}(\log_k x)^{-1+t}(\log_{k+1} x)^{c_1} \dots (\log_{k+s} x)^{c_s}$$

where $K \geqslant 0$ and $t > 0$, then we say W is in the divergence class in $F(a, b)$. The indicial function of W is the function on (a, b) defined by

$$IF(W)(\varphi) = \cos(\delta_{0k} t\varphi + \arg K)$$

where δ_{0k} is the Kronecker delta. Clearly $IF(W)$ has at most finitely many zeros unless $k > 0$ and K is purely imaginary (see [7, § 40, 100]).

If $N < x^{-1}$ is a non-constant logarithmic monomial, then by a simple computation, $IF(N'/N)$ is not identically zero.

2.6. NOTATION. Let Ω be an n-th order differential polynomial in y with coefficients in an LD_p over $F(a, b)$. We say that Ω is *NTPD* (non-trivial of positive degree) if at least one term of positive degree in the indeterminates has non-trivial coefficient. Let Ω be *NTPD*. Then Ω is unstable at M if for some $f \sim M$, $\Omega(f)$ is not $\sim \Omega(M)$. If M is a logarithmic monomial we say M is a critical monomial of Ω. If kM is a critical monomial of Ω for every non-zero constant k, then M is a parametric monomial of Ω (see [2, § 2, 3, 4, 14]).

2.7. NOTATION. Let $\alpha^* = (\alpha_0, \ldots, \alpha_n)$ be an $(n+1)$-tuple of non-negative integers. By f_{α^*} is meant $f_{\alpha_0\alpha_1\ldots\alpha_n}$. By y^{α^*} is meant $y^{\alpha_0}(y')^{\alpha_1} \ldots \ldots (y^{(n)})^{\alpha_n}$. If $\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_n)$, then by Φ^{α_n} is meant $(\varphi_0)^{\alpha_0}(\varphi_1)^{\alpha_1} \ldots \ldots (\varphi_n)^{\alpha_n}$. Define

$$d(\alpha^*) = \alpha_0 + \alpha_1 + \ldots + \alpha_n \quad \text{and} \quad w(\alpha^*) = \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n .$$

For simplicity we will write

$$f_{\alpha^*} = f(\alpha_0, \alpha_1, \ldots, \alpha_n) \qquad \text{and} \qquad f_s = f(0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

where 1 is in the s-th place. Let $\Omega(y) = \sum f_{\alpha^*} y^{\alpha^*}$. Let $\Omega[*, \alpha] =$ maximum over α^* of $\alpha d(\alpha^*) + \delta_0(f_{\alpha^*}) - w(\alpha^*)$. Let $\Psi(u, v)$ be the transform of $\Omega(y)$ under the change of variables $x = e^u$ and $y = ve^{\alpha u}$. Then the differential polynomial $\exp(-\Omega[*, \alpha]u)\Psi(u, v)$ is denoted $[\alpha; \Omega](v)$ or simply $[\alpha; \Omega]$. By induction define $[\alpha_i; [\alpha_{i-1}, \ldots, \alpha_0; \Omega]]$. By $[M, i, \Omega]$ is meant $[\delta_{i-1}(M), \ldots, \delta_0(M); \Omega]$ (see [2, § 7]).

2.8. NOTATION. Let $u \in LDF$ and let $y = c \exp \int u$. Define $\psi_n(u)$ by $y^{(n)} = \psi_n(u)y$. Clearly $\psi_0(u) = 1$ and by induction

$$\psi_{n+1}(u) = u\psi_n(u) + (\psi_n(u))' .$$

By $\Psi(u)$ is meant $(\psi_0(u), \ldots, \psi_n(u))$.

2.9. DEFINITION. Let r be a non-negative integer. Let $L_r = x \log x \ldots \log_{r-1} x$ and $L_0 = 1$. Let $V_1, V_2, \ldots, V_n$ be logarithmic monomials of logarithmic rank $\leqslant r-1$, such that

$$L_r^{-1} \lessapprox V_1 \lessapprox V_2 \lessapprox \cdots \lessapprox V_n .$$

Let $W_j \sim V_j$ and $A_j = -(L_r W_j)^{-1}$ for $j = 1, 2, \ldots, n$. Let $P(x, -)$ be an algebraic differential operator and let $P(x, y)$, when written as a polynomial in $\Theta_r y = (y, \theta_r y, \ldots, \theta_r^n y)$ have the form $P(x, y) = \sum_{\alpha^*} P_{\alpha^*} \cdot (\Theta_r y)^{\alpha^*}$. Then $P(x, -)$ is normal with respect to $(W_1, W_2, \ldots, W_n, r)$ if

(a) $P(x, 0) \prec 1$ and $HLP(x, -) \sim \dot{W}(n, \ldots, 1)$

and

(b) $\alpha_j P_{\alpha^*} \lessapprox A_1 \ldots A_j = H_j$ for $j = 0, 1, \ldots, n$ and for all α^*.

Here, $HLP(x, -)$ is the homogeneous linear part of P and by $HLP \cdot (x, -) \sim \dot{W}(n, \ldots, 1)$ is meant that $(W_1, W_2, \ldots, W_n)$ is a factorization sequence for $HLP(x, -)$ (see [8, §§ 29, 87, 101 and 102]).

2.10. Definition. A homogeneous linear differential operator

$$\Lambda = A_0 + A_1 D + \ldots + A_n D^n$$

with coefficients in an LD_p is unimajoral if $\Lambda(1) \sim 1$ and $\Lambda(E) \prec 1$ for any $E \prec 1$.

2.11. Definition. We say that the sequence $(V_1, V_2, \ldots, V_n)$ is unblocked in I if no $IF(V_i)$ is identically zero on I.

3. Uniform hypotheses.

3.1. Hypotheses. Let $\Omega(y) = \sum f_{\alpha^*} y^{\alpha^*}$ be an n-th order differential polynomial with coefficients in an LDF over $F(a, b)$. Let $A = \{d(\alpha^*) \colon f_{\alpha^*}$ is not identically zero$\}$. Let $p = \max A$. Let $G(z) = \sum_{d(\alpha^*) = p} f_{\alpha^*}(\Psi(z))^{\alpha^*}$ (see § 2.7).

3.2. Lemma. Assume § 3.1. Assume that $\partial G / \partial z^{(n-1)}$ is not identically zero. Let N be a simple (see [2, § 28]) non-parametric critical monomial of G such that N is not a critical monomial of $\partial G / \partial z^{(n-1)}$. Then there exists a logarithmic monomial $Q(x)$ such that $QG^{(1)} \cdot$

$\cdot(N+Nz)$ ($G^{(1)}$ means the homogeneous linear part of G) is unimajoral with at least one factorization sequence $(V_1, V_2, \dots, V_{n-1})$. Also $QG(N) < 1$ and

$$Q \sim \left(\frac{\partial G(N+Nz)}{\partial z}(0)\right)^{-1}.$$

PROOF. Since N is not a critical monomial of $\partial G/\partial z^{(n-1)}$ and $\partial G/\partial z^{(n-1)}$ is not trivial, then $\partial G/\partial z^{(n-1)}(N)$ is not trivial. Thus, since N is a simple non-parametric critical monomial of G, the result follows by [2, § 40(*a*)]. This completes the proof.

3.3. LEMMA. Assume § 3.1 and the hypotheses and notation of Lemma 3.2. Let $\Lambda(z) = QG(N+Nz)$. Assume that $\Lambda(z)$ is normal with respect to $(V_1, V_2, \dots, V_{n-1}, r)$ for some r and that the sequence $(V_1, V_2, \dots, V_{n-1})$ is unblocked [8, § 98]. Then e.f.d. in $F(a, b)$, there exists a function $u_0 \sim N$ such that $G(u_0) = 0$.

PROOF. By [8, § 103], since Λ is normal with respect to $(V_1, V_2, \dots$ $\dots, V_{n-1}, r)$, it is also weak with respect to $(V_1, V_2, \dots, V_{n-1})$. Therefore by [2, § 41], $(V_1, V_2, \dots, V_{n-1})$ is an asymptotically steady type for Λ (see [8, §§ 88 and 117]). Hence by [2, § 43], e.f.d. in $F(a, b)$, there exists a function $u_0 \sim N$ such that $G(u_0) = 0$. This completes the proof.

4. Theorem.

Assume § 3.1. If $A - \{p\}$ is non-empty let $q = \max(A - \{p\})$ and $H(z) = \sum_{d(\alpha^*)=q} f^{\alpha^*}(\Psi(z))^{\alpha^*}$. Let N be a critical monomial of G such that

(i) $N > x^{-1}$,

(ii) $IF(N) > 0$ on (a, b),

(iii) if $A - \{p\}$ is non-empty then N is not a critical monomial of H,

and

(iv) N satisfies the hypotheses of Lemma 3.2.

Let $V_1, V_2, \ldots, V_{n-1}$ be the functions defined in Lemma 3.2. Assume that

(v) the hypotheses of Lemma 3.3 are satisfied,

(vi) $V_1 \prec V_2 \prec \cdots \prec V_{n-1}$,

(vii) $V_2 \succ x^{-1}$,

(viii) V_1 is not $\sim -N'/N$ and if $V_1 \approx N'/N$ say $N'/NV_1 \sim \delta$, then $IF((1+\delta)V_1) \not\equiv 0$,

and

(ix) if $V_m \approx N$ for some m, then with $V_m \sim \sigma N$, $\sigma \notin \{(q-p)(m+1)/(m+1-s)$ for $s=0,1,\ldots,m\}$ and $IF((\sigma+p-q)N) \not\equiv 0$.

Then under these hypotheses, e.f.d. in $F(a, b)$, there exists a function $u_0 \sim N$ such that, for x_0 in the domain of u_0, the equation $\Omega(y) = 0$ possesses solutions $y_c^* \sim c \exp \int_{x_0}^{x} u$ for every $c \neq 0$. The solutions y_c^* have the properties:

(A) For every real a, $y_c^* \succ x^a$.

(B) For each $c \neq 0$ there is a function $W_c \sim N$ such that y_c^* is of the form $\exp \int W_c$.

PROOF. By Lemma 3.3, e.f.d. in $F(a, b)$, there exists a function $u_0 \sim N$ such that $G(u_0) = 0$. Let $u_0 = N + Nw_0$. Let I be any open subinterval of (a, b) such that u_0 exists in $F(I)$. By Lemma 5.1, for any non-zero c with $y_c = c \exp \int_{x_0}^{x} u$ and for any real a,

$$y_c \succ x^a \qquad \text{in } F(I)\,. \tag{1}$$

If $A = \{p\}$, we are done since $\Omega(y_c) = y_c^p G(u_0) = 0$. Then if K is a value of $\log c$, we have $y_c = \exp(K + \int u_0)$ which is of the correct form.

Assume that $A - \{p\}$ is not empty. For Q given in Lemma 3.2 write $\Omega(y) = 0$ in the form,

$$Q \sum_{d(\alpha^*)=p} f_{\alpha^*} y^{\alpha^*} = -Q \sum_{d(\alpha^*)<p} f_{\alpha^*} y^{\alpha^*}\,. \tag{2}$$

Let c be non-zero. Then $y_c^{(j)} = y_c\psi_j(u_0)$. Let $y = y_c(1+v)$ and multiply by y_c^{-p}. Then (2) becomes

$$(13) \qquad \sum g_{\gamma^*}v^{\gamma^*} = \sum h_{\gamma^*}v^{\gamma^*} .$$

Specifically,

$$y^{(n)} = \sum_{k=0}^{n} \binom{n}{k} y_c^{(k)} (1+v)^{n-k} ,$$

so by § 2.8,

$$(4) \qquad Qy_c^{-p}\Omega(y) = Q\sum_{\alpha^*} f_{\alpha^*} \prod_{j=0}^{n} \left[\sum_{m=0}^{j} \binom{j}{m} (\psi_m(u_0))(1+v)^{(j-m)} \right]^{\alpha_j} y_c^{d(\alpha^*)-p} .$$

We place those terms in (4) for which $d(\alpha^*) = p$ into the left hand side of (3) and the rest into the right hand side of (3). Then since since each f_{α^*} is $<$ a power of x and u_0 and $\psi_m(u_0)$ are $<$ a power of x for any m, we have $h_{\alpha^*} = y_c^{d(\alpha^*)-p}E_{\alpha^*}$ where $E_{\alpha^*} < x^\beta$ for some β. It follows from (1) that

(5) $\qquad h_{\alpha^*}$ is trivial in $F(I)$ for each α^* .

Similarly,

(6) $\qquad g_{\alpha^*}$ is $<$ some power of x in $F(I)$.

From (3) and (4),

$$(7) \qquad \sum g_{\gamma^*}v^{\gamma^*} = Q\sum_{d(\alpha^*)=p} f_{\alpha^*} \prod_{j=0}^{n} \left[\sum_{m=0}^{j} \binom{j}{m} \psi_m(u_0)(1+v)^{j-m} \right]^{\alpha_j} .$$

We compute $g(0, 0, \ldots, 0)$. From (7)

$$(8) \qquad g(0,0,\ldots,0) = Q\sum_{d(\alpha^*)=p} f_{\alpha^*}(\Psi(u_0))^{\alpha^*} = QG(u_0) = 0 .$$

Recall the notation in § 2.7. From (7)

$$(9) \qquad g_s = Q\sum_{d(\alpha^*)=p} f_{\alpha^*} \left[\prod_{j=0}^{s-1} (\psi_j(u_0))^{\alpha_j} \right] \cdot$$

$$\cdot \left[\sum_{j=s}^{n} \prod_{\substack{m=s \\ m\neq j}}^{n} (\psi_m(u_0))^{\alpha_m} (\psi_j(u_0))^{\alpha_j - 1} \alpha_j \binom{j}{s} \psi_{j-s}(u_0) \right] .$$

Since $G(u) = \sum_{d(\alpha^*)=p} f_{\alpha^*}(\Psi(u_0))^{\alpha^*}$, then

$$\frac{\partial G}{\partial u^{(s-1)}}\bigg|_{u=u_0} = \sum_{d(\alpha^*)=p} f_{\alpha^*} \prod_{j=0}^{s-1} (\psi_j(u_0))^{\alpha_j} \cdot$$

$$\cdot \left[\sum_{j=s}^{n} \prod_{\substack{m=s \\ m\neq j}}^{n} (\psi_m(u_0))^{\alpha_m} \alpha_j (\psi_j(u_0))^{\alpha_j - 1} \times \left(\frac{\partial \psi_j(n)}{\partial u^{(s-1)}}\bigg|_{u=u_0} \right) \right].$$

Thus by (9) and Lemma 5.5,

$$g_s = Q \frac{\partial G}{\partial u^{(s-1)}}\bigg|_{u=u_0} \qquad \text{for } s \geqslant 1 \tag{10}$$

and

$$g_0 = pQ \sum_{d(\alpha^*)=p} f^{\alpha^*}(\Psi(u_0))^{\alpha^*} = pQG(u_0) = 0\,. \tag{11}$$

We have assumed that $QG(N + Nw)$ is normal with respect to $(V_1, V_2, \ldots, V_{n-1}, r)$. As in § 2.9, let $QG(N + Nw) = P(x, w)$ as an operator in $\Theta_r y$. As in [8, p. 68], we write

$$P(x, w) = P_0(x) + \sum_{j=0}^{n-1} C_j(x) \Lambda_j w + \sum \{P(k^*, r, x, w) \colon d(k^*) \geqslant 2\}\,,$$

where $P(k^*, r, x, w) = P_{k^*}(x)(\Theta_r w)^{k^*}$, $\Lambda_0 = I$ the identity operator, $\Lambda_j = \check{V}_j \ldots \check{V}_1$, $C_j(x) \prec 1$ for $j = 0, 1, \ldots, n-2$ and $C_{n-1}(x) \sim 1$. Note that $\check{V}$ means the operator $1 - V^{-1}D$. As in § 2.9, $P(x, 0) \prec 1$ so $P_0(x) \prec 1$.

We also write $\Lambda_j = \sum_{k=0}^{j} B_{jk} D^k$. Recall that $u_0 = N + Nw$, so that with $w = (z - N)N^{-1}$ we have,

$$Q \frac{\partial G}{\partial z^{(r)}}\bigg|_{z=u_0} = \frac{\partial P(x, (z-N)N^{-1})}{\partial z^{(r)}}\bigg|_{z=u_0} = \tag{12}$$

$$= \sum_{k=r}^{n-1} \frac{\partial P}{\partial (\Lambda_k w)}\bigg|_{w=w_0} \sum_{m=r}^{k} \frac{\partial (\Lambda_k w)}{\partial w^{(m)}}\bigg|_{w=w_0} \frac{\partial w^{(m)}}{\partial z^{(r)}}\,.$$

Define $H_k = \partial P / (\partial \Lambda_k w)\,|_{w=w_0}$. Then by Lemma 5.2 and 5.3 and by

(10), (12) gives

$$(13)\qquad g_{s+1}=\sum_{k=s}^{n-1} H_k \sum_{m=s}^{k} B_{km}\binom{m}{s}(N^{-1})^{(m-s)} \qquad \text{for } s=0,1,\dots,n-1\,.$$

From [8, p. 68],

$$(14)\qquad H_k < 1 \qquad \text{for } k<n-1 \text{ and } H_{n-1}\sim 1\,.$$

We assert that

$$(15)\qquad g_{s+1}\sim(-1)^s(V_1\dots V_sN)^{-1} \qquad \text{for } s=1,2,\dots,n-1\,.$$

Recall that $L_r^{-1}\approx V_1 < V_2 < \dots < V_{n-1}$ and $V_2 > x^{-1}$. For $m > s \geqslant 1$, by Lemma 5.3, $(N^{-1})^{(m-s)} \lessapprox x^{-m+s}N^{-1}$. Also by Lemma 5.2, $B_{km}\sim \sim(-1)^m(V_1\dots V_m)^{-1}$. Therefore by these estimates, (14), and the fact that $x^{-1}V_k < 1$ for $k\geqslant 2$, we have for $1\leqslant s < k\leqslant m$,

$$\binom{m}{s}H_kB_{km}(N^{-1})^{(m-s)} < (V_1\dots V_s)^{-1}N^{-1}\,.$$

For the case $m=s$, the terms in (13) are $L=\sum_{k=s}^{n-1}H_kB_{ks}N^{-1}$ which is easily shown to be $\sim(-1)^s(V_1\dots V_s)^{-1}N^{-1}$. Write $g_{s+1}=L+M$ where each term of M has $1\leqslant s<m$. Assertion (15) follows.

We assert that

$$(16)\qquad g_1\sim(1+T)N^{-1} \qquad \text{with } T=N'/NV_1\,.$$

From (13), $g_1=\sum_{k=0}^{n-1}H_k\left(\sum_{m=0}^{k}B_{km}(N^{-1})^{(m)}\right)$. Define

$$\varphi_1=\sum_{k=0}^{n-1}H_kB_{k0}N^{-1}\,,\qquad \varphi_2=\sum_{k=1}^{n-1}H_kB_{k1}N^{-1} \qquad\text{and}\qquad \varphi_3=g_1-\varphi_1-\varphi_2\,.$$

Recall that by Lemma 5.2, $B_{k0}=1$. Hence by (14), $\varphi_1\sim N^{-1}$. Again by Lemma 5.2, $B_{k1}\sim -V_1^{-1}$. Thus by (14), $\varphi_2\sim V_1^{-1}N^{-2}N'=TN^{-1}$. Let $H_kB_{km}(N^{-1})^{(m)}$ be an arbitrary term in φ_3. As in Lemma 5.3, $(N^{-1})^{(m)}\lessapprox x^{-m+1}(N^{-1})'$. By Lemma 5.2, $B_{km}\sim(-1)^mV_1\dots V_m^{-1}$. Hence by (14), we have for $2\leqslant m<k$, $H_kB_{km}(N^{-1})^m < V_1^{-1}N^{-2}N'=TN'$. Therefore $\varphi_3 < TN^{-1}$ and the assertion follows.

Let $0^* = (0, 0, \ldots, 0)$. Since $q = \max(A - \{p\})$, by (4) we have,

$$h_{0^*} = -Q \sum_{t=0}^{q} \Gamma_t(u_0) y^{t-p} \tag{17}$$

where $\Gamma_t(z) = \sum_{d(\alpha^*)=t} f_{\alpha^*}(\Psi(z))^{\alpha^*}$. By assumption, N is not a critical mono- monomial of $\Gamma_q(z) = H(z)$, so by [2, § 5], $\Gamma_q(N) \neq 0$ and since

$$u_0 = N + Nw_0 \sim N , \quad \Gamma_q(u_0) \sim \Gamma_q(N) .$$

Then $\Gamma_q(N)$ lies in some LDF (namely the field generated by the original field and the set of logarithmic monomials of rank $\leqslant$ the rank of N). Since $\Gamma_q(N) \neq 0$, there exists a monomial B such that $\Gamma_q(N) \sim B$. Thus

$$\Gamma_q(u_0) \sim B \quad \text{in } F(I) . \tag{18}$$

For each $t \leqslant q$, $\Gamma_t(u_0)$ is $<$ a power of x. Hence by (18), $\Gamma_q(u_0)^{-1} \cdot \Gamma_t(u_0) <$ a power of x for $t < q$. Since $y_c^{q-t} > x^a$ for all real a by (1), for $t < q$ we have $\Gamma_q(u_0)^{-1}\Gamma_t(u_0) < y_c^{q-t}$ and hence, $\Gamma_t(u_0)y_c^{t-q} < \Gamma_q(u_0)y_c^{q-p}$ for $t < q$. From this fact, (17) and (18), it follows that $h_{0^*} \sim -QBy_c^{q-p}$.

Let $s_{\alpha^*} = g_{\alpha^*} - h_{\alpha_*}$. Then (3) becomes

$$\sum s_{\alpha^*} v^{\alpha^*} = 0 . \tag{19}$$

By (8), $g_{0^*} = 0$ and so

$$s_{0^*} = -h_{0^*} \sim QBy_c^{q-p} . \tag{20}$$

From (5) and (11), s_0 is trivial. Also, by (5), (15) and (16),

$$s_1 \sim (1 + T)N^{-1} \tag{21}$$

and

$$s_{k+1} \sim (-1)^k (V_1 \ldots V_k)^{-1} N^{-1} \quad \text{for } k = 1, 2, \ldots, n-1 .$$

Consider the equation

$$\sum_{k=1}^{n} s_k v^{(k)} + s_{0^*} = 0 . \tag{22}$$

Let U be a logarithmic monomial so that $U \sim (1+T)$. Divide (22) by UN^{-1} to get $\sum_{k=1}^{n} s_k U^{-1} N v^{(k)} = -s_{0*} U^{-1} N$. By (21),

$$s_k U^{-1} N \sim (-1)^{k-1} (V_1 \dots V_{k-1} U)^{-1} \quad \text{for } k \geqslant 2 .$$

Write $s_k U^{-1} N = R_{k-1} U_{k-1}$, where $R_k \sim 1$ and $U_k = (-1)^k (UV_1 \dots V_k)^{-1}$ for $k \geqslant 2$. Set $R_0 = s_1 U^{-1} N \sim 1$. Let $v' = u$ and consider the operator Λ given by $\Lambda(u) = \sum_{k=1}^{n} u^{(k-1)} s_k U^{-1} N$. Then $\Lambda(u) = \sum_{k=1}^{n-1} R_k U_k u^{(k)} + R_0 u$. Since u_0 is in an LD, then by (7) each $g_{\alpha*}$ is a finite linear combination of elements in an LD_m for some m. Each $h_{\alpha*}$ is trivial by (5). The set of functions $M_1 F_1 + \dots + M_k F_k + t$, with the M_i's monomials, the F_k's in an LD_m and t trivial, is an LD_s for some $s \geqslant m$ by [7, § 53]. Thus, the $s_{\alpha*}$ lie in an LD. Clearly then the R_k's all lie in an LD.

By 2.2 and assumption (vii), $V_1 T = N'/N \precsim x^{-1} \prec V_2$. Thus by assumption (vi),

$$V_1 \precsim (1+T) V_1 \approx U V_1 \prec V_2 . \tag{23}$$

Let $R(z) = \sum_{k=1}^{n-1} R_k U_k z^k + R_0$. By Lemma 5.6, $R(z)$ possesses a sequence of critical monomials $M_1, M_2, \dots, M_{n-1}$ where $M_1 \sim UV_1$ and $M_k \sim V_k$ for $k = 2, \dots, n-1$.

By assumption (v), the sequence $V_1, V_2, \dots, V_{n-1}$ is unblocked on (a, b). Hence $IF(M_k) \neq 0$ for $k = 2, \dots, n-1$ on (a, b). If $T \prec 1$, then $M_1 \sim V_1$ and $\mathrm{IF}(M_1) \neq 0$ on (a, b). If $T \succ 1$, then $M_1 \sim TV_1 = N'/N$ and N is non-constant so by 2.5, $IF(N'/N)$ is not identically zero. Thus, $IF(M_1)$ is not identically zero on (a, b). If $T \approx 1$, then by assumption (viii) with $T \sim \delta$, $IF\big((1+\delta)V_1\big)$ is not identically zero. In any case, there is an open interval $J \subset I$ on which the sequence $(M_1, M_2, \dots, M_{n-1})$ is unblocked.

It follows from Lemma 5.7 and (23) that Λ possesses a factorization sequence $(M_1, M_2, \dots, M_{n-1})$. Also by assumption (vi) and the unblockedness, the sequence $(M_1, M_2, \dots, M_{n-1})$ is separated in the sense of Chamberlain [6, § 2]. Therefore by [6, § 3], there exist functions $s \sim 1$ and $W_k \sim W_k$ for $k = 1, 2, \dots, n-1$ such that $\Lambda = s\dot{W}_{n-1} \dots \dot{W}_1$.

Let $H = -QBNU^{-1}$. By (20) we may write $\Lambda(u) = -s_{0*} U^{-1} N$ as

$$\Lambda(u) = EHy_c^{q-p} \quad \text{where } E \sim 1 . \tag{24}$$

We will apply Lemma 5.8. Now y_c^{q-p} is of the form $\exp \int (q-u)u_0$. Let $V = (q-p)u_0$. Then $W_1, \dots, W_{n-1}$, V and H are $\sim$ to logarithmic monomials. Since $IF(N) > 0$, $IF((q-p)u_0) < 0$. Clearly $V > x^{-1}$.

We assert that W_i is not $\sim V$ for each i and $IF(W_i - V) \not\equiv 0$. For $i \geqslant 2$, there are three possibilities. If $W_i < V$, then $W_i - V \sim -V$. Since $IF(-V) > 0$, then $IF(W_i - V) > 0$. If $W_i < V$, then $W_i - V \sim \sim W_i$ and since $IF(V_i) \not\equiv 0$, then $IF(W_i) \not\equiv 0$ and so $IF(W_i - V) \not\equiv 0$. If $W_i \approx N$, say $W_i \sim \sigma N$, then $W_i - V \sim (\sigma + p - q)N$ and by assumption (ix), $IF((\sigma + p - q)N) \not\equiv 0$. For $i = 1$, $W_1 \sim UV_1 \sim (1 + T)V_1$. If $T < 1$, then $W_1 \sim V_1$ and the assertion holds as for $i \geqslant 2$. If $T \gtrapprox 1$ then $W_1 \approx TV_1 = N'/N \lessapprox x^{-1} < N$. Thus $W_1 < V$ and so $W_1 - V \sim \sim -V$. Then $IF(-V) > 0$ implies $IF(W_i - V) \not\equiv 0$.

All of the hypotheses of Lemma 5.8 are satisfied. Hence, e.f.d. in $F(I)$, equation (24) possesses a solution $z_0 = R_1 y_c^{q-p}$ where

$$R_1 \sim \frac{-V_1 \dots V_{n-1} QBN}{(UV_1 - V)(V_2 - V) \dots (V_{n-1} - V)} . \tag{25}$$

Clearly R_1 is $\sim$ a logarithmic monomial and, since $IF(V) \not\equiv 0$ and $V > x^{-1}$, by Lemma 5.9, the equation $v' = z_0$ possesses a solution, e.f.d. in $F(I)$, of the form

$$v_0 = R_2 y_c^{r-p} \qquad \text{with } R_2 \sim R_1 V^{-1} . \tag{26}$$

Therefore v_0 is a solution of (22).

By (1) since $q < p$, v_0 is trivial and so

$$v_0^m \text{ is trivial for } m \geqslant 1 . \tag{27}$$

Let J be any open subinterval of I for which v_0 exists in $F(J)$. A simple computation gives $v_0' = Mv_0$ where $M = (q-p)v_0 + R_2'/R_2$. From 2.2, $R_2'/R_2 \lessapprox x^{-1} < u_0$ and so

$$M \sim (q-p)u_0 \sim (q-p)N \text{ and hence } M \sim V . \tag{28}$$

In the notation of (2.8), (22) evaluated at v_0 is

$$\sum_{k=1}^{n} s_k(\psi_k(M))v_0 + s_{0*} = 0 . \tag{29}$$

Let $v = v_0(1+u)$. Then

$$v^{(j)} = \sum_{k=0}^{j} \binom{j}{k} (\psi_k(M))\, v_0(1+u)^{j-k} .$$

Insertion into (22) and division by v_0 yields

$$\sum s_{\alpha^*} v_0^{d(\alpha^*)-1} \prod_{j=0}^{n} \left[\sum_{k=0}^{j} \binom{j}{k} (\psi_k(M))(1+u)^{(j-k)} \right]^{\alpha_j} = 0 . \tag{30}$$

Denote this equation

$$T(u) = \sum t_{\gamma^*} u^{\gamma^*} = 0 . \tag{31}$$

We now proceed to estimate the t_{γ^*}'s. By (5) and (6),

(32) $\qquad s_{\alpha^*}$ is $<$ a power of x in $F(I)$ for all α^*.

From (28) and the definition of $\psi_j(M)$, $\psi_j(M)$ is $\sim$ a logarithmic monomial for any j. Hence,

(33) $\qquad \psi_j(M)$ is $<$ a power of x in $F(J)$.

Now by (30) and (31),

$$\begin{aligned} t_{0*} &= \sum s_{\alpha^*} v_0^{d(\alpha^*)-1} (\Psi(M))^{\alpha^*} = \\ &= v_0^{-1} \left[\sum_{k=1}^{n} s_k(\psi_k(M)) v_0 + s_{0*} \right] + s_0 + \sum_{d(\alpha^*)>1} s_{\alpha^*} v_0^{d(\alpha^*)-1} (\Psi(M))^{\alpha^*} . \end{aligned} \tag{34}$$

By (27), (32) and (33), the last term in (34) is trivial in $F(J)$. Since s_0 is also trivial, it follows from (29) and (34) that t_{0*} is trivial in $F(J)$.

Clearly,

$$t_j = \left. \frac{\partial T}{\partial u^{(j)}} \right|_{u=0} \qquad \text{for } j = 0, 1, \dots, n . \tag{35}$$

Thus by (30), (31) and (35),

$$\begin{aligned} t_0 &= \sum s_{\alpha^*} v_0^{d(\alpha^*)-1} d(\alpha^*) (\Psi(M))^{\alpha^*} = \\ &= \sum_{d(\alpha^*)=1} s_{\alpha^*} (\Psi(M))^{\alpha^*} + \sum_{d(\alpha^*)>1} d(\alpha^*) s_{\alpha^*} v_0^{d(\alpha^*)-1} (\Psi(M))^{\alpha^*} . \end{aligned} \tag{36}$$

By (27), (32) and (33) the last term of (36) is trivial in $F(J)$. The first term is precisely $\sum_{k=0}^{n} s_k \psi_k(M)$, and hence by (29), (36) becomes

$$t_0 = - s_{0*} v_0^{-1} + q_0 , \tag{37}$$

where q_0 is trivial in $F(J)$.

For $s \geqslant 1$, by (30), (31) and (35)

$$t_s = \sum s_{\alpha*} \prod_{j=0}^{s-1} (\psi_j(M))^{\alpha_j} \cdot \left[\sum_{j=s}^{n} \prod_{\substack{m=s \\ m \neq j}}^{n} (\psi_m(M))^{\alpha_m} (\psi_j(M))^{\alpha_j - 1} \alpha_j \binom{j}{s} (\psi_{j-s}(M)) \right] v_0^{d(\alpha^*)-1} . \tag{38}$$

As before, the terms for which $d(\alpha^*) > 1$ are trivial in $F(J)$ by (27), (32) and (33). In view of 2.2 and 2.8, $\psi_j(M) \sim M^j$ for each j. Now from (38), for $s \geqslant 1$,

$$t_s = \sum_{j=s}^{n} s_j \binom{j}{s} (\psi_{j-s}(M)) + q_s$$

with q_s trivial and therefore by (21),

$$t_s = \sum_{j=s}^{n} F_{sj} (-1)^{j-1} (V_1 \dots V_{j-1})^{-1} N^{-1} \binom{j}{s} M^{j-s} + q_s \tag{39}$$

for $s = 2, 3, \dots, n$ with $F_{sj} \sim 1$ for $j = s, \dots, n$

and

$$t_1 = \sum_{j=2}^{n} F_{1j} (-1)^{j-1} (V_1 \dots V_{j-1})^{-1} N^{-1} \binom{j}{1} M^{j-1} + F_{11}(1+T) N^{-1} + q_1 \tag{40}$$

with $F_{1j} \sim 1$ for $j = 1, \dots, n$.

For $d(\gamma^*) \geqslant 2$, $t_{\gamma*}$ is trivial in $F(J)$, since the only terms in (30) which contribute have $d(\alpha^*) \geqslant 2$, and by (27), (32) and (33) such coefficients are trivial in $F(J)$.

We have definite information about the coefficients of $T(u)$ except for t_0. It follows from (20), (26), and (37) that $t_0 \sim - QBR_2^{-1}$ which, from (25) and (26), becomes

$$(41) \qquad t_0 \sim \frac{V(UV_1 - V)(V_2 - V) \dots (V_{n-1} - V)}{V_1 V_2 \dots V_{n-1} N}$$

where $V = (q-p)u_0 \sim (q-p)N$. Clearly t_0 is $\sim$ a logarithmic monomial say R. Divide (31) by R to get

$$(42) \qquad \sum a_{\alpha^*} u^{\alpha^*} = 0 \text{ with } a_0 \sim 1,\ a_s = t_s R^{-1} \text{ for } s = 1, \dots, n \text{ and } a_{\alpha^*} \text{ trivial for } d(\alpha^*) \neq 1 .$$

For simplicity, let $W_1 = (1 + T)V_1 \sim UV_1$ and $W_k = V_k$ for $k = 2, \dots$ $\dots, n-1$. It then follows from (39), (40), (41) and (42) that for $s \geqslant 1$

$$(43) \qquad a_s = \sum_{j=s}^{n} E_{sj}(-1)^{j-1}\binom{j}{s}\frac{W_j \dots W_{n-1} M^{j-s}}{V(W_1 - V) \dots (W_{n-1} - V)} + p_s ,$$

where p_s is trivial in $F(J)$, $E_{sj} \sim 1$ for each pair (s, j) and the empty product $W_0 \dots W_1$ is 1.

To proceed, we must estimate the coefficients a_s. This is done in four cases which are:

(*a*) $W_{n-1} < N$,

(*b*) $N < W_1$,

(*c*) $W_m < N < W_{m+1}$ for some $1 \leqslant m < n-2$

and

(*d*) $W_m \approx N$ for some m.

Case (*a*). Recall that $V \sim (q-p)N$. Thus by (23) and assumption, $W_1 < W_2 < \dots < W_{n-1} < N$. Then, $W_i - V \sim -(q-p)N$ for $i \leqslant$ $\leqslant n-1$ and so, by (28), (43) becomes $a_s = \sum_{j=s}^{n} G_{sj} + p_s$ with

$$G_{sj} \sim (-1)^{j-1}\binom{j}{s} W_j \dots W_{n-1}((q-p)N)^{j-s-n} \qquad \text{for } j \geqslant s \geqslant 1 .$$

Now $W_i < N$ and so $W_i N^{-1} < 1$. Hence, $G_{sj} < N^{-s}$ for $j = s, \dots, n-1$

and $G_{sn} \sim \binom{n}{s}(q-p)^{-s}N^{-s}$. Thus

$$a_s \sim G_{sn} \quad \text{for } s \geqslant 1 . \tag{44}$$

Let $\Phi(y) = \sum_{k=0}^{n} a_k y^{(k)}$ and $A(z) = \sum_{k=0}^{n} a_k z^k$. By (42) and (44),

$$A(z) = \sum_{k=0}^{n} A_k \binom{n}{k} (q-p)^{-k} N^{-k} z^k \text{ where } A_k \sim 1 .$$

That all the A_k's lie in an LD follows from the same argument used for the R_k's. Clearly $A(z)$ has the sequence of critical monomials $(U_1, U_2, \ldots, U_n)$ where $U_k = (p-q)N$ (see the proof of Lemma 5.6). Thus by Lemma 5.7, $(U_1, U_2, \ldots, U_n)$ is a factorization sequence for Φ. Hence,

$$\Phi(y) = \dot{U}_n \ldots \dot{U}_1 y + \sum_{k=0}^{n} P_k \dot{U}_k \ldots \dot{U}_1 y . \tag{45}$$

where $P_k < 1$ for each k. Equation (42) may be written

$$\sum a_{\alpha^*} u^{\alpha^*} = a_{0^*} + \Phi(y) + \sum_{d(\alpha^*) \geqslant 2} a_{\alpha^*} u^{\alpha^*} = 0 . \tag{46}$$

Now by (42), a_{0^*} and a_{α^*} for $d(\alpha^*) \geqslant 2$ are trivial in $F(J)$. Also, $IF \cdot \cdot ((p-q)N) > 0$ on J. We may then apply Lemma 5.10. Hence, equation (46) possesses a solution $u^* < 1$ in $F(J)$. Thus $v^* = v_0(1 + u^*)$ is a solution of (19) and finally $y_c^* = y_c(1 + v^*)$ is a solution of $\Omega(y) = 0$. It is clear from the nature of I and J that such a y_c^* exists e.f.d. in $F(a, b)$. Since v_0 is trivial by (27), then v^* is trivial. So $y_c^* \sim y_c$ and part (A) follows from (1). Since $y_c^* = y_c(1 + v^*)$, then $(y_c^*)' = y_c^*(u_0 + (v^*)'/(1 + v^*))$. Now v^* is trivial. Therefore, $(v^*)'$ is trivial and hence $(y_c^*)' = y_c^* W_c$ where $W_c \sim u_0$. For x_1 a point in the domain of W_c and for some non-zero constant K we have $y_c^* = K \exp \int_{x_1}^{x} W_c$. Note that y_c^* is non-zero and thus K is non-zero. Then for any value of $\log K$ we have $y_c^* = \exp\left(\log K + \int_{x_1}^{x} W_c\right)$, which is of the form $\exp \int W_c$. This proves part (B) and concludes the proof for Case (a).

Case (b). We first note that $N < W_1$ implies that both $N < V_1$ and $T < 1$. If $T \gtrapprox 1$, then $W_1 = (1 + T)V_1 \approx TV_1 = N'/N < N$, contrary to assumption. Similarly if $V_1 \lessapprox N$, then $W_1 = (1 + T)V_1 = V_1 + N'/N \lessapprox N$. This is again a contradiction. Hence we have $W_i - V \sim W_i$ for each i. Now $W_i \sim V_i$, so by (28) and (43), $a_s = \sum_{j=s}^{n} G_{sj} + p_s$ for $s \geqslant 1$ where

$$G_{sj} \sim (-1)^{j-1}\binom{j}{s}(V_1 \dots V_{j-1})^{-1}((q-p)N)^{j-s-1} \qquad \text{for } j \geqslant s\,.$$

Since $N < V_i$ for all i, $V_i^{-1}N < 1$ and so $G_{sj} < (V_1 \dots V_{s-1})^{-1}N^{-1}$ for $j \geqslant s+1$ and $G_{ss} \sim (-1)^{s-1}(V_1 \dots V_{s-1})^{-1}((q-p)N)^{-1}$. It follows that

$$a_s \sim G_{ss} \qquad \text{for } s \geqslant 1\,. \tag{47}$$

Recall that $a_0 \sim 1$. Therefore in view of (47)

$$A(z) = \sum_{k=1}^{n} A_k(-1)^{k-1}(V_1 \dots V_{k-1})^{-1}((q-p)N)^{-1}z^k + A_0$$

where $A_k \sim 1$ for each k. Clearly the A_k's lie in an LD. Let $U_1 = (p-q)N$ and $U_k = V_{k-1}$ for $k \geqslant 2$. By following the proof of Lemma 5.6, $A(z)$ has the sequence of critical monomials $(U_1, U_2, \dots, U_n)$. Hence by Lemma 5.7, Φ has the factorization sequence $(U_1, U_2, \dots, U_n)$. Equations (45) and (46) are valid for this sequence. Now $IF((p-q)N) > 0$ and no $IF(V_i)$ is identically zero. Thus for any open subinterval K of J on which these indicial functions are nowhere zero, equation (46) possesses a solution $u^* < 1$ in $F(K)$ by Lemma 5.10. Then $v^* = v_0(1 + u^*)$ solves (19) and $y_c^* = y_c(1 + v^*)$ solves $\Omega(y) = 0$. The remainder of the proof is as in Case (a).

Case (c). Since for some m, $W_m < N < W_{m+1}$ and $V \sim (q-p)N$, we have $W_i - V \sim (p-q)N$ for $i \leqslant m$ and $W_i - V \sim W_i$ for $i > m$. Then from (28) and (43), $a_s = \sum_{j=s}^{n} G_{sj} + p_s$ for $s \geqslant 1$ where

$$G_{sj} \sim (-1)^{j-1-m}\binom{j}{s}(W_j \dots W_{n-1})(W_{m+1} \dots W_{n-1})^{-1}((q-p)N)^{j-s-1-m}\,.$$

Clearly $W_i N^{-1} < 1$ for $i \leqslant m$ and $W_i N^{-1} > 1$ for $i > m$. Hence $G_{sj} < < N^{-s}$ for $s \leqslant j \leqslant m$,

$$G_{s,m+1} \sim \binom{m+1}{s} ((q-p)N)^{-s} \quad \text{and} \quad G_{sj} \prec N^{-s} \quad \text{for } j > m+1 .$$

It follows that

$$a_s \sim G_{s,m+1} \quad \text{for } s \leqslant m . \tag{48}$$

When $s > m$, $G_{sj} < (W_{m+1} \dots W_{s-1})^{-1} N^{-m-1}$ for $j > s$ and

$$G_{ss} \sim (-1)^{s-1-m} (W_{m+1} \dots W_{s-1})^{-1} ((q-p)N)^{-m-1} .$$

Since $m \geqslant 1$ and $W_k = V_k$ for $k \geqslant 2$, then

$$a_s \sim G_{ss} \quad \text{for } s > m . \tag{49}$$

From (48) and (49) we have

$$A(z) = \sum_{k=0}^{m+1} A_k \binom{m+1}{k} ((q-p)N)^{-k} z^k +$$
$$+ ((q-p)N)^{-m-1} \sum_{k=m+2}^{n} A_k (-1)^{k-m-1} (V_{m+1} \dots V_{k-1})^{-1} z^k$$

where $A_k \sim 1$. Clearly the A_k's lie in an LD. From the proof of Lemma 5.6, $A(z)$ has the sequence of critical monomials $(U_1, U_2, \dots, U_n)$ with $U_k = (p-q)N$ for $k \leqslant m+1$ and $U_k = V_{k-1}$ for $k > m+1$. Since $N > x^{-1}$, $U_2 > x^{-1}$ so by Lemma 5.7, Φ has a factorization sequence $(U_1, U_2, \dots, U_n)$. Now $IF((p-q)N) > 0$ and no $IF(V_k)$ is identically zero. Thus on any open subinterval K of J on which these indicial functions are nowhere zero, there exists a solution u^* of (46) where, by Lemma 5.10, $u^* < 1$ in $F(J)$. Hence, $v^* = v_0(1+u^*)$ solves (19) and $y_c^* = y_c(1+v^*)$ solves $\Omega(y) = 0$. The remainder of the proof is as in Case (a).

Case (d). Here $W_m \approx N$ for some m. We separate momentarily the possibility that $m = 1$. So assume that $m > 1$. By (ix), if $W_m = V_m \sim \sigma N$, then $(\sigma + p - q)$ is not zero. Since $W_{m-1} < W_m < W_{m+1}$ and $V \sim (q-p)N$, then $W_i - V \sim -(q-p)N$ if $i < m$, $W_m - V \sim \sim (\sigma + p - q)N$ and $W_i - V \sim W_i$ for $i > m$.

Hence from (28) and (43), $a_s = \sum_{j=s}^{n} G_{sj} + p_s$ where

$$G_{sj} \sim (-1)^{j-m} \binom{j}{s} (W_j \dots W_{n-1})(W_{m+1} \dots W_{n-1})^{-1} \cdot$$
$$\cdot ((\sigma + p - q) N)^{-1} ((q-p) N)^{j-s-m} .$$

The relations $W_i N^{-1} < 1$ for $i < m$, $W_i^{-1} N < 1$ for $i > m$ and $W_m \cdot \cdot N^{-1} \sim \sigma$ are valid. For $s > m$, $G_{sj} < (W_{m+1} \dots W_{s-1})^{-1} N^{-m-1}$ for $j > s$ and

$$G_{ss} \sim (-1)^{s-m} (W_{m+1} \dots W_{s-1})^{-1} ((q-p)N)^{-m} ((\sigma + p - q) N)^{-1} .$$

Hence,

(50) $$a_s \sim G_{ss} \quad \text{for } s > m .$$

When $s \leqslant m$, then $G_{sj} < N^{-s}$ for $j < m$ or $j > m+1$,

$$G_{sm} \sim \binom{m}{s} \sigma ((q-p) N)^{-s} (\sigma + p - q)^{-1}$$

and

$$G_{s,m+1} \sim -\binom{m+1}{s} ((q-p) N)^{-s} (\sigma + p - q)^{-1} (q-p) .$$

If follows that $G_s \approx G_{s,m+1}$. However by (ix),

(51) $$\binom{m}{s}\sigma - \binom{m+1}{s}(q-p) = \binom{m}{s}\left(\sigma - \frac{m+1}{m+1-s}(q-p)\right) \neq 0 .$$

Let λ_{sm} be the left hand side of (51). Thus

(52) $$a_s \sim \lambda_{sm} (q-p)^{-s} (\sigma + p - q)^{-1} N^{-s} \quad \text{for } s \leqslant m .$$

For the case $m = 1$, since $TV_1 = N'/N$, then necessarily $W_1 = = V_1 + N'/N$ is $\sim V_1$ and hence $V_1 \approx N$. Let $V_1 \sim \sigma N$. Then by (ix), $(\sigma + p - q)$ is non-zero. Hence, $W_1 - V \sim (\sigma + p - q) N$. Also, $W_i - V \sim W_i$ for $i \geqslant 2$. Thus (50) holds even for $m = 1$ and so too, (52) is valid for $m = 1$.

We proceed as in the previous cases. From (50) and (52) we have

$$A(z) = A_0 + \sum_{k=1}^{m} A_k \lambda_{km}(q-p)^{-k}(\sigma + p - q)^{-1} N^{-k} z^k +$$

$$+ \sum_{k=m+1}^{n} A_k(-1)^{k-m}(V_{m+1} \dots V_{k-1})^{-1}(q-p)^{-m}(\sigma + p - q)^{-1} N^{-m-1} z^k$$

where $A_k \sim 1$. Clearly the A_k's lie in an LD. From the proof of Lemma 5.6, $A(z)$ has the sequence of critical monomials $(U_1, U_2, \dots \dots, U_n)$ where $U_k = c_k N$ for $k \leqslant m+1$ and $U_k = V_{k-1}$ for $k > m+1$ and where c_k is a non-zero root of

$$(53) \qquad F(v) = 1 + \sum_{k=1}^{m} \lambda_{km}(q-p)^{-k}(\sigma + p - q)^{-1} v^k - \\ - (q-p)^{-m}(\sigma + p - q)^{-1} v^{m+1}.$$

This last espression is the form of $F_i(v)$ in Lemma 5.6. Multiply (53) by $(\sigma + p - q)(q-p)^m$ and simplify to get

$$(54) \qquad (\sigma + p - q)(q-p)^m F(v) = \big(\sigma - (v + q - p)\big)(v + q - p)^m .$$

Clearly then from (54), $c_1 = \sigma + p - q$ and $c_k = p - q$ for $2 \geqslant k \geqslant m+1$. By Lemma 5.7, $(U_1, U_2, \dots, U_n)$ is a factorization sequence for Φ. Since $IF\big((\sigma + p - q)N\big)$ is not identically zero, $IF\big((p-q)N\big) > 0$ and no $IF(V_i)$ is identically zero, it follows that for any open subinterval K of J on which these indicial functions are nowhere zero, the equation (46) possesses a solution $u^* \prec 1$ in $F(K)$. This is a consequence of Lemma 5.10. Hence, $v^* = v_0(1 + u^*)$ solves (19) and $y_c^* = y_c(1 + v^*)$ solves $\Omega(y) = 0$. The remainder of the proof is as in Case (a). This completes the proof of the theorem.

5. Supplementary Lemmas.

5.1. Lemma. Let $N \succ x^{-1}$ and $IF(N) > 0$ on I. Let $u_0 \sim N$ exist on I and for each non-zero c let $y_c = c \exp \int u_0$. Then for each real a, $y_c \succ x^a$ in $F(I)$.

PROOF. This is a consequence of [4, Lemma 4, p. 132].

5.2. Lemma. Let $L_p = x \log x \dots \log_{p-1} x$. Let $V_1, V_2, \dots, V_n$ be logarithmic monomials of rank $\leqslant p-1$ such that $L_p^{-1} \lesssim V_1 < V_2 < \dots$ $\dots < V_n$ and $x^{-1} < V_2$. Let $\Lambda_j = \dot{V}_j \dots \dot{V}_1$ and $\Lambda_0 = I$ with $\dot{W} = 1 - W^{-1}D$. Let $\Lambda_j = \sum_{k=0}^{j} B_{jk} D^k$. Then

$$B_{km} \sim (-1)^m (V_1 \dots V_m)^{-1} \tag{1}$$

for $m \leqslant k \leqslant n$ and with the empty product taken as 1 and

$$\frac{\partial \Lambda_k(w)}{\partial w^{(m)}} = B_{km} \qquad \text{for } m \leqslant k \leqslant n\,. \tag{2}$$

PROOF. Part (1) is proved by induction on both indices. Part (2) is obvious.

5.3. Lemma. Let $z = N + Nw$ where N is in an LD_r for some r and N is not trivial. Then

$$\frac{\partial w^{(m)}}{\partial z^{(k)}} = \binom{m}{k} (N^{-1})^{(m-k)} \lesssim x^{-(m-k)} N^{-1} \qquad \text{for } m \geqslant k\,.$$

PROOF. This is a straightforward computation using 2.2.

5.4. LEMMA. Let $D^n = \sum_{j=1}^{n} D(r, n, j) \theta_r^j$ for $n \geqslant 1$. Then

$$D(r, n, n) = L_r^{-n}\,, \tag{1}$$

and

$$D(r, n, j) \lesssim x^{-(n-j)} \mathrm{L}_r^{-j} \qquad \text{for } 1 \leqslant j \leqslant n\,, \tag{2}$$

where $L_r = x \log x \dots \log_{r-1} x$.

PROOF. The proof is by induction on n.

5.5. Lemma. Let $u \in LDF$ and let $y = c \exp \int u$. Define $\psi_n(u)$ by $y^{(n)} = \psi_n(u) y$. Then

$$(\psi_n(u))' = \sum_{k=1}^{n} \binom{n}{k} \psi_{n-k}(u) u^k \qquad \text{for } n \geqslant 1 \tag{1}$$

and

$$(2) \qquad \frac{\partial \psi_n(u)}{\partial u^{(k-1)}} = \binom{n}{k} \psi_{n-k}(u) \qquad \text{for } 1 \leqslant k \leqslant n .$$

PROOF. Statement (1) is proved by induction. Statement (2) is also proved by induction by using (1).

5.6. Lemma. Let $V_1, V_2, \ldots, V_n$ be logarithmic monomials of rank $\leqslant p-1$ such that $L_p^{-1} \lesssim V_1 \lesssim V_2 \lesssim \cdots \lesssim V_n$. Let $U_j = (-1)^j (V_1 \ldots V_j)^{-1}$. Let $G(z) = \sum_{j=1}^{n} E_j U_j z^j + E_0$ where the E_i's all lie in an LD_q for some q and $E_i \sim 1$ for each i. Then if $M_1, M_2, \ldots, M_n$ is the sequence of critical monomials of G, the M_i can be arranged so that $M_i \approx V_i$. Furthermore, if $V_1 \prec V_2 \prec \ldots \prec V_n$, then $M_i \sim V_i$ for $i = 1, 2, \ldots, n$.

PROOF. Assume that

$$V_1 \approx V_2 \approx \ldots \approx V_{k_0} \prec V_{k_0+1} \approx \ldots \approx V_{k_1} \prec \ldots \prec V_{k_{\sigma-1}+1} \approx \ldots \approx V_{k_\sigma} = V_n .$$

Let $c_i = V_i]V_i[^{-1}$, $(c_1 \ldots c_0)^{-1} = 1$ and $k_{-1} = 0$. Let

$$F_i(v) = \sum_{j=k_{i-1}}^{k_i} (-1)^j v^j (c_1 \ldots c_j)^{-1} \qquad \text{for } i = 0, 1, \ldots, \sigma .$$

For fixed i, let $W_i =]V_{k_i}[$ and let $a_{i1}, a_{i2}, \ldots, a_{ir}$ be the r distinct non-zero roots of $F_i(v)$ with respective multiplicities $p_{i1}, \ldots, p_{ir}$. Clearly, $\sum_{j=1}^{r} p_{ij} = k_i - k_{i-1}$.

CLAIM. $G(z)$ has critical monomial $a_{ij} W_i$ of multiplicity p_{ij} for $j = 1, 2, \ldots, r$. Since $W_i V_j^{-1} \prec 1$ for $j > k_i$, $W_i V_j^{-1} \succ 1$ for $j \leqslant k_{i-1}$ and $W_i V_j^{-1} \approx 1$ for $k_{i-1} < j \leqslant k_i$ and $E_j U_j W_i^j = (-1)^j E_j (W_i V_1^{-1}) \ldots (W_i V_j^{-1})$, it follows that

$$E_j U_j W_i^j \prec E_{k_i} U_{k_i} W_i^{k_i} \qquad \text{for } j > k_i ,$$

$$E_j U_j W_i^j \prec E_{k_{i-1}} U_{k_{i-1}} W_i^{k_{i-1}} \qquad \text{for } j < k_{i-1}$$

and

$$E_j U_j W_i^j \approx E_{k_{i-1}} U_{k_{i-1}} W_i^{k_{i-1}} \approx E_{k_i} U_{k_i} W_i^{k_i} \qquad \text{for } k_{i-1} \leqslant j \leqslant k_i .$$

Hence $[W_i, p-1, G] = F_i(v) + R_i(v)$ where $R_i(v)$ is trivial in $\log_p F(a, b)$. Each non-zero root a_{ij} of $F_i(v)$ has multiplicity p_{ij}. Thus, $a_{ij}W_i$ is a critical monomial of G of multiplicity p_{ij}. This proves the claim.

The first part of the lemma follows from the claim. Let $V_1 < < V_2 < \dots < V_n$. Then $F_i(v) = (-1)^{i-1}v^{i-1}(c_1 \dots c_{i-1})^{-1}(1 - c_i^{-1}v)$. The only non-zero root of $F_i(v)$ is c_i with multiplicity one. The result follows. This completes the proof.

5.7. Lemma. Let $V_1, V_2, \dots, V_n$ be logarithmic monomials of rank $\leqslant p-1$ such that $L_p^{-1} \lessapprox V_1 \lessapprox V_2 \lessapprox \dots \lessapprox V_n$. Let $U_j = (V_1 \dots V_j)^{-1}$ for $j = 1, \dots, n$. Let

$$A = \sum_{j=1}^{n} E_j U_j D^j + E_0 \quad \text{and} \quad G(z) = \sum_{j=0}^{n} E_j U_j z^j + E_0 ,$$

where the E_j's lie in an LD_q for some q and where $E_j \sim 1$ for each j. Let $V_2 > x^{-1}$. Assume that $(M_1, M_2, \dots, M_n)$ is the sequence of critical monomials of $G(z)$ with $M_1 \lessapprox M_2 \lessapprox \dots \lessapprox M_n$. Then A is unimajoral and $(M_1, M_2, \dots, M_n)$ is a principal factorization sequence for A (see [8, §§ 28 and 29]).

Proof. Let $r \geqslant \max(p-1, q)$. By Lemma 5.4,

$$D^j = \sum_{k=1}^{j} D(r+1, j, k)\theta_{r+1}^k$$

and hence,

$$A = \sum_{j=1}^{n} E_j U_j \sum_{k=0}^{j} D(r+1, j, k)\theta_{r+1}^k + E_0 .$$

Interchange the order of summation and put $B_k = \sum_{j=k}^{n} E_j U_j D(r+1, j, k)$ to get $A = \sum_{k=1}^{n} B_k \theta_{r+1}^k + E_0$.

We assert that $B_k \sim U_k L_{r+1}^{-k}$ for each k. For $j > k$, by Lemma 5.4, $D(r+1, j, k) \lessapprox x^{-(j-k)} L_{r+1}^{-k}$. Therefore, since $E_m \sim 1$ for all m and $(V_j x)^{-1} < 1$ for $j \geqslant 2$, we have

$$E_j U_j D(r+1, j, k) \lessapprox U_j x^{-(j-k)} L_{r+1}^{-k} \lessapprox$$
$$\lessapprox U_k L_{r+1}^{-k} (V_{k+1} \dots V_j)^{-1} x^{-(j-k)} < U_k L_{r+1}^{-k} .$$

Now by Lemma 5.4, $D(r+1, k, k) = L_{r+1}^{-k}$ and since $E_k \sim 1$, we have $E_k U_k D(r+1, k, k) \sim U_k L_{r+1}^{-k}$. Then, $B_k \sim U_k L_{r+1}^{-k}$. This proves the assertion.

Since $\theta_{r+1} = L_{r+1}D$, $\Lambda = \sum_{k=1}^{n} B_k(L_{r+1}D)^k + E_0$. The $(r+1)$-characteristic polynomial (see [8, § 42]) of Λ is $C^*_{r+1}\Lambda(z) = \sum_{k=1}^{n} B_k(L_{r+1}z)^k + E_0$. From the definition of U_k, it is clear that $B_k \lessapprox 1$. Hence, Λ is strongly unimajoral of rank $r+1$. Thus by [8, § 20(a)], Λ is unimajoral.

Let $F_k = B_k L_{r+1}^k$. Clearly $F_k \sim U_k$. We write $F_k = G_k U_k$ with $G_k \sim 1$. Then $C^*_{r+1}\Lambda(z) = \sum_{k=1}^{n} G_k U_k z^k + E_0$. Now $G(z)$ has the sequence of critical monomials $M_1, M_2, \ldots, M_n$ with $M_1 \lessapprox M_2 \lessapprox \ldots \lessapprox M_n$. One notices that in the proof of Lemma 5.6, the perturbations $E_k \sim 1$ do not alter the critical monomials of G. It is clear that $C^*_{r+1}\Lambda(z)$ has critical monomials $M_1, M_2, \ldots, M_n$.

As a result of Lemma 5.6, $M_k \approx \,]V_k[$ for each k. Each of the V_k's lie in the divergence class, and thus so do the M_k's. (That is, $(\delta_0(M_k), \delta_1(M_k), \ldots)$ is lexicographically greater than $(-1, -1, \ldots)$.) The coefficients of Λ all lie in some LD and Λ is unimajoral, hence by [8, § 44], $(M_1, M_2, \ldots, M_n)$ is a principal factorization sequence for Λ. This completes the proof.

5.8. Lemma. Let $W_1, W_2, \ldots, W_n$, H and V be functions which are asymptotic to logarithmic monomials in some $F(I)$. Let $V \succ x^{-1}$ and W_i be not $\sim V$ for each i. Let $IF(V) \neq 0$ and $IF(V - W_i) \neq 0$ for each i. Let w_0 be a function of the form $\exp \int V$. Then, e.f.d. in $F(I)$, the equation $\dot{W}_n \ldots \dot{W}_n y = Hw_0$ possesses a solution $v_0 = Rw_0$ where

$$R \sim W_1 W_2 \ldots W_n H[(W_1 - V)(W_2 - V) \ldots (W_n - V)]^{-1}.$$

Proof. Consider the equation $\dot{W} z = Hw_0 = z_0$ where W is not $\sim V$ and $IF(V - W) \neq 0$ (W in the divergence class). Let $z = -xWz_0u$. Then with

$$Hw_0 = H \exp \int V = z_0,$$

$$z' = -(Wz_0 + xW'z_0 + xH'H^{-1}z_0 + xVz_0)u - xWz_0u'.$$

Substitute z' into the equation and divide by xz_0 to get $u' + Uu = x^{-1}$ where $U = V - W + H'H^{-1} + W'W^{-1} + x^{-1}$. Since $H' \lessapprox Hx^{-1}$ and

$W' \lessapprox Wx^{-1}$ by 2.2 and $V - W \gtrapprox V > x^{-1}$, then $U \sim V - W > x^{-1}$. In some element of $F(I)$, U is nowhere zero. Hence, we may write $u + u'U^{-1} = (xU)^{-1}$. Clearly, $IF(U) \not\equiv 0$ and $(xU)^{-1} < 1$. Let J be any open subinterval of I on which $IF(U)$ is nowhere zero. It follows from [6, p. 271, Lemma δ] that there is a function $u_1 < 1$ in $F(J)$ such that $u_1 + u_1' U^{-1} = (xU)^{-1}$. Since $u_1 < 1$, then also $xu_1' < 1$. Hence $u_1' U^{-1} < (xU)^{-1}$ and so $u_1 \sim (xU)^{-1}$. Clearly then, $w_1 = -xWu_1z_0$ solves $\dot{W}z = z_0$. Then $w_1 = sw_0$ where $s = -xWu_1H$. It follows that $s \sim WH(W - V)^{-1}$.

To solve $\dot{W}_n \dots \dot{W}_1 y_0 = Hw_0$, proceed iteratively. Clearly $\dot{W}_n z = Hw_0$ is of the form solved above. Thus, e.f.d. in $F(I)$, $\dot{W}_n z = Hw_0$ possesses a solution $v_n = R_n w_0$ where $R_n \sim W_n H(W_n - V)^{-1}$. Now R_n is $\sim$ a logarithmic monomial. Hence, $\dot{W}_{n-1} z = R_n w_0$ is also of the form solved. Proceed in this fashion to get that $\dot{W}_n \dots \dot{W}_1 y = Hw_0$ possesses, e.f.d. in $F(I)$, a solution $v_0 = Rw_0$ where R is as desired. This completes the proof.

5.9. Lemma. Let V and R be $\sim$ logarithmic monomials in some $F(I)$. Let $V > x^{-1}$ and $IF(V) \not\equiv 0$. Let w_0 be a function of the form $\exp \int V$. Then, e.f.d. in $F(I)$, the equation $v' = Rw_0$ possesses a solution $u_0 = Sw_0$ where $S \sim RV^{-1}$.

Proof. This is a consequence of [3, p. 19, Lemma 10(b)].

5.10. Lemma. Let $\Phi(y) = \dot{W}_n \dots \dot{W}_1 y + \sum_{j=0}^{n} E_j \dot{W}_j \dots \dot{W}_1 y$ where $W_1, W_2, \dots, W_n$ are in the divergence class and $E_i < 1$ in some $F(I)$ for $i = 0, 1, \dots, n$. Let $\Psi(y) = a_{0^*} + \Phi(y) + \sum_{d(\alpha^*) \geq 2} a_{\alpha^*} y^{\alpha^*}$ where a_{0^*} and a_{α^*} with $d(\alpha^*) \geq 2$ are trivial in $F(I)$. Then for any open subinterval J of I on which no $IF(W_i)$ has a zero, the equation $\Psi(y) = 0$ possesses at least one solution $y_1 < 1$ in $F(J)$.

Proof. Let $\Lambda_k(y) = \dot{W}_k \dots \dot{W}_1 y$. Then $y^{(j)}$ is a linear polynomial in $\Lambda_1 y, \Lambda_2 y, \dots, \Lambda_j y$ with each coefficient $<$ a power of x. Thus, with $\Lambda_0 y = y$,

$$\Psi(y) = a_{0^*} + \Lambda_n(y) + \sum_{j=0}^{n} E_j \Lambda_j(y) + \sum_{d(\beta^*) \geq 2} b_{\beta^*} (\Lambda^*(y))^{\beta^*}$$

where $\Lambda^*(y) = (\Lambda_0(y), \Lambda_1(y), \dots, \Lambda_n(y))$. Since the a_{α^*}'s are trivial for $d(\alpha^*) \geq 2$, the b_{β^*}'s are also trivial in $F(I)$. By the remark after

[8, § 99.7], (99.6) and (99.7) are sufficient for a strong factorization sequence. These are satisfied here, so $(W_1, W_2, \ldots, W_n)$ is a strong factorization sequence for $\Psi(y)$ (see also [8, § 88(*b*)]). Hence if J is any open subinterval of I on which no $IF(W_i)$ has a zero, the equation $\Psi(y) = 0$ possesses at least one solution $y_1 \prec 1$ in $F(J)$ by [8, § 99, Part 5]. This completes the proof.

6. An example.

Let

$$\Omega(y) = x^{-1}y^{n+1}y''' - (1 + 4x^{-1})y^n y' y'' + (4 + 9x^{-1})y^{n+1}y' - \\ - (3 + 6x^{-1})y^{n+2} + \psi(x) y^{r-m-p-q}(y')^m (y'')^p (y''')^q + \Lambda(y)$$

where n is a non-negative integer, r is any non-negative integer $\leqslant n + 1$, m, p and q are any non-negative integers such that $m + p + q \leqslant r$, $\psi(x)$ is any finite sum of logarithmic monomials and Λ is any third order differential polynomial with coefficients in an LDF over $F(-\pi, \pi)$ each of whose terms has total degree $\leqslant r - 1$ in y, y', y'' and y'''. In this case

$$G(z) = -(1 + 3x^{-1})z^3 - (1 + x^{-1})zz' + x^{-1}z'' + (4 + 9x^{-1})z - 3 - 6x^{-1} ,$$

$$\partial G/\partial z'' = x^{-1} \quad \text{and} \quad H(z) = \psi(x) z^m (z' + z^2)^p (z'' + 3zz' + z^3)^q .$$

By using [2, § 26], we find that $G(z)$ has the three simple, non-parametric critical monomials $M_1 = 1$, $M_2 = (13^{\frac{1}{2}} - 1)/2$ and $M_3 = -M_2$. By using [2, §§ 21 and 26], we find that H has the critical monomial x^{-1} if $p \neq 0$, the critical monomials x^{-1} and $2x^{-1}$ if $q \neq 0$ and no critical monomials if $p = q = 0$. Clearly, $\partial G/\partial z''$ has no critical monomials.

Since $IF(M_1) = IF(M_2) = 1$ and $IF(M_3) = -1$, only M_1 and M_2 satisfy (ii). Consider first the monomial M_1. Then

$$\Lambda_1(z) = G(M_1 + M_1 z) = -(1 + 3x^{-1})(z + 1)^3 - \\ - (1 + x^{-1})(zz' + z') + x^{-1}z'' + (4 + 9x^{-1})(z + 1) - 3 - 6x^{-1} .$$

Since $\partial A_1(0)/\partial z = 1$, $Q = 1$ in Lemma 3.2. Thus,

$$HLP(QA_1(z)) = x^{-1}z'' - (1 + x^{-1})z' + z .$$

By Lemmas 5.6 and 5.7, $HLP(QA_1(z))$ has a factorization sequence (V_1, V_2) with $V_1 = 1$ and $V_2 = x$. An easy computation shows that $QA_1(z)$ is normal with respect to $(V_1, V_2, 1)$. Since $IF(V_1) = 1$ and $IF(V_2) = \cos 2\theta$, (V_1, V_2) is unblocked.

Clearly, $V_1 < V_2$ and $x^{-1} < V_2$. Since $V_1 = M_1$, $\sigma = 1$ and $1 \notin \{r - n - 2, 2(r - n - 2)\}$. Finally

$$IF((\sigma + n + 2 - r)M_1) = IF(n + 3 - r) = 1 .$$

Thus, e.f.d. in $F(-\pi, \pi)$, the equation $\Omega(y) = 0$ possesses a one parameter family of solutions $y_c^* = \exp \int W_c$ where $W_c \sim 1$.

Now consider the monomial $M_2 = \gamma$. Then

$$A_2(z) = G(\gamma + \gamma z) = -(1 + 3x^{-1})\gamma^3(z + 1)^3 - (1 + x^{-1})\gamma^2(zz' + z') + \\ + x^{-1}\gamma z'' + (4 + 9x^{-1})\gamma(z + 1) - 3 - 6x^{-1} .$$

Thus, $\partial A_2(0)/\partial z = -9x^{-1}(\gamma^3 - \gamma) + 4\gamma - 3\gamma^3$. Since $-\gamma^3 + 4\gamma - 3 = 0$, then $Q_2 = (9 - 8\gamma)^{-1}$. By Lemmas 5.6 and 5.7, $HLP(Q_2\, A_2(z))$ has a factorization sequence $(\gamma^{-2}(9 - 8\gamma), \gamma x) = (W_1, W_2)$. That $Q_2A_2(z)$ is normal with respect to $(W_1, W_2, 1)$ is again straightforward. Since W_1 is real and non-zero, then $IF(W_1) = \pm 1$. Also, $IF(W_2) = \cos 2\theta$. Now $W_1 \approx 1$ so $W_1 \approx \gamma$ and then $\sigma = \gamma^{-1}W_1$ which is not an integer. Now $IF(\sigma + (p - r)\gamma) = \pm 1$, so (ix) is satisfied. Thus, e.f.d. in $F(-\pi, \pi)$, the equation $\Omega(y) = 0$ has a one parameter family of solutions $y_c^* = \exp \int W_c$ with $W_c \sim \gamma$.

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