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Paola Unterholzner

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Algebraic and Relational Semantics for Tense Logics.

PAOLA UNTERHOLZNER (*)

The close connection between algebraic and relational models for modal logics is expressed mathematically with a duality between categories. Here we extend such duality to models for tense logics. We follow the exposition given in [8], thus reducing proofs only to these steps, which differ significantly from the modal case. In the second section, we also extend to tense logics a technique for proving completeness (the unravelling technique of [7]), and adapt it to obtain completeness of the tense logic of provability GL. We use here notations of [8].

Chapter 1

§ 1. We recall that a tense logic is a logic with two operators $\Box$ and $\boxminus$, usually known as necessity in the future and in the past, respectively, so that $\Box$ and $\boxminus$ both satisfy separately rules and axioms for the logic K, i.e.:

(1) (MN_1): $A/\Box A$,

(2) (MN_2): $A/\boxminus A$,

(3) $(\Box)$: $\Box(A \to B) \to \Box A \to \Box B$,

(4) $(\boxminus)$: $\boxminus(A \to B) \to \boxminus A \to \boxminus B$,

(*) Indirizzo dell'A.: Seminario Matematico, Università di Padova, Via Belzoni 7, 35100 Padova.

and the following formulas:

(5) $$A \rightarrow \Box \diamondminus A\,,$$

(6) $$A \rightarrow \boxminus \Diamond A\,,$$

(about tense logic see [12]).

As usual, the algebraic semantics for tense logic is given by tense algebras, which are triples $(\mathcal{A}, \tau, \varrho)$, with $\mathcal{A} = \langle A, +, \cdot, \nu, 0, 1\rangle$ a Boolean algebra, where $+$, $\cdot$, ν, 0, 1, are the symbols for join, meet, complementation, zero, one respectively, and τ and ϱ are unary operations satisfying:

(7) $$\tau 1 \quad = 1\,,$$

(8) $$\varrho 1 \quad = 1\,,$$

(9) $$\tau(x \cdot y) = \tau x \cdot \tau y\,,$$

(10) $$\varrho(x \cdot y) = \varrho x \cdot \varrho y\,,$$

(an operation with conditions (7) and (9) is also called a hemimorphism), and also:

(11) $$x \leqslant \tau\nu\varrho\nu x\,,$$

(12) $$x \leqslant \varrho\nu\tau\nu x\,.$$

Conditions (7)-(12) together are easily seen to be equivalent to the conditions (7)-(10) and:

(13) $$x + \tau y = 1 \quad \text{if and only if} \quad \varrho x + y = 1\,.$$

Let us observe that two operators τ, ϱ satisfying (13) were called conjugate in [4].

The relational semantics is given by triples $F = (X, r, \mathcal{C})$, where X is a set, r a binary relation, $\mathcal{C}$ a subalgebra of 2^X closed with respect to r^+ and $(r^{-1})^+$, where:

(14) $$r^+ C = \{x\colon\ xry \rightarrow y \in C\}\,.$$

We take $\mathcal{C}$ as a base of clopen (closed and open) subsets for a topology on X. The valuations on F are restricted to elements of $\mathcal{C}$, and this justifies our assumption of closure with respect to r^+ and $(r^{-1})^+$, since we put as usual:

$$x \Vdash \Box P \quad \text{if and only if} \quad xry \to y \Vdash P\,,$$

$$x \Vdash \boxminus P \quad \text{if and only if} \quad xr^{-1}y \to y \Vdash P\,.$$

§ 2. We shall extend to the tense case the following theorem on duality between algebras and frames, as stated in [8]:

THEOREM. The category of modal algebras and homomorphisms is dual to the category of descriptive frames and contractions.

Before giving a tense formulation of this theorem, let us observe that all our assumptions for $\Box$, and hence for τ, are indipendently true for $\boxminus$, and hence for ϱ.

In fact, axioms and rules for $\Box$ and $\boxminus$ are exactly the same.

In the proof of this theorem in the modal case, relations are seen as morphisms r between frames; more precisely, morphisms from $(X, \mathcal{C})$ to $(Y, \mathcal{U})$, with $(X, \mathcal{C})$ and $(Y, \mathcal{U})$ frames, are all relations $r \subseteq X \times Y$, so that:

$$\forall C \subseteq Y\,, \qquad C \in \mathcal{U} \to r^{-1}C \in \mathcal{C}\,,$$

where

$$r^{-1}C = \{x \in X\colon\ \exists y(xry \,\&\, y \in C)\}\,.$$

We observe that in the tense case, it is impossible to follow this way, because we must add to this last condition the following:

$$\forall C \subseteq Y\,, \qquad C \in \mathcal{U} \to rC \in \mathcal{C}\,,$$

with

$$rC = \{x \in X\colon\ \exists y(yrx \,\&\, y \in C)\}\,.$$

Obviously, this would compell r to be a morphism of a frame on itself. Thus we directly define functors between the categories of tense algebras with homomorphisms and tense frames with bicontractions; bicontractions are obtained extending to tense frames the concept of contraction (or p-morphism) between frames. A precise definition is:

DEF. 1.1. Let $(X, r, \mathcal{C})$, $(Y, s, \mathcal{U})$ be tense frames; a continuous function $c\colon X \to Y$ is a *bicontraction* if:

(i) $crx = scx$,

(ii) $cr^{-1}x = s^{-1}cx$, $\forall x \in X$.

As we will see, with this definition, the dual c^+, defined as c^{-1}, preserves both τ and ϱ.

For any tense algebra $(\mathcal{A}, \tau, \varrho)$, we now put:

(*a*) $\mathcal{A}_+ =$ the Stone space of $\mathcal{A}$,

(*b*) τ_+ (ϱ_+) is the relation: $S\tau_+T$ $(S\varrho_+T)$ if and only if $\tau a \in S$ $(\varrho a \in S) \to a \in T$, with $S, T \in \mathcal{A}_+$,

(*c*) $h_+ = h^{-1}$, for any homomorphism h of tense algebras.

Then we define $(\mathcal{A}, \tau, \varrho)_+ = (\mathcal{A}_+, \tau_+, \varrho_+)$ and the first step is to prove:

LEMMA 1.2. For any tense algebra $(\mathcal{A}, \tau, \varrho)$, $(\mathcal{A}, \tau, \varrho)_+$ is a tense frame.

PROOF. It is enough to show that $\tau_+ = \varrho_+^{-1}$, which follows from the assumptions linking τ with ϱ. In fact, by formulas $x \leqslant \tau\nu\varrho\nu x$, $x \leqslant \varrho\nu\tau\nu x$, it follows:

$S\tau_+^{-1}T$ if and only if $T\tau_+S$ (by definition of τ_+^{-1}), if and only if $\forall a \in A$, $\tau a \in T \to a \in S$ (by definition of τ_+), if and only if $\forall a \in A$, $a \in T \to \nu\varrho\nu a \in S$ (in fact $a \in T \to \tau\nu\varrho\nu a \in T$, but $\tau\nu\varrho\nu a \in T \to \nu\varrho\nu a \in S$), if and only if, $\forall a \in A$, $\varrho a \in S \to a \in T$, if and only if $S\varrho^+T$.

On the other side, for any tense frame $\mathcal{F} = (X, r, \mathcal{C})$, we put $\mathcal{F}^+ = (\mathcal{C}, r^+, (r^{-1})^+)$, where by $\mathcal{C}$ we intend the Boolean algebra $(\mathcal{C}, \cup, \cap, \setminus, \emptyset, X)$.

For any relation r, r^+ is a hemimorphism (see [8]), thus we need only to prove:

$$C \subseteq r^+ \setminus (r^{-1})^+ \setminus C\,,$$

$$C \subseteq r^{-1^+} \setminus r^+ \setminus C\,, \qquad \text{for any } C \in \mathcal{C}\,.$$

These facts follow easily from the identities $r^{-1^+}C = \setminus r \setminus C$ and $r^+C = \setminus r^{-1} \setminus C$ (see [8], Lemma 1).

Let us now call *Bimal the category of tense algebras and homomorphisms, Bifra the category of tense frames and bicontractions*, and let us consider Bifra*, i.e. the subcategory of compact frames, Hausdorff, with r a strongly continuous relation (see [8]). Then the following theorem holds:

THEOREM 1.3. *Bimal and Bifra* are dual one to each other.*

It is obviously impossible to have a duality with Bifra, because the dual of a tense algebra is always compact.

The proof of this theorem is obtained modifying the proof of duality in the modal case (see [8]), according to the observation above, using Lemma 1.2 and the fact that each property true for $\Box$, or τ, or r, is separately true for $\boxminus$, ϱ, r^{-1}.

§ 3. A relevant semantic consequence of the duality theorem is the proof of completeness of first order semantics, also in the tense case.

COROLLARY 1.4. Let $\mathcal{F} = (X, r, \mathcal{C})$ be a tense frame; then $\mathcal{F}$ is equivalent to $\mathcal{F}^+$.

PROOF. Let nt be the natural translation from the set of formulas into the set of terms in our first order language (see [8]). Let us now suppose that, for any valuation α, $\mathrm{val}_\alpha^{\mathcal{F}} P = X$; but $\mathrm{val}_\alpha^{\mathcal{F}} P = X$, if and only if $(\mathrm{nt}\, P)_{\mathcal{F}^+}(\alpha p_1, \ldots, \alpha p_n) = 1$, where $p_1, \ldots, p_n$ are all variables occurring in P, if and only if $\mathrm{val}_\alpha^{\mathcal{F}^+} P = X$.

For any modal algebra $\mathcal{A}$, we call $L\mathcal{A} = \{P: P \text{ holds in } \mathcal{A}\}$ the *logic* of $\mathcal{A}$, and similarly, for any frame $\mathcal{F}$, $L\mathcal{F} = \{P: \mathcal{F} \models P\}$ the logic of $\mathcal{F}$. Also we say that two models are *equivalent* if they give the same logic. A frame $\mathcal{F}$ is said to be *for* the logic L if $L \subseteq L\mathcal{F}$.

COROLLARY 1.5. Let $\mathcal{A}$ be a tense algebra; then $\mathcal{A}$ is equivalent to $\mathcal{A}_+$.

PROOF. It is sufficient to observe that, since $\mathcal{A} \cong \mathcal{A}_+^+$, then $L\mathcal{A}_+ = L\mathcal{A}_+^+ = L\mathcal{A}$. In fact, $\mathcal{A}_+$ is a frame, and then for $\mathcal{A}_+$ Corollary 1.4 holds.

These corollaries are essential to prove the following:

THEOREM 1.6. For any normal tense logic L, $\vdash_L P$ if and only if $\models^{\mathcal{F}} P$, for all first order frames $\mathcal{F}$ for L.

PROOF. The necessity follows from the definition of a frame for L. For the converse, let us observe that $\vdash_L P$ if and only if $\mathrm{nt}\, P = 1 \in \in \mathrm{Id}\; TA_L$ (where TA_L is the class of all L-tense algebras, i.e. the class of all algebras which satisfy the identities, which are the translation, by nt, of formulas valid in L).

So, if $\not\vdash_L P$, then by the completeness of algebraic semantics, there exists a L-tense algebra $\mathcal{A}$, so that $\mathrm{nt}\, P = 1 \notin \mathrm{Id}\, \mathcal{A}$, i.e. $P \notin L\mathcal{A}$. But $L\mathcal{A} = L\mathcal{A}_+$, therefore $P \notin L\mathcal{A}^+$, i.e. $\not\models_L P$, because $\mathcal{A}_+$ is a frame in which P is false.

Let us observe that this is not completeness with respect to the frames with the discrete topology. For example, let us consider the logic with the following axioms:

(a) $\Box \Diamond P \to \Diamond \Box P$,

(b) $\boxminus (\boxminus P \to P) \to \boxminus P$.

It is well known that, if $\mathcal{F} = (X, r, \mathcal{T})$ is a frame with r transitive, (a) is equivalent to the fact that r is upper unbounded, (b) is equivalent to the well-foundedness of r and implies irriflexivity. Let us consider a set S, with both S and its complement cofinal, S upper unbounded and let $x_0 \in S$. Then $x_0 \not\Vdash \Box \Diamond P \to \Diamond \Box P$, and hence the logic with these axioms has no models with the discrete topology. It is however consistent, because $(N, <, \pi)$, where π contains all finite and cofinite subsets of N, is a first order model for that logic.

The results of duality are useful in many applications, for example in the following semantic lemmas, whose proof is an extension of the modal case (see [8]).

LEMMA 1.6 (*Bicontractions-lemma*). Let $\mathcal{F} = (X, r, \mathcal{T})$, $\mathcal{G} = (Y, s, \mathcal{U})$ be tense frames. Let $c\colon \mathcal{F} \to \mathcal{G}$ be a bicontraction, c onto. Then, for any set of formulas Γ and formula P, $\Gamma \models^{\mathcal{F}} P \to \Gamma \models^{\mathcal{G}} P$, (where $\models$ is the strong consequence relation; see [8]). In particular $L\mathcal{F} \subseteq L\mathcal{G}$.

LEMMA 1.7. Let $\mathcal{F} = (X, r, \mathcal{T})$, $\mathcal{G} = (Y, s, \mathcal{U})$ be tense frames. Let $\mathcal{F}$ be a subframe of $\mathcal{G}$. Then $\Gamma \models^{\mathcal{F}} P \to \Gamma \models^{\mathcal{G}} P$, and in particular $L\mathcal{F} \subseteq L\mathcal{G}$.

Chapter 2

§ 1. Now we extend to the tense case the technique introduced in [7] and there called « unravelling technique », and we apply it to the proof of some completeness theorems.

The result of the usual « unravelling » is the transformation of models for logics in equivalent frames, which are generated trees. In the tense case, we will not obtain trees, but more complex structures, which we call « generated nets ».

Let now $\mathcal{M} = (X, r, \alpha)$ be a model of a tense logic L, and $x_0 \in X$. We define the new structure $\mathcal{M}^* = (X^*, r^*, \alpha^*)$ as follows:

i) X^* will consist of sequences $\langle x_0 \varepsilon_0 x_1 \varepsilon_1 \dots x_m \varepsilon_m x_n \rangle$, where $x_i \in X$, for $0 \leqslant i \leqslant n$, and for any i, $0 \leqslant i \leqslant n$, $\varepsilon_i = r$ or $\varepsilon_i = r^{-1}$.

ii) $\langle x_0 \varepsilon_0 x_1 \dots \varepsilon_{m-1} x_m \rangle \varepsilon^* \langle x_0 \varepsilon_0 \dots x_m \varepsilon_m x_n \rangle$ if and only if $x_m \varepsilon x_n$ and $\langle x_0 \varepsilon_0 \dots \varepsilon_{n-1} x_n \rangle = \langle x_0 \varepsilon_0 \dots x_m \rangle \varepsilon x_n$.

iii) for atomic formulas P_i, $\langle x_0 \varepsilon_0 \dots \varepsilon_{n-1} x_n \rangle \in \mathrm{val}^*_\alpha P_i$, if and only if $x_n \in \mathrm{val}_\alpha P_i$.

By induction on the length of formulas, we can show the following:

Lemma 2.8. $\langle x_0 \varepsilon_0 \dots \varepsilon_{n-1} x_n \rangle \Vdash^* P$, if and only if $x_n \Vdash P$, for any formula P.

The result is in conclusion a model equivalent to $\mathcal{M}$ and based on a frame generated by x_0, and with a net structure. In this way, we obtain completeness proofs for several tense logics with no difficulty. In fact, for any modal logic L we define the *tense extension* L_T as the minimal tense logic whose axioms on $\Box$ are the same as in L. More explicitely:

Def. 2.9. We call tense extension of a modal logic L the logic:

$$L_T = L + \begin{cases} A \to \Box \diamondminus A\,, \\ A \to \boxminus \Diamond A\,, \\ MN_2\colon\ A/\boxminus A\,, \\ (\boxminus)\colon\ \boxminus(A \to B) \to \boxminus A \to \boxminus B\,. \end{cases}$$

THEOREM 2.10. If L is proved to be complete using the unravelling technique, then L_T is also complete.

PROOF. First, if $L_T \not\models P$, there exists and $x_0 \in L_T$, with $x_0 \not\Vdash P$; by Lemma 2.8, there exists a sequence $\langle x_0 \rangle$ in the generated net, so that $\langle x_0 \rangle \not\Vdash^* P$.

Conversely, L_T is valid in the generated nets, whose generated subtrees satisfy the same properties of L (with the term subtree, we call any part of the net, which is isomorphic with a tree). This fact depends on our definition of L_T, in which the only axioms with the operator $\boxminus$ are those valid in every tense logic.

§ 2. Let us now consider the logic $K4_T$, which is the extension, in the sense of Def. 2.9, of the modal logic $K4$.

Rules and axioms of $K4_T$ are:

(1) Tautologies,

(2) $\Box(A \to B) \to \Box A \to \Box B$,

(3) $\Box A \to \Box\Box A$,

(4) MP: $A, A \to B/B$,

(5) MN_1: $A/\Box A$,

(6) MN_2: $A/\boxminus A$,

(7) $A \to \Box \diamondminus A$,

(8) $A \to \boxminus \Diamond A$.

For $K4$ the following theorem holds:

THEOREM. $K4$ is complete with respect to the class of transitive trees (see [7]).

Similarly, we prove that:

THEOREM 2.11. $K4_T$ is complete with respect to the class of transitive nets.

PROOF. By Theorem 2.10 and the fact that $K4_T$ is valid in every transitive net.

An other case is that of the tense extension of GL, the modal logic obtained adding $\Box(\Box A \to A) \to \Box A$ (Löb formula) to the minimal normal modal logic K. It is by now well-known that GL is complete with respect to transitive and terminal frames (a frame is said to be terminal if it contains no infinite chains $x_0 r x_1 r x_2 \ldots$) or also with respect to finite irriflexive trees (see [2] or [9]).

We can prove with a modification of the technique above, that GL_T is also complete. Note however that GL_T does not have the finite model property: $\boxminus(\boxminus A \to A) \to \boxminus A$ is false, but only in a lower unbounded frame.

THEOREM 2.12. GL_T is complete with respect to the class of transitive terminal nets.

PROOF. If $GL_T \vdash P$, then $K^0 \models P$ ($K^0 =$ the class of irriflexive, transitive, terminal, lower unbounded nets), because GL_T is valid in this kind of nets.

For the converse, we suppose that $GL_T \nvdash P$. Let $\mathcal{M}$ be a model for GL_T, and $x_0 \in \mathcal{M}$, $x_0 \not\Vdash P$. We generate, by induction, starting from x_0, a net, which falsifies P, in the following way: let the first sequence be $\langle x_0 \rangle$; if $\langle x_0 \varepsilon_0 \ldots \varepsilon_{n-1} x_n \rangle$ is a sequence, we consider the set $S = \{Q\colon Q$ is a subformula of $P\}$. If there exists some $Q \in S$, with $x_n \not\Vdash \Box Q$ (if this Q does not exist, the sequence is complete), there exists some x_t, with $x_n r x_t$ and $x_n \Vdash \neg Q \wedge \Box Q$ (by Löb formula). The new sequence will be the $\langle x_0 \varepsilon_0 \ldots \varepsilon_{n-1} x_n r x_t \rangle$.

The unravelling technique will be applied, starting from x_n, to falsify subformulas of P with $\boxminus$, which are not true in x_n.

Let us observe that our steps regarding the future are finite; in fact, the number of subformulas of P is finite, in particular formulas with $\Box$, and by application of Löb formula, when $x_n \not\Vdash \Box Q$, there exists a point x_m, with $x_n r x_m$ and $x_m \Vdash \Box Q$.

We obtain, by this way, an irriflexive net, which we close transitively. Moreover, $\langle x_0 \rangle \not\Vdash^* P$.

By induction on the structure of formulas, we prove in fact that $\langle x_0 \varepsilon_0 \ldots \varepsilon_{n-1} x_n \rangle \Vdash^* P$ if and only if $x_n \Vdash P$.

The step for atomic formulas is by definition iii).

Inductive steps with $\neg$, $\wedge$ are easy.

If $x_n \Vdash \Box P$, then by definition, for any x_t, if $x_n r x_t$, then $x_t \Vdash P$; this, by inductive hypothesis, gives that for each σ', $\sigma r \sigma'$ implies $\sigma' \Vdash^* P$, where $\sigma = \langle x_0 \varepsilon_0 \ldots \varepsilon_{n-1} x_n \rangle$, $\sigma' = \langle x_0 \varepsilon_0 \ldots \varepsilon_{n-1} x_n r x_t \rangle$. But then $\sigma \Vdash^* \Box P$.

If $x_n \Vdash\!\!\!\!/ \Box P$, then there exists a x_t, with $x_n r x_t$ and such that $x_t \Vdash \Box P \wedge \neg P$; by inductive hypothesis there exists a sequence σ', with $\sigma r \sigma' \wedge \sigma' \Vdash^* \neg P$, where $\sigma' = \langle x_0 \varepsilon_0 \dots \varepsilon_{n-1} x_n r x_t \rangle$; but then $\sigma \Vdash\!\!\!\!/^* \Box P$.

$x_n \Vdash \boxminus P$ if and only if for each x_t, if $x_t r x_n$ then $x_t \Vdash P$, if and only if for each σ', if $\sigma r^{-1} \sigma'$ then $\sigma' \Vdash^* P$, where $\sigma = \langle x_0 \varepsilon_0 \dots \varepsilon_{n-1} x_n \rangle$, $\sigma' = \langle x_0 \varepsilon_0 \dots \varepsilon_{n-1} x_n r^{-1} \dots r^{-1} x_t \rangle$, if and only if $\sigma \Vdash^* \boxminus P$.

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