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Existentially Closed $L\mathfrak{X}$ -Groups.

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1. Introduction.

Let \mathfrak{Y} be a class of groups. A \mathfrak{Y} -group G is *existentially closed* (e.c.) in \mathfrak{Y} , if every system of finitely many equations and inequations with coefficients from G , which has a solution in some \mathfrak{Y} -supergroup of G , can already be solved in G . If \mathfrak{Y} is closed under forming subgroups ($\mathfrak{Y} = S\mathfrak{Y}$), and if \mathfrak{Y} is *inductive* (i.e., unions of ascending chains of \mathfrak{Y} -groups are again \mathfrak{Y} -groups), then every \mathfrak{Y} -group of cardinality \aleph is contained in an e.c. \mathfrak{Y} -group of cardinality $\max\{\aleph, \aleph_0\}$ (cf. J. Hirschfeld and W. H. Wheeler [8], Proposition I.1.3).

The aim of this paper is the investigation of the structure of e.c. $L\mathfrak{X}$ -groups, where \mathfrak{X} is a group class satisfying certain closure properties and where $L\mathfrak{X}$ denotes the class of all locally- \mathfrak{X} -groups. This is motivated by results of P. Hall [3] (cf. also O. H. Kegel and B. A. F. Wehrfritz [13], Chapter 6) and B. Maier [17] about e.c. groups in the classes $L\mathfrak{F}$ of all locally finite groups and $L\mathfrak{F}_p$ of all locally finite p -groups. For any class \mathfrak{X} the class $L\mathfrak{X}$ is inductive. If $\mathfrak{X} = S\mathfrak{X}$, it can be shown by transfinite induction on the cardinality of the $L\mathfrak{X}$ -groups, that $L\mathfrak{X}$ is the smallest inductive group class containing \mathfrak{X} .

We will always consider classes \mathfrak{X} satisfying

- (1) $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$ (\mathfrak{X} is closed under forming subgroups and extensions; cf. D. J. S. Robinson [21], § 1.1),

as well as some of the following properties:

- (2) cartesian powers of finitely generated (f.g.) \mathfrak{X} -groups are \mathfrak{X} -groups;

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(2') direct powers of f.g. \mathfrak{X} -groups are \mathfrak{X} -groups;

(3) $\mathfrak{X} = Q\mathfrak{X}$ (\mathfrak{X} is closed under forming homomorphic images).

$\mathfrak{X} = S\mathfrak{X}$ implies, that \mathfrak{X} contains the cyclic group C_m of order $m \in \mathbb{N} \cup \{\infty\}$, if and only if there exists an \mathfrak{X} -group containing an element of order m . We denote by $\pi_{\mathfrak{X}}$ the set of all primes p such that $C_p \in \mathfrak{X}$. Then torsion elements of $L\mathfrak{X}$ -groups are always $\pi_{\mathfrak{X}}$ -elements.

Examples for \mathfrak{X} are provided by the following classes:

- \mathfrak{S}_{π} : class of all soluble groups, whose torsion-elements are π -elements ((1), (2); $L\mathfrak{X} = L\mathfrak{S}_{\pi}$);
- \mathfrak{S} : class of all soluble groups ((1), (2), (3); $L\mathfrak{X} = L\mathfrak{S}$);
- \mathfrak{P}_{π} : class of all (periodic) π -groups ((1), (2'), (3); $L\mathfrak{X} = \mathfrak{P}_{\pi}$);
- $L\mathfrak{F}_{\pi}$: class of all locally finite π -groups ((1), (2), (3); $L\mathfrak{X} = L\mathfrak{F}_{\pi}$);
- $L(\mathfrak{F}_{\pi} \cup \mathfrak{S})$: class of all locally finite-soluble π -groups ((1), (2), (3); $L\mathfrak{X} = L(\mathfrak{F}_{\pi} \cup \mathfrak{S})$).

The greatest problem when investigating e.c. $L\mathfrak{X}$ -groups is the construction of supergroups, in which useful systems of equations and inequations are solvable and which are still contained in the class $L\mathfrak{X}$. Our main tool will be the embedding of e.c. $L\mathfrak{X}$ -groups into certain subgroups of regular wreath products; in Section 4 a new technique for the embedding of countable groups into unrestricted wreath products will be developed.

By P. Hall [3], Theorem 1 (cf. also A. Macintyre and S. Shelah [16], §1.4) and by B. Maier [17], Satz 2 there exist up to isomorphism exactly one countable, e.c. $L\mathfrak{F}$ -group ULF and exactly one countable, e.c. $L\mathfrak{F}_p$ -group E_p . Both are verbally complete (P. Hall [3], §2.1; B. Maier [17], Satz 7). In Section 2 we will therefore pursue the question, which e.c. $L\mathfrak{X}$ -groups are verbally complete. Assuming (1) it will be shown that every periodic, e.c. $L\mathfrak{X}$ -group is verbally complete (Theorem 2.1), whereas (1) and (2) imply, that an e.c. $L\mathfrak{X}$ -group G is verbally complete, if and only if for every $g \in G$ and every $r \in \mathbb{N}$ there exists an $h \in G$ such that $h^r = g$ (Theorem 2.7). The latter holds for example, if $\pi_{\mathfrak{X}}$ is the set of all primes (Theorem 2.8).

In any verbally complete group H we have $V(H) = H$ for every verbal subgroup $V(H)$. But the rule V on the class of all groups is functorial and coradical (cf. p. 202 for definitions), and assuming (1)

and (2) we can show, that $\mathfrak{r}(G) = G$ for a functorial, coradical rule \mathfrak{r} on $L\mathfrak{X}$ and for every e.c. $L\mathfrak{X}$ -group G , if there exists a finite \mathfrak{X} -group F such that $\mathfrak{r}(F) \neq 1$. Correspondingly, (1) and (2') imply $\mathfrak{r}(G) = 1$ for a functorial, radical rule \mathfrak{r} on $L\mathfrak{X}$, if there exists a finite \mathfrak{X} -group F with $\mathfrak{r}(F) \neq F$ (Theorem 3.4). Hence, the behaviour of these rules for e.c. $L\mathfrak{X}$ -groups depends heavily on their behaviour for finite \mathfrak{X} -groups. In particular, the Baer-radical of every e.c. $L\mathfrak{X}$ -group is trivial, if (1), (2') and $\pi_{\mathfrak{X}} \neq \emptyset$ hold, and the Hirsch-Plotkin-radical is trivial too for (1), (2') and $|\pi_{\mathfrak{X}}| \geq 2$ (Theorem 3.1). Section 3 will end with some remarks about characteristically simple, e.c. $L\mathfrak{X}$ -groups.

In Section 4 we will study the structure of the normal subgroups of countable, e.c. $L\mathfrak{X}$ -groups. There, we will always assume (1), (2) and (3). The restriction to countable groups arises from the use of the above-mentioned new embedding technique into unrestricted wreath products. However, in the case that \mathfrak{X} is a class of locally finite groups, we will be able to show in a subsequent paper, that some of the results of Section 4 even hold for e.c. $L\mathfrak{X}$ -groups of arbitrary cardinality.

The first result (Theorem 4.7) is the most important one: Every countable, e.c. $L\mathfrak{X}$ -group G has a unique chief series, i.e., the lattice of all normal subgroups is totally ordered by inclusion. The normal closures $\langle g^G \rangle$, $g \in G$, are exactly the normal subgroups occurring as the groups M in the chief factors M/N . If K is a non-trivial normal subgroup of G with $K \neq \langle g^G \rangle$ for all $g \in G$, then K is e.c. in G , i.e., every system of finitely many equations and inequations with coefficients from K , which is solvable in G , can already be solved in K (Theorem 4.8). In particular, every automorphism of K , which is induced by an inner automorphism of G , is locally inner. In Theorems 4.9 and 4.10 certain results on how G acts on its chief factors by conjugation will be obtained.

During the rest of Section 4 we will assume in addition to (1), (2), (3), that for every $g \in G$ there exists a verbal subgroup of the normal closure $\langle g^G \rangle$, which is different from $\langle g^G \rangle$. This holds for example, if $\mathfrak{X} = \mathfrak{S}$ or $\mathfrak{X} = L(\mathfrak{F}_\pi \cap \mathfrak{S})$, since the chief factors of locally soluble groups are always abelian. Under these conditions we will prove a second remarkable fact (Theorem 4.11). The order-type of the unique chief series of G is the order of the rationals, i.e., the chief factors M/N of G can be indexed with the rationals in such a way, that $q_1 < q_2 \Leftrightarrow M_{q_1} < N_{q_2}$ for all $q_1, q_2 \in \mathbb{Q}$. Hence, the structure of the normal subgroup lattice of G can be described as follows. Every

$q \in \mathbf{Q}$ corresponds with a chief factor M_q/N_q , while every $r \in \mathbf{R} \setminus \mathbf{Q}$ corresponds with a normal subgroup K_r satisfying $q_1 < r < q_2 \Leftrightarrow M_{q_1} < < K_r < N_{q_2}$. A further result is that for any proper subnormal subgroup S of G there exists a chief factor M/N such that $N \leq S \leq M$. Hence the defect of the subnormal subgroups of G is bounded by 2, if \mathfrak{X} is a class of locally soluble groups.

Section 5 contains some remarks about the automorphism group of a countable, e.c. $L\mathfrak{X}$ -group G . Assuming (1) we will construct for every possible order 2^{\aleph_0} locally inner automorphisms of this order for G (Theorem 5.1). Moreover, the existence of locally inner automorphisms of finite order, whose product has infinite order, will be established. This implies that the amalgamation property fails in the class of all countable, e.c. $L\mathfrak{X}$ -groups, if every \mathfrak{X} -group is periodic. Finally—assuming (1), (2) and (3)—we will show that periodic automorphisms of a countable, e.c. $L\mathfrak{X}$ -group, which centralize every chief factor, are locally inner (Theorem 5.3).

In contrast to the situation for $\mathfrak{X} = L\mathfrak{F}_p$ or $\mathfrak{X} = L\mathfrak{F}$ it will be noted in Section 6, that there exist 2^{\aleph_0} non-isomorphic, countable, e.c. $L\mathfrak{S}_\pi$ -groups, if π is an infinite set of primes.

The results of this paper were part of the author's doctoral thesis [15] at Albert-Ludwigs-Universität Freiburg i. Br.

NOTATION. As far as basic definitions are concerned the reader is referred to B. Huppert [10] and D. J. S. Robinson [21].

The following symbols will be used:

C_m	cyclic group of order m ;
E_p	B. Maier's unique countable, e.c. $L\mathfrak{F}_p$ -group;
ULF	P. Hall's unique countable, e.c. $L\mathfrak{F}$ -group;
$\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$	sets of natural numbers, integers, rationals, real numbers;
$ \mathcal{G} , o(g)$	cardinality of the group \mathcal{G} , order of the element g ;
\leq, \trianglelefteq	subgroup, normal subgroup;
$U \cong G$	U is isomorphic to a subgroup of G ;
$\langle A \rangle$	group generated by the set A ;
$[x, y]$	commutator ($= x^{-1}y^{-1}xy$);
$C_G(U), N_G(U)$	centralizer and normalizer of U in G ;

$Z(G), G'$	centre of G , commutator subgroup of G ;
$U \hookrightarrow G$	embedding of U into G ;
$\theta \upharpoonright U$	restriction of the map θ to U ;
$\text{Kern } \varrho, \text{Im } \varrho$	kernel and image of the homomorphism ϱ ;
$f \equiv y$	the map f takes the value y everywhere.

2. Verbal completeness.

A group G is said to be *verbally complete*, if for every non-trivial word $w(x_1, \dots, x_r)$ and for every element $g \in G$ there exist $g_1, \dots, g_r \in G$ such that $g = w(g_1, \dots, g_r)$. By P. Hall [3], § 2.1 and B. Maier [17], Satz 7 the groups ULF and E_p are verbally complete. Using Lemma 7 of P. Hall [3] we can show that every periodic, e.c. $L\mathfrak{X}$ -group is verbally complete.

THEOREM 2.1. *Let $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$. Then for every torsion-element g of an e.c. $L\mathfrak{X}$ -group G and for every non-trivial word $w(x_1, \dots, x_r)$ there exist $g_1, \dots, g_r \in G$ such that $g = w(g_1, \dots, g_r)$. In particular, every periodic, e.c. $L\mathfrak{X}$ -group is verbally complete.*

In the proof of Theorem 2.1 we will make use of the *unrestricted (regular) wreath product* $A \text{ Wr } B$ of two groups A and B ; if A^B denotes the group of all maps $f: B \rightarrow A$ (under component-wise multiplication $(f \cdot g)(b) = f(b) \cdot g(b)$), then the group $A \text{ Wr } B$ is the set $\{(b, f): b \in B, f \in A^B\}$ with multiplication rule $(b_1, f_1)(b_2, f_2) = (b_1 b_2, f_1^{b_2} f_2)$, where $f_1^{b_2}(b) = f_1(b_2 b)$. Hence the wreath product $A \text{ Wr } B$ is a semi-direct product of the *base-group* $\{(1, f): f \in A^B\} \cong A^B$ by the *top group* $\{(b, 1): b \in B\} \cong B$. As usual, the *diagonal* of $A \text{ Wr } B$ is $\{(1, f_a): a \in A\}$, where $f_a(b) = a$ for all $b \in B$, and the *1-component* of $A \text{ Wr } B$ is $\{(1, g_a): a \in A\}$, where

$$g_a(b) = \begin{cases} a & \text{if } b = 1, \\ 1 & \text{else.} \end{cases}$$

For $U \triangleleft A$ and $V \triangleleft B$ the subgroup $\{(b, f): b \in V, f(B) \subseteq U, f(b') = 1 \text{ for all } b' \in B \setminus V\} \cong U \text{ Wr } V$ of $A \text{ Wr } B$ will also be denoted by $U \text{ Wr } V$.

Now suppose $N \trianglelefteq G$ and let $\theta: G \rightarrow G/N$ be the canonical epi-

morphism. Then there exist so-called Krasner-Kaloujnine-embeddings $G \hookrightarrow N \text{Wr } G/N$, which are given as follows: Choose a counter-map $\theta^*: G/N \rightarrow G$, i.e., a map θ^* assigning to each element of G/N a preimage under θ ; then a Krasner-Kaloujnine-embedding $\sigma: G \hookrightarrow N \text{Wr } G/N$ is defined by $g\sigma = (g\theta, f_\sigma)$, where $f_\sigma(h\theta) = [(gh)\theta\theta^*]^{-1} \cdot g \cdot h\theta\theta^*$.

PROOF OF THEOREM 2.1. Fix any $g \in G$ of finite order m and any word $w(x_1, \dots, x_\nu) \neq 1$. Then Lemma 7 of P. Hall [3] ensures the existence of a finite, nilpotent $\pi_{\mathfrak{X}}$ -group $A = \langle a_1, \dots, a_\nu \rangle$ such that $w = w(a_1, \dots, a_\nu)$ is an element of order m in the centre $Z(A)$ of A . Because of $w \in Z(A)$ a Krasner-Kaloujnine-embedding $\sigma: A \hookrightarrow W_1 = \langle w \rangle \text{Wr } A/\langle w \rangle$ carries w into the diagonal of W_1 . Identifying $w \in \langle w \rangle$ with $g \in G$ we can regard W_1 as the subgroup $\langle g \rangle \text{Wr } A/\langle w \rangle$ of $W_2 = G \text{Wr } A/\langle w \rangle$ and σ as an embedding of A into W_2 . Now identify G with the diagonal of W_2 in the obvious way. Because of $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$ the group W_2 is an $L\mathfrak{X}$ -supergroup of the e.c. $L\mathfrak{X}$ -group G , and the equation $g = w(x_1, \dots, x_\nu)$ has the solution $a_1\sigma, \dots, a_\nu\sigma$ in W_2 . \square

Although the class $L(\mathfrak{F}_\pi \cap \mathfrak{R})$ of all locally finite-nilpotent π -groups is not closed with respect to forming extensions, Theorem 2.1 has the following

COROLLARY 2.2. *Every e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{R})$ -group is verbally complete.*

PROOF. Let G be an e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{R})$ -group and choose $w(x_1, \dots, x_\nu) \neq 1$. By B. Maier [17], Satz 5 the group G is a direct product of e.c. $L\mathfrak{F}_p$ -groups G_p , $p \in \pi$. Hence for any $g \in G$ there exist $p_1, \dots, p_k \in \pi$ and $g_i \in G_{p_i}$ such that $g = g_1 \dots g_k$. By Theorem 2.1 we get $h_{i1}, \dots, h_{i\nu} \in G_{p_i}$ with $g_i = w(h_{i1}, \dots, h_{i\nu})$ for $1 \leq i \leq k$. Now $g = w(h_{11} \dots h_{k1}, \dots, h_{1\nu} \dots h_{k\nu})$. \square

Next we will consider group classes containing also non-periodic groups.

THEOREM 2.3. *Let $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$; in the case $C_\infty \in \mathfrak{X}$ let \mathfrak{X} satisfy (2). Then every element of an e.c. $L\mathfrak{X}$ -group is a commutator.*

PROOF. By Theorem 2.1 we can restrict our attention to the case that $C_\infty \in \mathfrak{X}$. Let G be any e.c. $L\mathfrak{X}$ -group, let Σ denote the local system of all f.g. subgroups of G , and define W to be the union of the subgroups

$S \text{ Wr } C_\infty$, $S \in \Sigma$, of the wreath product $G \text{ Wr } C_\infty$. Then W is an $L\mathfrak{X}$ -group. By identifying G with the diagonal of $G \text{ Wr } C_\infty$ we have $G \leq W$. Because of P. M. Neumann [20], Corollary 5.3 every element of the base group of $S \text{ Wr } C_\infty$, $S \in \Sigma$, is a commutator. Hence the equation $g = [x_1, x_2]$ has a solution for every $g \in G$ in the $L\mathfrak{X}$ -supergroup W of G . \square

If $V = \{w_i(x_1, \dots, x_r) : i \in I\}$ is a non-empty set of non-trivial words, then $V(G) = \langle w_i(g_1, \dots, g_r) : i \in I, g_k \in G \rangle$ is called the *verbal subgroup* of the group G generated by V . Every verbal subgroup is characteristic. For example, all terms of the lower central series and the derived series of G are verbal subgroups (generated by a single word). Theorem 2.1 gives us $G = V(G)$ for every periodic, e.c. $L\mathfrak{X}$ -group G and all non-trivial sets V of words. This can be generalized as follows.

THEOREM 2.4. *Let \mathfrak{X} satisfy (1), (2) and $\pi_{\mathfrak{X}} \neq \emptyset$. Then $G = V(G)$ for every e.c. $L\mathfrak{X}$ -group G and all non-trivial sets V of words.*

PROOF. By Theorem 2.1 every torsion-element of G is contained in $V(G)$. Hence it suffices to prove Lemma 2.5. \square

LEMMA 2.5. *Let $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$ and $p \in \pi_{\mathfrak{X}}$; in the case $C_\infty \in \mathfrak{X}$ let \mathfrak{X} satisfy (2). Then every finite subset of an e.c. $L\mathfrak{X}$ -group G is contained in the normal closure of an element of order p in G . In particular, every element of G is a product of elements of order p in G .*

PROOF. Identify G with the 1-component of the $L\mathfrak{X}$ -group $W = G \text{ Wr } C_p$. By H. Neumann and J. Wiegold [19], Lemma 2.9 the commutator subgroup G' of G is contained in the normal closure of the top group of W , and by Theorem 2.3 we have $G' = G$. \square

With respect to Theorems 2.1, 2.3, and 2.4 we pose the

QUESTION. *Is every e.c. $L\mathfrak{X}$ -group verbally complete, if \mathfrak{X} satisfies (1) and (2)?*

We now try to approach an answer to this question by using the idea of the proof of Theorem 2.1. At first we need a pendant to P. Hall [3], Lemma 7 for elements of infinite order.

LEMMA 2.6. *For every non-trivial word $w(x_1, \dots, x_r)$ there exists a torsion-free, nilpotent group $A = \langle a_1, \dots, a_r \rangle$ such that every factor*

of the lower central series of A is free abelian of finite rank and such that $w(a_1, \dots, a_r)$ is a non-trivial element in the last term of the lower central series of A .

PROOF. Consider the free group F in generators x_1, \dots, x_r . By well-known theorems of W. Magnus and E. Witt (cf. B. Huppert and N. Blackburn [11], Theorems VIII.11.12 and VIII.11.15) the intersection of the terms $F = F_1 \supseteq F_2 \supseteq \dots$ of the lower central series of F is trivial, while the factors F_k/F_{k+1} are free abelian of finite rank. Choose $l \in \mathbb{N}$ with $w(x_1, \dots, x_r) \in F_l \setminus F_{l+1}$, and let $A = F/F_{l+1}$, $a_i = x_i F_{l+1}$. \square

A group G is π -divisible (for a set π of primes), if for every $g \in G$ and every π -number $r \in \mathbb{N}$ there exists an $h \in G$ such that $h^r = g$. In the case that π is the set of all primes we call G divisible. Clearly, every verbally complete group is divisible. Assuming (1) and (2) we can now prove a converse for e.c. $L\mathfrak{X}$ -groups.

THEOREM 2.7. *Let \mathfrak{X} satisfy (1) and (2). Then for every e.c. $L\mathfrak{X}$ -group G the following three conditions are equivalent:*

- (a) G is verbally complete.
- (b) G is divisible.
- (c) For every $g \in G$ of infinite order and for every $r \in \mathbb{N}$ there exists an $h \in G$ such that $h^r = g$.

PROOF. We only have to show that (c) implies (a). Fix $g \in G$ and $w(x_1, \dots, x_r) \neq 1$. If g has finite order, then Theorem 2.1 ensures the existence of $g_1, \dots, g_r \in G$ such that $g = w(g_1, \dots, g_r)$. Otherwise, let A be the group given by Lemma 2.6. Denote the last non-trivial term of the lower central series of A by A^* . Then $w = w(a_1, \dots, a_r) \in A^* \setminus 1$. As a subgroup of A^* the preimage T in A^* of the torsion-subgroup of $A^*/\langle w \rangle$ is free abelian of finite rank, i.e., $T = \bigotimes_{i=1}^s \langle t_i \rangle$, where $o(t_i) = \infty$.

Suppose, $s \geq 2$. Since $T/\langle w \rangle$ is periodic, there exist integers m_1, m_2, r_1, r_2 with $t_1^{m_1} = w^{r_1}$ and $t_2^{m_2} = w^{r_2}$. Hence $w^{r_1 r_2} \in \langle t_1 \rangle \cap \langle t_2 \rangle = 1$, contradicting $o(w) = \infty$. Therefore, $T = \langle t \rangle$ is infinite cyclic, and $w = t^r$ for some $r \in \mathbb{N}$.

By (c) there exists an $h \in G$ with $h^r = g$. Now, proceeding as in the proof of Theorem 2.1, we can identify G with the diagonal of $W = G \text{ Wr } A/T$ and find a Krasner-Kaloujnine-embedding $\sigma: A \hookrightarrow$

$\hookrightarrow T \text{ Wr } A/T = \langle h \rangle \text{ Wr } A/T \leq W$ such that $h = t\sigma$. Hence the equation $g = w(x_1, \dots, x_r)$ has the solution $a_1\sigma, \dots, a_r\sigma$ in W , and we only have to show that $\langle G, A\sigma \rangle$ is an $L\mathfrak{X}$ -subgroup of W .

Because of $o(g) = \infty$ we have $C_\infty \in \mathfrak{X}$. Since A^*/T is torsion-free and f.g., A/T has a finite series with free abelian factors of finite rank. Hence, $\mathfrak{X} = P\mathfrak{X}$ implies that A/T is an \mathfrak{X} -group. Now, let W_0 be a f.g. subgroup of $\langle A\sigma, G \rangle$. Then there exist $h_1, \dots, h_k \in G$ such that $W_0 \leq \langle A\sigma, h_1, \dots, h_k \rangle$. Since $a\sigma = (aT, f_a)$ for every $a \in A$, where $f_a(A/T) \subseteq \langle h \rangle$, it follows that $W_0 \leq V = \langle h, h_1, \dots, h_k \rangle \text{ Wr } A/T \leq W$. But V is an \mathfrak{X} -group by (1) and (2). \square

Theorem 2.7 shows that the main problem when trying to prove the verbal completeness of a non-periodic, e.c. $L\mathfrak{X}$ -group G is the solution of the equation $g = x^r$ for every torsion-free $g \in G$ and every $r \in \mathbb{N}$. If we identify G with the diagonal of $W = G \text{ Wr } C_r$, such a solution is given in W by the element (c, f) , where $C_r = \langle c \rangle$ and

$$f(c^i) = \begin{cases} g & \text{for } i = 0, \\ 1 & \text{for } 1 \leq i \leq r - 1. \end{cases}$$

But under which conditions will $\langle G, (c, f) \rangle$ be an $L\mathfrak{X}$ -group? Assuming (1) this will be the case, if $\pi_{\mathfrak{X}}$ contains every prime.

THEOREM 2.8. *Let \mathfrak{X} satisfy (1) and (2). If $\pi_{\mathfrak{X}}$ is the set of all primes, then every e.c. $L\mathfrak{X}$ -group is verbally complete.*

Otherwise, the above method only enables us to show that every e.c. $L\mathfrak{X}$ -group is $\pi_{\mathfrak{X}}$ -divisible. However, the following theorem should be noted at this point.

THEOREM 2.9. *Denote by $L\mathfrak{N}_\pi$ the class of all locally nilpotent groups, whose torsion-subgroup is a π -group, and let π' be the set of all primes not contained in π . Then every e.c. $L\mathfrak{N}_\pi$ -group is π' -divisible.*

PROOF. By M. I. Kargapolov [12], Theorem every $L\mathfrak{N}_\pi$ -group can be embedded into a π' -divisible $L\mathfrak{N}_\pi$ -group. This holds in particular for e.c. $L\mathfrak{N}_\pi$ -groups. \square

Unfortunately, the proof of Kargapolov's theorem cannot be transferred to other group classes, since it makes use of the fact, that every f.g. \mathfrak{N}_π -group is residually a finite π -group (cf. K. W. Gruenberg [2]).

3. Some characteristic subgroups.

Assuming (1), (2) and $\pi_{\mathfrak{X}} \neq \emptyset$, Theorem 2.4 states that $V(G) = G$ for every e.c. $L\mathfrak{X}$ -group G and every non-trivial set V of words. Since the verbal subgroups are characteristic, we will now pursue the question, whether there exist other rules C attaching to each $L\mathfrak{X}$ -group H a characteristic subgroup $C(H)$ such that $C(G) = G$ or $C(G) = 1$ for every e.c. $L\mathfrak{X}$ -group G . At first, B. Maier's result [17], Hilfsatz 2 saying that the Fitting-radical of E_p is trivial can be generalized as follows.

THEOREM 3.1. *Let \mathfrak{X} satisfy (1), (2') and $\pi_{\mathfrak{X}} \neq \emptyset$. Then the Baer-radical of every e.c. $L\mathfrak{X}$ -group G is trivial. The Hirsch-Plotkin-radical of G is trivial too, if $|\pi_{\mathfrak{X}}| \geq 2$.*

In the proof of Theorem 3.1 we will make use of the *restricted (regular) wreath product* $A \text{ wr } B$ of two groups A and B , which is just the subgroup $\{(b, f) : b \in B, f(b') \neq 1 \text{ for finitely many } b' \in B\}$ of the unrestricted wreath product $A \text{ Wr } B$. All remarks of p. 195 concerning notation apply correspondingly. We will also use the following two lemmata.

LEMMA 3.2. *Let \mathfrak{X} satisfy (1) and (2'). If \mathfrak{X} contains the finite group F , then every torsion element g of an e.c. $L\mathfrak{X}$ -group G is contained in a characteristically simple, locally finite subgroup U_g of G , which contains an isomorphic image of F and has trivial Baer-radical.*

PROOF. Consider the $L\mathfrak{X}$ -supergroup $G \times F$ of G . Since G is an e.c. $L\mathfrak{X}$ -group, this gives us an embedding $\varrho_1: \langle g \rangle \times F \hookrightarrow G$ with $g\varrho_1 = g$. Let $H = \text{Im } \varrho_1$.

Next, we will construct a finite subgroup G_0 of G with $H \leq G_0'$. Let $e = g_0, g_1, \dots, g_n$ be the elements of H . Clearly, $H_0 = H$ is a finite subgroup of G with $g_0 \in H_0'$. Suppose by induction, that we have found a finite subgroup H_{k-1} of G with $H \leq H_{k-1}$ and $g_0, \dots, g_{k-1} \in H_{k-1}'$ for some $k \leq n$. Because of P. Hall [3], Lemma 7, there exists a finite, nilpotent $\pi_{\mathfrak{X}}$ -group $A = \langle a_1, a_2 \rangle$ such that $a = [a_1, a_2]$ is an element of order $o(g_k)$ in the centre of A . Proceeding as in the proof of Theorem 2.1 we can identify G with the diagonal of $W_1 = G \text{ Wr } A / \langle a \rangle$ and find an embedding $\sigma: A \hookrightarrow \langle g_k \rangle \text{ Wr } A / \langle a \rangle \leq W_1$ with $a\sigma = g_k$. But W_1 is an $L\mathfrak{X}$ -supergroup of G containing the finite subgroup

$H_{k-1} \text{Wr } A/\langle a \rangle$ satisfying $\langle H_{k-1}, a_1\sigma, a_2\sigma \rangle \leq H_{k-1} \text{Wr } A/\langle a \rangle$ and $g_k = [a_1\sigma, a_2\sigma]$. This gives us an embedding $\varrho_2: H_{k-1} \text{Wr } A/\langle a \rangle \hookrightarrow G$ with $g'\varrho_2 = g'$ for all $g' \in H_{k-1}$. Let $H_k = \text{Im } \varrho_2$. Then $g_0, \dots, g_{k-1} \in H'_{k-1} \leq H'_k$, and $g_k = [a_1\sigma\varrho_2, a_2\sigma\varrho_2] \in H'_k$ as well as $H \leq H_{k-1} \leq H_k$. Finally, the above construction yields $H \leq H'_n$. Let $G_0 = H_n$.

Now, we will construct a chain $G_0 \leq G_1 \leq G_2 \leq \dots$ of finite subgroups of G with $G_n \cong G_0 \text{ wr } (G_{n-1} \text{ wr } G_0)$, where G_{n-1} is the 1-component of the top group 1 wr $(G_{n-1} \text{ wr } G_0) \cong G_{n-1} \text{ wr } G_0$ of G_n . Assume by induction, that G_{n-1} has already been obtained. Identify G in the natural way with the 1-component of the top group of $W_2 = G_0 \text{ wr } (G \text{ wr } G_0)$. By (1) and (2'), W_2 is an $L\mathfrak{X}$ -supergroup of G containing the finite subgroup $G_0 \text{ wr } (G_{n-1} \text{ wr } G_0)$. Since G is an e.c. $L\mathfrak{X}$ -group, this gives us G_n , and the chain is constructed.

Let U_σ be the commutator subgroup of the union $V = \bigcup_{n \in \mathbb{N}} G_n$.

Since every G_n is finite, U_σ must be locally finite. Moreover, it can be shown, that V is isomorphic to the generalized restricted (regular) wreath product $\text{Wr } G_0^Z$, which has been introduced by P. Hall in [4] (cf. F. Leinen [15], § II.2 for details). Therefore, by P. Hall [4], Theorem D, the group U_σ is characteristically simple. We also have $g \in H \leq G'_0 \leq V' = U_\sigma$ and $F \leq H \leq U_\sigma$. Finally, P. Hall [4], Theorem B yields that the Baer-radical of V is trivial, and since every subnormal subgroup of U_σ is subnormal in V , the Baer-radical of U_σ is contained in the Baer-radical of V . \square

LEMMA 3.3. *Let \mathfrak{X} satisfy (1), (2') and $\pi_{\mathfrak{X}} \neq \emptyset$. Then the normal closure of any non-trivial element of an e.c. $L\mathfrak{X}$ -group G contains a non-trivial element of finite order.*

PROOF. Let $p \in \pi_{\mathfrak{X}}$ and $C_p = \langle c \rangle$. Identify G with the top group of $W = (C_p \text{ wr } G) \in L\mathfrak{X}$. If $f_c: G \rightarrow C_p$ denotes the map given by

$$f_c(g') = \begin{cases} c & \text{for } g' = 1, \\ 1 & \text{else,} \end{cases}$$

then $[(g, 1), (1, f_c)] = (1, f_g)$ for every $g \in G \setminus 1$, where

$$f_g(g') = \begin{cases} c & \text{for } g' = 1, \\ c^{-1} & \text{for } g' = g^{-1}, \\ 1 & \text{else.} \end{cases}$$

Hence $(1, f_g)$ is an element of order p in the normal closure of g in W . This can be expressed by two equations and one inequation with coefficient $g \in G$. \square

PROOF OF THEOREM 3.1. Denote by B and N the rules assigning to each group H its Baer-radical $B(H)$ and its Hirsch-Plotkin-radical $N(H)$. Since $B(G) \trianglelefteq G$ and $N(G) \trianglelefteq G$, Lemma 3.3 yields that it suffices to show that $B(G)$ and $N(G)$ cannot contain any non-trivial torsion element. Therefore, choose $g \in G \setminus 1$ of finite order. By Lemma 3.2 we obtain a subgroup U_g of G with $g \in U_g$ and $B(U_g) = 1$. Hence, $\langle g \rangle \cap B(G) \leq U_g \cap B(G) \leq B(U_g) = 1$ and $g \notin B(G)$.

Now let $p, q \in \pi_{\mathfrak{X}}$ be distinct primes. Then Lemma 3.2 gives a characteristically simple, locally finite subgroup U_g of G containing g and an isomorphic image of $F = C_p \times C_q \in \mathfrak{X}$. Suppose, $N(U_g) = U_g$. Then U_g is the direct product of its primary components, and by choice of F there are two different, non-trivial primary components in U_g . But this contradicts the characteristic simplicity of U_g . Hence $N(U_g) \neq U_g$ and $N(U_g) = 1$. Now $\langle g \rangle \cap N(G) \leq U_g \cap N(G) \leq N(U_g) = 1$. \square

Let \mathfrak{Y} be an arbitrary class of groups, and let \mathfrak{r} be a rule assigning to each \mathfrak{Y} -group S a subgroup $\mathfrak{r}(S)$ of S . Then the rule \mathfrak{r} on \mathfrak{Y} is said to be *radical* (*coradical*), if $\mathfrak{r}(T) \cap S \leq \mathfrak{r}(S)$ (resp. $\mathfrak{r}(S) \leq \mathfrak{r}(T)$) for any two \mathfrak{Y} -groups S and T with $S \leq T$. The rule \mathfrak{r} on \mathfrak{Y} is *functorial*, if $(\mathfrak{r}(S))\alpha = \mathfrak{r}(S\alpha)$ for any isomorphism $\alpha: S \rightarrow S\alpha$ of the \mathfrak{Y} -group S . (In this case $\mathfrak{r}(S)$ is characteristic in S .) Examples for functorial, radical rules on the class of all groups are the rules B , N and Z_n , $n \in \mathbb{N}$, assigning to each group H the Baer-radical $B(H)$, the Hirsch-Plotkin-radical $N(H)$ and the n -th term $Z_n(H)$ of the upper central series. Examples for functorial, coradical rules on the class of all groups are the rules V (V a non-trivial set of words) assigning to each group H the verbal subgroup $V(H)$.

THEOREM 3.4. *Let \mathfrak{X} satisfy (1) and (2'), and let \mathfrak{r} be a rule on the class $L\mathfrak{X}$.*

(a) *If \mathfrak{r} is functorial and radical, and if there exists a finite \mathfrak{X} -group F with $\mathfrak{r}(F) \neq F$, then $\mathfrak{r}(G) = 1$ for every e.c. $L\mathfrak{X}$ -group G .*

(b) *In the case $C_\infty \in \mathfrak{X}$ let \mathfrak{X} satisfy (2) too. If \mathfrak{r} is functorial and coradical, and if there exists a finite \mathfrak{X} -group F with $\mathfrak{r}(F) \neq 1$, then $\mathfrak{r}(G) = G$ for every e.c. $L\mathfrak{X}$ -group G .*

PROOF. (a) Since r is functorial, Lemma 3.3 yields that it is enough to show that $r(G)$ contains no non-trivial element of finite order. Let $g \in G \setminus 1$ be any torsion element. By Lemma 3.2 the element g is contained in a characteristically simple subgroup U_g of G , which contains an isomorphic copy \tilde{F} of F too. Hence $r(U_g) \cap \tilde{F} \leq r(\tilde{F}) \neq \tilde{F}$. Therefore $r(U_g) \neq U_g$ (otherwise $\tilde{F} = r(U_g) \cap \tilde{F} \leq r(\tilde{F}) \neq \tilde{F}$ would be a contradiction). Since U_g is characteristically simple, this implies $r(U_g) = 1$. But then $r(G) \cap \langle g \rangle \leq r(G) \cap U_g \leq r(U_g) = 1$ and $g \notin r(G)$.

(b) By Lemma 2.5 we only have to show that $r(G)$ contains every torsion element of G . Let $g \in G \setminus 1$ be of finite order. Again g is contained in a characteristically simple subgroup U_g of G with $F \cong \tilde{F} \leq U_g$. Then $1 \neq r(\tilde{F}) \leq r(U_g)$. Hence $U_g = r(U_g)$ and $g \in U_g = r(U_g) \leq r(G)$. \square

If \mathfrak{X} is a class of locally finite groups, V. Stingl has shown in [22], Satz 9 and Bemerkung 10 (a), that (because of $\mathfrak{X} = S\mathfrak{X}$) every functorial, radical (coradical) rule on the class of all finite \mathfrak{X} -groups can be continued to a functorial, radical (coradical) rule r on $L\mathfrak{X}$ in such a way that

$$(*) \quad r(H) = \bigcup_{S \in \Sigma_H} \bigcap_{S \leq T \in \Sigma_H} (S \cap r(T)) \quad \left(\text{resp. } r(H) = \bigcup_{S \in \Sigma_H} r(S) \right)$$

for every $L\mathfrak{X}$ -group H , where Σ_H denotes the local system of all finite subgroups of H . Examples for such rules are:

- $O_\pi(H)$: the maximal normal π -subgroup of H ;
- $S(H)$: the maximal normal, locally soluble subgroup of H ;
- $N(H)$: the maximal normal, locally nilpotent subgroup of H ;
- $O^\pi(H)$: the minimal normal subgroup of H with $H/O^\pi(H)$ a π -group;
- $\hat{S}(H)$: the minimal normal subgroup of H with $H/\hat{S}(H)$ a locally soluble group;
- $\hat{N}(H)$: the minimal normal subgroup of H with $H/\hat{N}(H)$ a locally nilpotent group. (cf. V. Stingl [22], Folgerung 11).

In this case Theorem 3.4 has the

COROLLARY 3.5. *Let \mathfrak{X} be a class of locally finite groups satisfying (1) and (2'), and let r be a functorial, radical (coradical) rule on $L\mathfrak{X}$ satis-*

fyng (*). Then either $r(G) = 1$ or $r(G) = G$ holds for every e.c. $L\mathfrak{X}$ -group G .

PROOF. If there exists a finite \mathfrak{X} -group F with $r(F) \neq F$ (resp. $r(F) \neq 1$), then Theorem 3.4 yields $r(G) = 1$ (resp. $r(G) = G$) for every e.c. $L\mathfrak{X}$ -group G . Otherwise (*) implies

$$r(G) = \bigcup_{S \in \Sigma_G} \bigcap_{S \leq T \in \Sigma_G} (S \cap r(T)) = \bigcup_{S \in \Sigma_G} \bigcap_{S \leq T \in \Sigma_G} (S \cap T) = \bigcup_{S \in \Sigma_G} S = G$$

$$\left(\text{resp. } r(G) = \bigcup_{S \in \Sigma_G} r(S) = \bigcup_{S \in \Sigma_G} 1 = 1 \right). \quad \square$$

In [17], Satz 7 B. Maier has shown that E_p is characteristically simple, and by O. H. Kegel and B. A. F. Wehrfritz [13], Theorem 6.1 every e.c. $L\mathfrak{F}$ -group is simple. With regard to these facts and the above statements we pose the

QUESTION. Under which conditions is an e.c. $L\mathfrak{X}$ -group, where \mathfrak{X} is a class of locally finite groups, characteristically simple?

Using the classification of all finite, simple groups in connection with O. H. Kegel and B. A. F. Wehrfritz [13], Theorem 4.8 it can be shown easily, that locally finite, e.c. $L\mathfrak{X}$ -groups are in general not simple: If \mathfrak{X} satisfies (1), and if there exists such a simple group, then $\pi_{\mathfrak{X}}$ must be the set of all primes (cf. F. Leinen [15], Satz III.2.8).

With regard to the above question it should also be noted, that S. Thomas has constructed in [23] 2^{\aleph_1} non-isomorphic, e.c. groups of cardinality \aleph_1 in each of the classes $L\mathfrak{F}_{\pi}$ and $L(\mathfrak{F}_{\pi} \cap \mathfrak{S})$, which admit only locally inner automorphisms. Each of these groups is characteristically simple, if and only if it is simple. Therefore, the above question can in general only have a positive answer for countable, e.c. $L\mathfrak{X}$ -groups.

However, we can obtain characteristically simple $L\mathfrak{X}$ -groups by considering the e.c. objects in the class $(L\mathfrak{X})^{\text{Aut}}$ of all pairs (G, A) , where G is an $L\mathfrak{X}$ -group and where $A \leq \text{Aut}(G)$. In this class inclusion is defined as follows: $(G, A) \subseteq (H, B)$, if and only if $G \leq H$ and $A \leq B$, where A leaves G invariant and acts faithfully on G . If $\mathfrak{X} = S\mathfrak{X}$, then every $(L\mathfrak{X})^{\text{Aut}}$ -pair is contained in an e.c. $(L\mathfrak{X})^{\text{Aut}}$ -pair by J. Hirschfeld and W. H. Wheeler [8], Proposition I.1.3. Here, (G, A) is e.c. in $(L\mathfrak{X})^{\text{Aut}}$, if every system of finitely many equations and inequations with coefficients from $G \cup A$ and unknowns, which are group elements

or automorphisms, can be solved in (G, A) , whenever it has a solution in some $(L\mathfrak{X})^{\text{Aut}}$ -superpair of (G, A) .

For the investigation of e.c. $(L\mathfrak{X})^{\text{Aut}}$ -pairs (G, A) we have to construct $L\mathfrak{X}$ -supergroups H of G with the property, that the automorphisms from A can be continued to automorphisms of H in a way, which gives an embedding $A \hookrightarrow \text{Aut}(H)$. Fortunately, all wreath product constructions made in Sections 2 and 3 can still be used, since C. H. Houghton shows in [9], 3.1 and 3.2 how to continue automorphisms when identifying G with the 1-component, diagonal or top group of a wreath product. Therefore, all theorems of Sections 2 and 3 hold literally for groups G occurring in an e.c. $(L\mathfrak{X})^{\text{Aut}}$ -pair (G, A) . Moreover, by embedding G in a natural way into the generalized restricted wreath product $\text{Wr } G^{\mathbb{Z}}$ (cf. P. Hall [4]), the following theorem can be proved (for details cf. F. Leinen [15], Section V).

THEOREM 3.6. *Let \mathfrak{X} satisfy (1) and (2'); in the case $C_\infty \in \mathfrak{X}$ let \mathfrak{X} satisfy (2) too. Then every group G occurring in an e.c. $(L\mathfrak{X})^{\text{Aut}}$ -pair (G, A) is characteristically simple.*

In particular, the assumptions of Theorem 3.6 ensure that every $L\mathfrak{X}$ -group H can be embedded into a characteristically simple $L\mathfrak{X}$ -group G with $|G| = \max\{|H|, \aleph_0\}$. But in general a group G occurring in an e.c. $(L\mathfrak{X})^{\text{Aut}}$ -pair (G, A) is not an e.c. $L\mathfrak{X}$ -group.

4. Chief factors and normal subgroups of the countable groups.

During this section \mathfrak{X} will always be a group class satisfying (1), (2) and (3). Please note, that $\mathfrak{X} = S\mathfrak{X} = Q\mathfrak{X}$ implies $\pi_{\mathfrak{X}} \neq \emptyset$. In the case $C_\infty \in \mathfrak{X}$ the set $\pi_{\mathfrak{X}}$ will even contain every prime. In particular, Theorems 2.1 and 2.8 yield, that every e.c. $L\mathfrak{X}$ -group G is verbally complete. Hence, G has no proper normal subgroup with soluble factor group. Correspondingly, the Baer-radical of G is trivial by Theorem 3.1, and G contains no non-trivial, soluble subnormal subgroup. Also, G cannot have a proper subgroup of finite index or a non-trivial, finite normal subgroup (cf. proof of Theorem 4.11 (b)).

The aim of this section is to find out more about the normal subgroups of e.c. $L\mathfrak{X}$ -groups G . If $N \trianglelefteq G$, then G can be embedded into $N \text{ Wr } G/N$ by a Krasner-Kaloujnine-embedding. Unfortunately, this wreath product will in general not be an $L\mathfrak{X}$ -group. But since especially $N \text{ Wr } G/N$ contains solutions for many systems of equations and

inequations, we will try to modify the Krasner-Kaloujnine-embedding in order to obtain an embedding of G into an $L\mathcal{X}$ -subgroup of a similar wreath product.

The following generalization can be found in G. Higman [7]. Let $\theta: G \rightarrow H$ be a group homomorphism with $N = \text{Kern}\theta$. Then $\theta^*: H \rightarrow G$ is a countermap for θ , if $(g\theta h)\theta^* \theta = g\theta \cdot h\theta^* \theta$ for all $g \in G$ and all $h \in H$. Choosing such a countermap θ^* (this is always possible) the map $\sigma: G \rightarrow N \text{ Wr } H$, defined by $g\sigma = (g\theta, f_g)$, where $f_g(h) = [(g\theta h)\theta^*]^{-1} \cdot g \cdot h\theta^*$ for all $h \in H$, becomes a so-called *standard embedding*.

Next, denote by $G\lambda G$ the split extension of G by itself, where G acts on itself via conjugation, i.e., let $G\lambda G = \{(g, h) : g \in G, h \in G\}$, where multiplication is given by $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1^g h_2)$. If $U, V \leq G$ with $V \leq N(U)$, we write $U\lambda V$ for the subgroup $\{(v, u) : v \in V, u \in U\}$ of $G\lambda G$. Now we can establish our modification of the Krasner-Kaloujnine-embedding.

CONSTRUCTION 4.1. *Let $G = \bigcup_{n \in \mathbb{N}} G_n$, where $\{G_n\}_{n \in \mathbb{N}}$ is an ascending chain of subgroups of G , let $\theta: G \rightarrow H$ be a homomorphism with $N = \text{Kern}\theta$, and let $\sigma_1: G_1 \hookrightarrow (G_1 \cap N) \text{ Wr } H$ be the standard embedding with respect to a countermap $\theta_1^*: H \rightarrow G_1$ for $\theta \upharpoonright G_1$. Identifying $(G_1 \cap N) \text{ Wr } H$ in the natural way with the subgroup $((G_1 \cap N)\lambda 1) \text{ Wr } H$ of the wreath product $(G\lambda G) \text{ Wr } H$ we will continue σ_1 to an embedding $\sigma: G \hookrightarrow W = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{ Wr } H] \leq (G\lambda G) \text{ Wr } H$.*

Therefore, we follow the proof of G. Higman [7], Lemma 1 in order to obtain successively countermaps $\theta_{n+1}^*: H \rightarrow G_{n+1}$ for $\theta \upharpoonright G_{n+1}$, $n \in \mathbb{N}$, such that the map $\omega_n: H \rightarrow G_{n+1}$, given by $h\theta_{n+1}^* = h\theta_n^* \cdot h\omega_n$ (for all $h \in H$), is constant on each of the cosets $G_n \theta \cdot h$, $h \in H$. Then we define inductively maps $p^n: H \rightarrow G_n$, $n \in \mathbb{N}$, by $p^1 \equiv 1$ and $p^{n+1}(h) = (h\omega_n)^{-1} \cdot p^n(h)$ for all $h \in H$. Denoting the standard embedding of G_n into $(G_n \cap N) \text{ Wr } H$ with respect to θ_n^* by $\sigma_n: g \mapsto (g\theta, f_g^n)$ we can now define σ as follows.

For every $g \in G_n$, $n \in \mathbb{N}$, let $g\sigma = (g\theta, s_g^n) \in (G_n \lambda G_n) \text{ Wr } H$, where $s_g^n(h) = ([p^n(g\theta h)]^{-1} \cdot p^n(h), [p^n(h)]^{-1} \cdot f_g^n(h) \cdot p^n(h))$ for all $h \in H$.

Straightforward calculations yield, that $\sigma: G \rightarrow W$ is a well-defined embedding with $\sigma \upharpoonright G_1 = \sigma_1$. Also, assuming (1) and (2), W will be an $L\mathcal{X}$ -subgroup of $(G\lambda G) \text{ Wr } H$, if H is an $L\mathcal{X}$ -group and if the groups G_n , $n \in \mathbb{N}$, are f.g. \mathcal{X} -groups. Therefore, we can use Construction 4.1 in order to investigate the structure of countable, e.c. $L\mathcal{X}$ -groups.

We will now establish some technical lemmata, which will be needed for later applications of Construction 4.1.

LEMMA 4.2. *In the notation of 4.1 the following statements hold for each $g \in G_n \setminus N$:*

(a) *If $g\theta$ has infinite order, then $g\sigma$ is conjugate in $(G_n \lambda G_n) \text{Wr } H$ to $(g\theta, 1)$.*

(b) *If $g\theta$ has finite order m , and if T is a right transversal of $\langle g\theta \rangle$ in H , then the element $z = (1, s) \in (G_n \lambda G_n) \text{Wr } H$, where $s((g\theta)^r t) = s_r^n(t)$ for all $t \in T$, $0 \leq r \leq m - 1$, conjugates $g\sigma$ onto $(g\theta, \tilde{s}_g)$, where*

$$\begin{aligned} \tilde{s}_g(h) &= \begin{cases} (1, [p^n(t)]^{-1} \cdot f_{g^m}^n(t) \cdot p^n(t)) & \text{for } h = (g\theta)^{m-1}t, t \in T, \\ 1 & \text{else,} \end{cases} \\ &= \begin{cases} (1, (t\theta_1^*)^{-1} \cdot g^m \cdot t\theta_1^*) & \text{for } h = (g\theta)^{m-1}t, t \in T, \\ 1 & \text{else.} \end{cases} \end{aligned}$$

PROOF. (b) is shown by straightforward calculation. Similarly, (a) can be proved by choosing a right transversal T of $\langle g\theta \rangle$ in H and $z = (1, s) \in (G_n \lambda G_n) \text{Wr } H$, where $s((g\theta)^r t) = s_r^n(t)$ for all $t \in T$, $r \in \mathbb{Z}$; then $(g\sigma)^z = (g\theta, 1)$. \square

LEMMA 4.3. (a) *If $b \in B \setminus 1$ is of infinite order, then every element x of the base group of $W = A \text{Wr } B$ is contained in the normal closure $\langle (b, 1)^W \rangle$.*

(b) *If $b \in B \setminus 1$ is of finite order m , if T is a right transversal of $\langle b \rangle$ in B , and if $(b, f) \in W = A \text{Wr } B$ satisfies $f(b') = 1$ for all $b' \in B \setminus T$, then every commutator $x = [x_1, x_2]$ of two elements x_1, x_2 of the base group of W is contained in the normal closure $\langle (b, f)^W \rangle$.*

PROOF. (a) is a consequence of P. M. Neumann [20], Lemma 5.2. For the proof of (b) choose $x_1 = (1, f_1), x_2 = (1, f_2)$ and decompositions $f_i = f_{i1} \dots f_{im}$, where $f_{ij}(b') = 1$ for all $b' \in B \setminus b^j T$. Let $x_{ij} = (1, f_{ij})$. Then $x = \prod_{j=1}^m [x_{1j}, x_{2j}]$, and (b) can be shown by proving $[(b, f), x_{1j}], x_{2j}] = [x_{1j}, x_{2j}]$ for $2 \leq j \leq m$ and $[(b, f)^{(b,1)}, x_{11}], x_{12}] = [x_{11}, x_{12}]$. \square

COROLLARY 4.4. *In the notation of 4.1 the following holds: If X denotes the base group of $(G \lambda G) \text{Wr } H$, then for every $g \in G \setminus N$ and all $x_1, x_2 \in X \cap W$ the commutator $[x_1, x_2]$ is contained in the normal closure $\langle g\sigma^W \rangle$.*

PROOF. Choose $n \in \mathbb{N}$ such that $g \in G_n$ and $x_1, x_2 \in (G_n \lambda G_n) \text{Wr } H$. Then apply Lemmata 4.2 and 4.3. \square

LEMMA 4.5. *In the notation of 4.1 the following statements hold for any verbally complete group G :*

(a) *If $w(x_1, \dots, x_\nu)$ is a non-trivial word, and if $(1, (1, f))$ is an element of the base group of $(G\lambda 1) \text{Wr } H \leq (G\lambda G) \text{Wr } H$ such that $f(H)$ is finite, then there exist $v_1, \dots, v_\nu \in W$ with $(1, (1, f)) = w(v_1, \dots, v_\nu)$ and $v_1, \dots, v_\nu \in \langle g\sigma^w \rangle$ for every $g \in G \setminus N$.*

(b) *If V is a non-trivial set of words, and if $(1, s)$ is an element of the base group of $(G\lambda G) \text{Wr } H$ such that $s(H)$ is finite, then $(1, s) \in V \langle g\sigma^w \rangle$ for every $g \in G \setminus N$.*

PROOF. (a) By the verbal completeness there exist for every $g' \in f(H)$ elements $y_{\sigma'1}, \dots, y_{\sigma'\nu} \in G$ such that $g' = w(y_{\sigma'1}, \dots, y_{\sigma'\nu})$. For $1 \leq j \leq \nu$ define $f_j: H \rightarrow G$ by $f_j(h) = y_{\sigma'j}$, if $f(h) = g'$. Clearly, $f_j(H)$ is finite, and the $v_j = (1, (1, f_j)) \in W$ satisfy $(1, (1, f)) = w(v_1, \dots, v_\nu)$. If X denotes the base group of $(G\lambda G) \text{Wr } H$, the above argument shows, that each v_j is a commutator of two elements of $X \cap W$. Hence we have $v_1, \dots, v_\nu \in \langle g\sigma^w \rangle$ for each $g \in G \setminus N$ by Corollary 4.4.

(b) For $s = (f_1, f_2)$ we can find similarly elements $v_{11}, \dots, v_{1\nu}, v_{21}, \dots, v_{2\nu} \in \langle g\sigma^w \rangle$ such that $(1, (f_1, 1)) = w(v_{11}, \dots, v_{1\nu})$ and $(1, (1, f_2)) = w(v_{21}, \dots, v_{2\nu})$. \square

Before we start to investigate countable, e.c. $L\mathfrak{X}$ -groups by using the Construction 4.1, we need the notion of a chief series, which is due to P. Hall [5].

Let I be a totally ordered set. A family $\Sigma = \{(M_i, N_i) : i \in I\}$ of pairs of (normal) subgroups M_i, N_i of an arbitrary group G is a (normal) series of order-type I in G , if

- (a) $G \setminus 1 = \bigcup_{i \in I} (M_i \setminus N_i)$,
- (b) $N_i \triangleleft_{\neq} M_i$ for all $i \in I$, and
- (c) $M_k \leq N_i$ for all $i, k \in I$ with $k < i$.

The M_i/N_i are the factors of Σ . The normal series $\Sigma' = \{(K_j, L_j) : j \in J\}$ is called a normal refinement of Σ , if there exists for every $j \in J$ an $i \in I$ such that $N_i \leq L_j \leq K_j \leq M_i$. A chief series is a normal series, which coincides with each of its normal refinements. The factors

of a chief series are called *chief factors*. An application of Zorn's Lemma yields, that every normal series has a normal refinement, which is a chief series. In particular, every group G has at least one chief series.

If \bar{I} denotes the Dedekind-completion of I , we define the *Dedekind-completion* of the normal series $\Sigma = \{(M_i, N_i) : i \in I\}$ to be $\bar{\Sigma} = \{(K_i, L_i) : i \in \bar{I}\}$, where

$$(a) \quad K_i = \begin{cases} \bigcap_{\substack{j \in I \\ i < j}} N_j, & \text{if } i \text{ is not maximal in } \bar{I}, \\ G & \text{otherwise;} \end{cases}$$

$$(b) \quad L_i = \begin{cases} \bigcup_{\substack{j \in I \\ j < i}} M_j, & \text{if } i \text{ is not minimal in } \bar{I}, \\ 1 & \text{otherwise.} \end{cases}$$

The following lemma can be established easily.

LEMMA 4.6. *If $\bar{\Sigma} = \{(K_i, L_i) : i \in \bar{I}\}$ is the Dedekind-completion of the normal series $\Sigma = \{(M_i, N_i) : i \in I\}$ in G , then:*

- (a) $(K_i, L_i) = (M_i, N_i)$ for all $i \in I$,
- (b) $K_k \leq L_i$ for all $i, k \in \bar{I}$ with $k < i$, and
- (c) $K_i = L_i$ for all $i \in \bar{I} \setminus I$.

Moreover, the K_i and L_i are normal subgroups of G ; and if Σ refines a normal series Σ_0 in G , then every non-trivial, proper subgroup of G appearing in one of the pairs of Σ_0 is also contained in one pair of $\bar{\Sigma}$.

We are now able to formulate and prove the main result of this section.

THEOREM 4.7. *Let \mathfrak{X} satisfy (1), (2) and (3). If $M|N$ is a chief factor of a countable, e.c. $L\mathfrak{X}$ -group G , then there exist for every non-trivial word $w(x_1, \dots, x_r)$ and every $h \in N$ elements $g_1, \dots, g_r \in M$ such that $h = w(g_1, \dots, g_r)$. Moreover, $M = \langle g^\sigma \rangle$ for every $g \in M \setminus N$. In particular, G has exactly one chief series, and the normal subgroups of G form a chain.*

PROOF. Fix $h \in N$ and $g \in M \setminus N$. Choose an ascending chain $\{G_n\}_{n \in \mathbb{N}}$ of f.g. subgroups of G with $G_1 = \langle h \rangle$ and $G = \bigcup_{n \in \mathbb{N}} G_n$. Let $\theta: G \rightarrow G/N$ be the canonical epimorphism, and let $\sigma_1: G_1 \hookrightarrow G_1 \text{ Wr } G/N$

be the standard embedding with respect to the countermap $\theta_1^* \equiv 1$ for $\theta \upharpoonright G_1$. Applying Construction 4.1 we obtain an embedding $\sigma: G \hookrightarrow \hookrightarrow W = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{Wr } G/N]$, where $W \in L\mathfrak{X}$. Now $h\sigma = (1, (1, f))$ with $f \equiv h$. By Lemma 4.5 there exist $v_1, \dots, v_r \in \langle g\sigma^m \rangle$ such that $h\sigma = w(v_1, \dots, v_r)$. This can be expressed by $r + 1$ equations with coefficients $h\sigma, g\sigma \in G\sigma$. Since G is e.c. in $L\mathfrak{X}$, we obtain $g_1, \dots, g_r \in \langle g^\sigma \rangle \leq M$ with $h = w(g_1, \dots, g_r)$.

This works for every $h \in N$; hence $N \leq \langle g^\sigma \rangle$. But as a chief factor M/N is a minimal normal subgroup of G/N , and we get $M = \langle g^\sigma \rangle$.

Assume, there exist chief factors M_1/N_1 and M_2/N_2 with $g \in M_1 \setminus N_1$ and $g \in M_2 \setminus N_2$. Then $M_1 = \langle g^\sigma \rangle = M_2$ and $N_1 = \{h: \langle h^\sigma \rangle \not\leq \langle g^\sigma \rangle\} = N_2$. Hence, for every $g \in G$ there exists exactly one chief factor M/N such that $g \in M \setminus N$, and G must have a unique chief series Σ . If $1 \triangleleft K \triangleleft G$, then the unique chief series in G must refine the normal series $\{(G, K), (K, 1)\}$ in G . Therefore, K appears in one of the pairs of the Dedekind-completion $\overline{\Sigma}$, and the normal subgroups of G must form a chain (cf. Lemma 4.6). \square

Theorem 4.7 shows, that every countable, e.c. $L\mathfrak{X}$ -group G contains in a certain sense the least number of normal subgroups; for the normal closure $\langle g^\sigma \rangle$, $g \in G$, will always appear as the group M in some chief factor M/N (just refine $\{(G, \langle g^\sigma \rangle), (\langle g^\sigma \rangle, 1)\}$ to a chief series!).

In the case $\mathfrak{X} = L\mathfrak{S}$ Theorem 4.7 becomes trivial, since the e.c. $L\mathfrak{S}$ -groups are simple (cf. p. 204). Concerning the question posed on p. 204. Theorem 4.7 yields a necessary condition: If a countable, e.c. $L\mathfrak{X}$ -group G is characteristically simple, then G cannot have a proper minimal or a non-trivial maximal normal subgroup.

The next theorem will give us some information about the structure of the normal subgroups, which are not the normal closure of an element.

THEOREM 4.8. *Let \mathfrak{X} satisfy (1), (2) and (3). If M/N is a chief factor of a countable, e.c. $L\mathfrak{X}$ -group G , then every system of finitely many equations and inequations with coefficients from N , which is solvable in some $L\mathfrak{X}$ -supergroup of G , has already a solution in every verbal subgroup of M . Moreover, for every non-trivial $K \trianglelefteq G$ with $K \neq \langle g^\sigma \rangle$ for all $g \in G$ the following statements hold:*

(a) *Every system of finitely many equations and inequations with coefficients from K , which is solvable in some $L\mathfrak{X}$ -supergroup of G , has already a solution in K .*

(b) Every normal subgroup of K is a normal subgroup in G . In particular, K has exactly one chief series, and the normal subgroups of K form a chain.

(c) Each automorphism of K , which is induced by conjugation with some element from G , is locally inner.

PROOF. Let \mathcal{S} be a system of finitely many equations and inequations with coefficients $h_1, \dots, h_k \in N$, which is solvable in some $L\mathfrak{X}$ -supergroup of G . Since G is e.c. in $L\mathfrak{X}$, there exists a solution u_1, \dots, u_l in G . Choose $g \in M \setminus N$ and $w(x_1, \dots, x_r) \neq 1$. Let $\{G_n\}_{n \in \mathbb{N}}$ be an ascending chain of f.g. subgroups of G with $G_1 = \langle h_1, \dots, h_k \rangle$ and $G = \bigcup_{n \in \mathbb{N}} G_n$. Denote by $\theta: G \rightarrow G/N$ the canonical epimorphism and by $\sigma_1: G_1 \hookrightarrow G_1 \text{ Wr } G/N$ the standard embedding with respect to $\theta_1^* \equiv 1$ (for $\theta \upharpoonright G_1$). Apply Construction 4.1 in order to obtain an embedding $\sigma: G \hookrightarrow W = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{ Wr } G/N] \in L\mathfrak{X}$. A further embedding $\zeta: G \hookrightarrow W$ is given by $g' \zeta = (1, (1, f_{g'}))$, where $f_{g'} \equiv g'$ for all $g' \in G$. Then $h_r \zeta = h_r \sigma$ for $1 \leq r \leq k$. Now

(i) the system $\mathcal{S}\sigma$ with coefficients $h_1 \sigma, \dots, h_k \sigma \in G\sigma$ has the solution $u_1 \zeta, \dots, u_l \zeta \in W$, and

(ii) by Lemma 4.5 there exist $v_{ij} \in \langle g\sigma^w \rangle (1 \leq i \leq l, 1 \leq j \leq v)$ such that

(iii) $w(v_{i1}, \dots, v_{iv}) = u_i \zeta$ for $1 \leq i \leq l$.

The statements (i), (ii) and (iii) can be expressed as a system of finitely many equations and inequations with coefficients $g\sigma, h_1 \sigma, \dots, h_k \sigma \in G\sigma$, which is solvable in W . Hence there must already exist elements $g_1, \dots, g_l \in G$, which solve \mathcal{S} and which are contained in $V(\langle g^\sigma \rangle)$ for every set V of words containing $w(x_1, \dots, x_r)$. Clearly, $M = \langle g^\sigma \rangle$ by Theorem 4.7.

(a) Now let \mathcal{S} be a system of finitely many equations and inequations with coefficients from K , which is solvable in some $L\mathfrak{X}$ -supergroup of G . Denote the set of coefficients by K_0 . Because of $K \neq \langle g^\sigma \rangle$ for all $g \in G$ Theorem 4.7 implies $\langle K_0^\sigma \rangle \not\leq K$. If $h \in K \setminus \langle K_0^\sigma \rangle$, then $K_0 \subseteq \langle K_0^\sigma \rangle \leq N \leq M \leq K$ for the unique chief factor M/N of G with $h \in M \setminus N$. Hence, \mathcal{S} must have a solution in $M \leq K$ by the above argumentation.

(b) Let $L \trianglelefteq K$. For any $g \in \langle L^G \rangle$ there exist elements $g_1, \dots, g_k \in G$ and $l_1, \dots, l_k \in L$ with $g = l_1^{g_1} l_2^{g_2} \dots l_k^{g_k}$. Hence $g = l_1^{x_1} l_2^{x_2} \dots l_k^{x_k}$ is an equation with coefficients $g, l_1, \dots, l_k \in K$ and a solution in G . By (a) there exist $h_1, \dots, h_k \in K$ such that $g = l_1^{h_1} \dots l_k^{h_k} \in L$. Therefore, $L = \langle L^G \rangle \trianglelefteq G$.

(c) Fix $g \in G$. If K_0 is a finite subset of K , the equations $h^x = h^g, h \in K_0$, with coefficients $h, h^g \in K_0 \cup K_0^g \subseteq K$ form a finite system with solution $x = g$ in G . Hence, (a) yields an $v \in K$ such that $h^v = h^g$ for all $h \in K_0$. \square

With regard to Theorem 4.8. (c) it should be noted, that every countable, e.c. $L\mathfrak{X}$ -group G acts faithfully on each of its non-trivial normal subgroups N by conjugation: For Theorem 4.7 yields $C_G(N) \leq N$ or $N \leq C_G(N)$, and since the Baer-radical of G is trivial by Theorem 3.1, every abelian normal subgroup of G must be trivial; hence, $N = 1$ or $C_G(N) = 1$.

We will now investigate, how G acts on its chief factors by conjugation.

THEOREM 4.9. *Let \mathfrak{X} satisfy (1), (2) and (3), and let M/N be a chief factor of a countable, e.c. $L\mathfrak{X}$ -group G .*

(a) *If $M/N \not\leq Z(G/N)$, then $C_{G/N}(M/N) = Z(M/N)$, and M/N is infinite.*

(b) *Suppose, M/N is not torsion-free abelian, or every torsion-free divisible abelian group is an \mathfrak{X} -group. Denote by $\tau: G/N/C_{G/N}(M/N) \hookrightarrow \text{Aut}(M/N)$ the canonical embedding via conjugation and assume the existence of $x_1, x_2 \in M \setminus N$ with $\langle x_1 \rangle \cap N = 1 = \langle x_2 \rangle \cap N$. If there exists an $\alpha \in \text{Aut}(M/N)$ with $(x_1 N)\alpha = x_2 N$ such that the subgroup $\langle \alpha, \text{Im } \tau \rangle$ of $\text{Aut}(M/N)$ is an $L\mathfrak{X}$ -group, then there already exists $g \in G$ with $x_1^g = x_2$.*

(c) *Any two elements $x_1, x_2 \in mN$ ($m \in M \setminus N$) of equal order satisfying $\langle x_1 \rangle \cap N = 1 = \langle x_2 \rangle \cap N$ are conjugate in G .*

PROOF. Denote factor groups and cosets modulo N by bars.

(a) Theorem 4.7 yields $C_{\bar{G}}(\bar{M}) \leq \bar{M}$ or $\bar{M} \leq C_{\bar{G}}(\bar{M})$. In the first case we have $C_{\bar{G}}(\bar{M}) = Z(\bar{M})$. Now, assume $\bar{M} \leq C_{\bar{G}}(\bar{M})$. Fix $c \in G$ with $\bar{c} \in C_{\bar{G}}(\bar{M}) \setminus \bar{M}$. If M_1/N_1 is the chief factor of G satisfying $c \in M_1 \setminus N_1$, then $M \leq N_1 \leq M_1 = \langle c^G \rangle$ by Theorem 4.7. Fix $g \in G$ and $m \in M$. Then the equation $m^x = m^g$ with coefficients $m, m^g \in$

$\in \bar{M} \leq N_1$ has the solution $x = g$. Therefore Theorem 4.8 ensures the existence of $h \in \bar{M}_1$ such that $m^h = m^g$. Now, $\bar{h} \in \langle \bar{c}^g \rangle \leq \mathbf{C}_{\bar{c}}(\bar{M})$ and $\bar{m}^g = \bar{m}^h = \bar{m}$. This holds for every $g \in G$ and every $m \in \bar{M}$. Hence $\bar{M} \leq \mathbf{Z}(\bar{G})$.

In the case $\bar{M} \not\leq \mathbf{Z}(\bar{G})$ we have $\mathbf{C}_{\bar{c}}(\bar{M}) = \mathbf{Z}(\bar{M}) \neq \bar{G}$, and since G does not contain a proper normal subgroup of finite index, $\bar{G}/\mathbf{C}_{\bar{c}}(\bar{M}) \leq \leq \text{Aut}(\bar{M})$ yields $|\bar{M}| = \infty$.

(b) At first we will embed \bar{G} into an $L\mathfrak{X}$ -group H , in which the images of \bar{x}_1 and \bar{x}_2 are conjugate. In the case $\mathbf{C}_{\bar{c}}(\bar{M}) = 1$ we choose $H = \langle \alpha, \text{Im } \tau \rangle$ and the embedding $\tau: \bar{G} \hookrightarrow H$; then $(\bar{x}_1 \tau)^\alpha = \bar{x}_2 \tau$.

Now let $\mathbf{C}_{\bar{c}}(\bar{M}) \neq 1$. Denote by $\bar{M}\lambda \langle \alpha, \text{Im } \tau \rangle$ the subgroup of the holomorph of \bar{M} , which is generated by \bar{M} and $\langle \alpha, \text{Im } \tau \rangle$. If $\theta: \bar{G} \rightarrow G/\bar{M} = L$ denotes the canonical epimorphism, and if $\{G_n\}_{n \in \mathbb{N}}$ is an ascending chain of f.g. subgroups of G with $G_1 = \langle x_1, x_2 \rangle$ and $G = \bigcup_{n \in \mathbb{N}} G_n$, then the standard embedding $\sigma_1: \bar{G}_1 \hookrightarrow (\bar{G}_1 \cap \bar{M}) \text{Wr } L$

with respect to the countermap $\theta_1^* \equiv 1$ (for $\theta \upharpoonright \bar{G}_1$) can be continued to an embedding $\sigma: \bar{G} \hookrightarrow W = (\bar{M}\lambda \langle \alpha, \text{Im } \tau \rangle) \text{Wr } L$ by modifying Construction 4.1 in the following way: Instead of the maps $p^n: L \rightarrow G_n$ of Construction 4.1 use the maps $q^n: L \rightarrow \langle \alpha, \text{Im } \tau \rangle$, given by $q^n(l) = [(p^n(l) \mathbf{C}_{\bar{c}}(\bar{M}))\tau]$ for all $l \in L$; then define for all $g \in G_n$, $n \in \mathbb{N}$, the embedding $\sigma: \bar{G}_n \hookrightarrow W$ by $g\sigma = (g\theta, s_g^n) \in (\bar{M}\lambda \langle \alpha, \text{Im } \tau \rangle) \text{Wr } L$, where

$$\begin{aligned} s_g^n(l) &= ([q^n(\bar{g}\theta l)]^{-1} \cdot q^n(l), [f_{\bar{c}}^n(l)](q^n(l))) = \\ &= ([q^n(\bar{g}\theta l)]^{-1} \cdot q^n(l), (p^n(l))^{-1} \cdot f_{\bar{c}}^n(l) \cdot p^n(l)) \end{aligned}$$

for all $l \in L$.

By choice of θ_1^* we have $\bar{x}_i \sigma = (1, (1, f_{\bar{x}_i}^*))$ with $f_{\bar{x}_i}^* \equiv \bar{x}_i$ for $i = 1, 2$. Hence $(\bar{x}_1 \sigma)^{\alpha^*} = \bar{x}_2 \sigma$, where $\alpha^* = (1, s) \in W$ is given by $s \equiv (\alpha, 1)$. We choose $H = \langle \bar{G}\sigma, \alpha^* \rangle$ and have to show $H \in L\mathfrak{X}$.

Let H_0 be a finite subset of H ; choose $n \in \mathbb{N}$ with $H_0 \subseteq \langle \bar{G}_n \sigma, \alpha^* \rangle$. Then $H_0 \subseteq W_0 = (\bar{M}\lambda \langle \alpha, (\bar{G}_n \mathbf{C}_{\bar{c}}(\bar{M})/\mathbf{C}_{\bar{c}}(\bar{M}))\tau \rangle) \text{Wr } L \leq W$, and it suffices to show $W_0 \in L\mathfrak{X}$. The base group of W_0 is the split extension of a cartesian power of \bar{M} by a cartesian power of $\langle (\bar{G}_n \mathbf{C}_{\bar{c}}(\bar{M})/\mathbf{C}_{\bar{c}}(\bar{M}))\tau, \alpha \rangle$. Because of $\mathbf{C}_{\bar{c}}(\bar{M}) \neq 1$ we obtain from (a), that \bar{M} is abelian. If \bar{M} is not torsion-free, then \bar{M} must be elementary-abelian, and every cartesian power of \bar{M} is an \mathfrak{X} -group by (2). If \bar{M} is torsion-free, then \bar{M} must be divisible, and every cartesian power of \bar{M} is an \mathfrak{X} -group by assumption. Since (2) ensures, that every cartesian power of the f.g. \mathfrak{X} -group $\langle \alpha, (\bar{G}_n \mathbf{C}_{\bar{c}}(\bar{M})/\mathbf{C}_{\bar{c}}(\bar{M}))\tau \rangle$ is an \mathfrak{X} -group too, $\mathfrak{X} = P\mathfrak{X}$

yields that the base group of W_0 is an \mathfrak{X} -group. But then $W_0 \in L\mathfrak{X}$. Hence we have shown, that there exists an embedding of \bar{G} into an $L\mathfrak{X}$ -group H , in which the images of \bar{x}_1 and \bar{x}_2 are conjugate.

Denote by $\psi: G \rightarrow H$ the composition of the canonical epimorphism $G \rightarrow \bar{G}$ and the above embedding $\bar{G} \hookrightarrow H$. Then Construction 4.1 gives an embedding $\eta: G \hookrightarrow V = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{Wr } H]$, where $V \in L\mathfrak{X}$.

Because of $\langle x_1 \rangle \cap N = 1 = \langle x_2 \rangle \cap N$ and Lemma 4.2, the elements $x_i \eta$, $i = 1, 2$, are conjugate in $(G_1 \lambda G_1) \text{Wr } H \leq V$ to $(x_i \psi, 1) \in V$. Moreover, we have $(x_1 \psi, 1)^{(\alpha^*, 1)} = (x_2 \psi, 1)$. Hence, the equation $(x_1 \eta)^u = x_2 \eta$ with coefficients $x_1 \eta, x_2 \eta \in G\eta$ is solvable in V , and there must already exist $g \in G$ with $x_1^g = x_2$.

(c) Because of $x_1 N = mN = x_2 N$ we can follow the proof of (b) with $H = G/M$ and $\alpha^* = 1$. \square

It should be noted, that the assumption $\langle x \rangle \cap N = 1$ of Theorem 4.9 is always met, if $xN \in M/N$ has infinite order.

Every abelian chief factor M/N of a countable, e.c. $L\mathfrak{X}$ -group G is an elementary-abelian p -group or a torsion-free divisible abelian group. Hence, M/N is a vector space over $GF(p)$ or \mathbb{Q} . In the following theorem we will prove, that there exists for every coset mN , $m \in M \setminus N$, of the abelian chief factor M/N an element $y \in M$ with $yN = mN$ and $\langle y \rangle \cap N = 1$. Therefore Theorem 4.9 yields, that any two one-dimensional subspaces $\langle m_1 N \rangle$ and $\langle m_2 N \rangle$ of the vector space M/N are transposed onto one another by conjugation with an element of G/N , if and only if the image B of $G/N | C_{G/N}(M/N)$ under the canonical embedding into $\text{Aut}(M/N)$ is contained in an $L\mathfrak{X}$ -subgroup of $\text{Aut}(M/N)$, which contains an automorphism α such that $\langle m_1 N \rangle \alpha = \langle m_2 N \rangle$. In particular, G/N acts transitively on the one-dimensional subspaces of M/N , if and only if there exists an $L\mathfrak{X}$ -subgroup of $\text{Aut}(M/N)$, which contains B and acts transitively on M/N .

THEOREM 4.10. *Let \mathfrak{X} satisfy (1), (2) and (3), and let M/N be a chief factor of a countable, e.c. $L\mathfrak{X}$ -group G such that $V(M) \neq M$ for a non-trivial set V of words. Then $N = V(M)$, and the following holds:*

(a) *If $gN \in M/N \setminus 1$ is of infinite order, then every element contained in gN has infinite order, and any two elements of gN are conjugate in G .*

(b) *If $gN \in M/N \setminus 1$ is of finite order m , then gN contains for every $\pi_{\mathfrak{X}}$ -number \tilde{m} , which is a multiple of m , an element of order \tilde{m} ,*

and any two elements of order m , which are contained in gN , are conjugate in G .

PROOF. Clearly N is a maximal normal subgroup of M , while the normal subgroup $V(M)$ of M contains N by Theorem 4.8. Hence $N = V(M)$.

(a) Follows from Theorem 4.9. (c).

(b) Because of $G \leq G \times C_{\tilde{m}} \in L\mathfrak{X}$ the group G contains an element z of order \tilde{m} . Choose $1 \neq w(x_1, \dots, x_r) \in V$, and let $\{G_n\}_{n \in \mathbb{N}}$ be an ascending chain of f.g. subgroups of G with $G_1 = \langle g \rangle$ and $G = \bigcup_{n \in \mathbb{N}} G_n$. Denote by $\theta: G \rightarrow G/N$ the canonical epimorphism, and define the countermap $\theta_1^*: G/N \rightarrow G_1$ for $\theta \upharpoonright G_1$ via $((g\theta)^r t)\theta_1^* = g^r$ for $0 \leq r \leq m-1$, $t \in T$, where T is a right transversal of $G_1\theta$ in G/N . Then Construction 4.1 gives an embedding $\sigma: G \hookrightarrow W = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{Wr } G/N] \in L\mathfrak{X}$. Now $g\sigma = (g\theta, (1, f_g))$, where $f_g: G/N \rightarrow G_1$. Define $f, f_z: G/N \rightarrow G$ by $f_z \equiv z$ and $f(h) = z^{-1} \cdot f_g(h)$ for all $h \in G/N$. Then

$$(i) \quad g\sigma = (g\theta, (1, f_z)) \cdot (1, (1, f)),$$

$$(ii) \quad (g\theta, (1, f_z))^{\tilde{m}} = 1 \quad \text{and} \quad (g\theta, (1, f_z))^k \neq 1 \quad \text{for} \quad 1 \leq k \leq \tilde{m} - 1.$$

Moreover $f(G/N)$ is finite by choice of θ_1^* . Hence Lemma 4.5 yields

$$(iii) \quad v_1, \dots, v_r \in \langle g\sigma^w \rangle \quad \text{such that}$$

$$(iv) \quad (1, (1, f)) = w(v_1, \dots, v_r).$$

Now (i), (ii), (iii), and (iv) can be expressed as a system of finitely many equations and inequations with coefficient $g\sigma \in G\sigma$, which is solvable in W . Hence there must exist elements $\bar{g}, g_1, \dots, g_r \in G$ such that $o(\bar{g}) = \tilde{m}$ and $g_1, \dots, g_r \in \langle g\sigma^g \rangle$ and $g = \bar{g} \cdot w(g_1, \dots, g_r)$. It follows that $\bar{g} = g \cdot w(g_1, \dots, g_r)^{-1} \in g \cdot V(\langle g\sigma^g \rangle) = gN$. As in (a) any two elements of order m , which are contained in gN , are conjugate in G . \square

Please note, that the above condition $V(M) \neq M$ is equivalent to $V(M/N) = 1$ for every chief factor M/N of a countable, e.c. $L\mathfrak{X}$ -group.

THEOREM 4.11. *Let \mathfrak{X} satisfy (1), (2) and (3), and let G be a countable, e.c. $L\mathfrak{X}$ -group such that for every chief factor M/N of G there exists a non-trivial set of words V_M with $V_M(M) \neq M$. Then the following hold:*

(a) For any two chief factors M_1/N_1 and M_2/N_2 of G with $M_1 \triangleleft N_2$ and for any prime $p \in \pi_{\mathfrak{X}}$ there exists a chief factor M/N in G with $M_1 \triangleleft N \triangleleft M \triangleleft N_2$, which contains an element of order p .

(b) The order-type of the unique chief series of G is the order of the rationals.

(c) If M/N is a chief factor of G , then (a), (b), (c) of Theorem 4.8, hold for N in the role of K .

(d) If $|\pi_{\mathfrak{X}}| \neq 1$, then every central chief factor of G is torsion-free.

(e) If \mathfrak{X} contains every torsion-free, divisible, abelian group, then there exists no central chief factor in G .

(f) For every proper subnormal subgroup S of G there exists a chief factor M/N in G satisfying $N \triangleleft_{\neq} S \triangleleft_{\neq} M$.

PROOF. (a) By Theorem 4.7 we have $N_2 \triangleleft V_{M_1}(V_{M_2}(M_2))$. If $M_1 = N_2$ would hold, then Theorem 4.10 would yield $N_i = V_{M_i}(M_i)$ for $i = 1, 2$, and we would get the contradiction $N_2 \triangleleft V_{M_1}(V_{M_2}(M_2)) = V_{M_1}(N_2) = V_{M_1}(M_1) = N_1 \triangleleft M_1 = N_2$. Hence, $M_1 \triangleleft N_2$. Choose $g \in N_2 \setminus M_1$. Then, by Theorem 4.7, the unique chief factor M_1^*/N_1^* in G with $g \in M_1^* \setminus N_1^*$ satisfies $M_1 \triangleleft N_1^* \triangleleft M_1^* \triangleleft N_2$. Similarly, there exists a chief factor M_2^*/N_2^* in G such that $M_1 \triangleleft N_1^* \triangleleft M_1^* \triangleleft N_2^* \triangleleft M_2^* \triangleleft N_2$.

Fix $h \in M_1^* \setminus N_1^*$, and let $p \in \pi_{\mathfrak{X}}$ be any prime. Because of Lemma 2.5 there exists $x \in G$ of order p with $h \in \langle x^{\sigma} \rangle$. Therefore the system

$$x^p = 1, \quad x \neq 1, \quad h = x^{z_1} \dots x^{z_k}$$

with coefficient $h \in M_1^* \triangleleft N_2^*$ has a solution in G (for some $k \in \mathbb{N}$). By Theorem 4.8 there exists already a solution in M_2^* , i.e., there exists an $y \in M_2^*$ such that $h \in \langle y^{\sigma} \rangle$ and $o(y) = p$. Let M/N be the chief factor of G with $y \in M \setminus N$. Then $M_1 \triangleleft N_1^* \triangleleft M_1^* = \langle h^{\sigma} \rangle \triangleleft \langle y^{\sigma} \rangle = M \triangleleft M_2^* \triangleleft N_2$ and $M_1 \triangleleft N_1^* \triangleleft N \triangleleft M \triangleleft N_2$. But the element $yN \in M/N$ has order p .

(b) Suppose, G has a minimal normal subgroup M . Then $V_M(M) = 1$. Choose $p \in \pi_{\mathfrak{X}}$ and $1 \neq w(x_1, \dots, x_p) \in V_M$. By P. Hall [3], Lemma 7 there exists a finite p -group $A = \langle a_1, \dots, a_{2r} \rangle$ such that $1 \neq w([a_1, a_2], \dots, [a_{2r-1}, a_{2r}]) \in \mathbf{Z}(A)$. Identify G in the obvious way with the top group and A with the 1-component of the $L\mathfrak{X}$ -group $W = A \text{ Wr } G$. Because of Lemma 4.3 we have $[a_1, a_2], \dots, [a_{2r-1}, a_{2r}] \in$

$\in \langle g^w \rangle$ for some $g \in M \setminus 1$. Since G is e.c. in $L\mathfrak{X}$, there must already exist $g_1, \dots, g_r \in \langle g^a \rangle$ with $1 \neq w(g_1, \dots, g_r)$. This yields the contradiction $1 \neq w(g_1, \dots, g_r) \in V_M(\langle g^a \rangle) = V_M(M) = 1$. Hence G has no minimal normal subgroup.

Correspondingly, G has no maximal normal subgroup, since otherwise $V_G(G) \neq G$ would contradict the verbal completeness of G . Together with (a) we obtain, that the order-type of the unique chief series of G must be a dense linear order without endpoints. But the number of chief factors of G cannot be greater than the number of elements of G . Hence, the order-type is countable too. Now a well-known theorem of Cantor yields that the order-type must be the order of the rationals.

(c) Follows from (a).

(d) Suppose, there exists a central chief factor M/N in G , which is not torsion-free. Then $M/N \cong C_p$ for some $p \in \pi_{\mathfrak{X}}$. Because of $\mathfrak{X} = Q\mathfrak{X}$ and $|\pi_{\mathfrak{X}}| \neq 1$ there exists $q \in \pi_{\mathfrak{X}} \setminus \{p\}$. Denote by D the diagonal and by X the base group of $K = C_p \text{ Wr } C_q$. The top group Y of K is a q -group acting on the abelian p -group X via conjugation, and therefore $X = [Y, X] \times C_X(Y) = [Y, X] \times D$ (cf. H. Kurzweil [14], Satz 7.13). Here, $[Y, X]$ and D are nontrivial, Y -invariant subgroups of X . Let X_0 be a minimal Y -invariant subgroup of X , which is contained in $[Y, X]$. Because of $C_{X_0}(Y) \leq C_X(Y) \cap X_0 \leq D \cap [Y, X] = 1$, the group Y acts non-trivially on X_0 via conjugation. Let $U = \langle X_0, Y \rangle$ and $C_p \cong Z \leq X_0$.

Now, a standard embedding $\tau_1: G/N \hookrightarrow M/N \text{ Wr } G/M$ with respect to some countermap for the canonical epimorphism $\theta: G/N \rightarrow G/M$ maps the central subgroup M/N of G/N onto the diagonal of $M/N \text{ Wr } G/M$. Identifying M/N with Z we can regard $M/N \text{ Wr } G/M$ as a subgroup of $H = U \text{ Wr } G/M$ and τ_1 as an embedding into H . Define $\tau_2: U \hookrightarrow H$ by $v\tau_2 = (1, f_v)$, where $f_v \equiv v$ for all $v \in U$. Clearly, $H \in L\mathfrak{X}$.

By Theorem 4.10 there exists $g \in M \setminus N$ with order p . Let $\{G_n\}_{n \in \mathbb{N}}$ be an ascending chain of f.g. subgroups of G with $G_1 = \langle g \rangle$ and $G = \bigcup_{n \in \mathbb{N}} G_n$. Denote by $\theta: G \rightarrow H$ the composition of the canonical epimorphism $G \rightarrow G/N$ and the embedding $\tau_1: G/N \hookrightarrow H$. Then a standard embedding $\sigma_1: G_1 \hookrightarrow 1 \text{ Wr } H$ with respect to some countermap for $\theta \upharpoonright G_1$ can be continued to $\sigma: G \hookrightarrow W = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{ Wr } H] \in L\mathfrak{X}$ by Construction 4.1. Now, $G_1 \sigma = \langle (g\theta, 1) \rangle = \{(z\tau_2, 1) : z \in Z\} \leq V = \{(v\tau_2, 1) : v \in U\} \cong U$. The group V can be described by finitely

many equations and inequations with coefficients in $\langle g\sigma \rangle \subseteq G\sigma$. Hence, there must already exist an embedding $\varrho: U \hookrightarrow G$ with $Z\varrho = \langle g \rangle$.

But $X_0\varrho$ is a minimal normal subgroup of $U\varrho$. Therefore, $g \notin N \cap X_0\varrho \trianglelefteq U\varrho$ and $g \in M \cap X_0\varrho \trianglelefteq U\varrho$ imply $N \cap X_0\varrho = 1$ and $X_0\varrho \leq M$. In particular we have $M/N = (X_0\varrho)N/N$, and every coset mN , $m \in M \setminus N$, contains exactly one element from $X_0\varrho$. Since $Y\varrho$ acts non-trivially on $X_0\varrho$ via conjugation, G/N must act non-trivially on $M/N = (X_0\varrho)N/N$ via conjugation. This contradiction to $M/N \triangleleft Z(G/N)$ shows, that M/N has to be torsion-free.

(e) Suppose, there exists a central chief factor M/N in G . By assumption, \mathfrak{X} contains a torsion-free group. Hence, $\pi_{\mathfrak{X}}$ is the set of all primes. Now (d) implies, that M/N is torsion-free. But then M/N is isomorphic to the additive group of \mathbb{Q} , and $\alpha: xN \mapsto (xN)^2$ is a non-trivial automorphism of infinite order on M/N . Now Theorem 4.9 gives an element in G/N , which acts non-trivially on M/N by conjugation. This contradicts $M/N \triangleleft Z(G/N)$.

(f) Choose $S = S_m \triangleleft S_{m-1} \triangleleft \dots \triangleleft S_1 \triangleleft G$ with $m \geq 2$ minimal. Suppose, that there exists no chief factor M/N with $S_1 = M$. Then Theorem 4.8 yields $S_2 \triangleleft G$, in contradiction to the choice of m . Hence, $S_1 = M$ for some chief factor M/N . We will show $N \leq S_k$ for $1 \leq k \leq m$.

Assume by induction, that we have $N \leq S_k$ for some $k \leq m-1$. Suppose $S_{k+1} \leq N$. Then $S_{k+1} \trianglelefteq S_k$ gives $S_{k+1} \trianglelefteq N$, and (c) yields $S_{k+1} \trianglelefteq G$, in contradiction to the choice of m . Hence, there exists $x \in S_{k+1} \setminus N$.

Fix $g \in N$ and denote by $\theta: G \rightarrow G/N$ the canonical epimorphism. Let T be a right transversal of $\langle x\theta \rangle$ in G/N , and choose $1 \neq w(x_1, \dots, x_r) \in V_M$. From the verbal completeness of G we obtain elements $y_j, z_j \in G$, $1 \leq j \leq 2\nu$, such that $g_1 = w([y_1, y_2], \dots, [y_{2\nu-1}, y_{2\nu}])$ and $g_2 = w([z_1, z_2], \dots, [z_{2\nu-1}, z_{2\nu}])$ satisfy $g = [g_1, g_2]$. Because of (c) we can even assume $y_j, z_j \in N$ for $1 \leq j \leq 2\nu$. Now, let $\{G_n\}_{n \in \mathbb{N}}$ be an ascending chain of f.g. subgroups of G with $G_1 = \langle x \rangle$, $G_2 = \langle x, y_j, z_j: 1 \leq j \leq 2\nu \rangle$ and $G = \bigcup_{n \in \mathbb{N}} G_n$. In the case $o(x\theta) = \mu < \infty$ define a countermap θ_1^* for $\theta \upharpoonright G_1$ by

$$((x\theta)^r t)\theta_1^* = x^r \quad \text{for } 0 \leq r \leq \mu - 1 \text{ and all } t \in T;$$

otherwise choose θ_1^* arbitrarily. Now the standard embedding $\sigma_1: G_1 \hookrightarrow (G_1 \cap N) \text{ Wr } G/N$ with respect to θ_1^* can be continued to $\sigma: G \hookrightarrow W = \bigcup_{n \in \mathbb{N}} [(G_n \lambda G_n) \text{ Wr } G/N] \in L\mathfrak{X}$ by Construction 4.1.

First, we consider the case $o(x\theta) = \mu < \infty$. Then $x\sigma = (x\theta, (1, f_x))$, where

$$f_x(h) = \begin{cases} x^\mu & \text{if } h = (x\theta)^{\mu-1} \cdot t \text{ for some } t \in T, \\ 1 & \text{else.} \end{cases}$$

Hence, $f_x(H)$ is finite, and Lemma 4.5 yields elements

- (i) $v_1, \dots, v_\nu \in \langle x\sigma^W \rangle$ satisfying
- (ii) $x\sigma = (x\theta, 1) \cdot w(v_1, \dots, v_\nu)$.

Every $w = (1, s)$ from the base group of $(G_2 \lambda G_2) \text{Wr } G/N \leq W$ has a unique decomposition $w = w^{(1)} \dots w^{(\mu)}$, where $w^{(i)} = (1, s^{(i)})$ is given by

$$s^{(i)}(h) = \begin{cases} s(h) & \text{if } h = (x\theta)^i t \text{ for some } t \in T, \\ 1 & \text{else.} \end{cases}$$

It follows, that

- (iii) $[[x\sigma, g_1\sigma^{(i)}], g_2\sigma^{(i)}] = [g_1\sigma^{(i)}, g_2\sigma^{(i)}] = g\sigma^{(i)}$ for $1 \leq i \leq \mu - 1$ and
- (iv) $[[x\sigma^{(x\theta, 1)}, g_1\sigma^{(\mu)}], g_2\sigma^{(\mu)}] = [g_1\sigma^{(\mu)}, g_2\sigma^{(\mu)}] = g\sigma^{(\mu)}$.

Moreover, we have

- (v) $g_1\sigma^{(i)} = w([y_1\sigma^{(i)}, y_2\sigma^{(i)}], \dots, [y_{2\nu-1}\sigma^{(i)}, y_{2\nu}\sigma^{(i)}])$,
 $g_2\sigma^{(i)} = w([z_1\sigma^{(i)}, z_2\sigma^{(i)}], \dots, [z_{2\nu-1}\sigma^{(i)}, z_{2\nu}\sigma^{(i)}])$ for $1 \leq i \leq \mu$ and
- (vi) $[y_l\sigma^{(i)}, y_{l+1}\sigma^{(i)}], [z_l\sigma^{(i)}, z_{l+1}\sigma^{(i)}] \in \langle x\sigma^W \rangle$
 for $l = 1, 3, 5, \dots, 2\nu - 1$ and $1 \leq i \leq \mu$.

The statements (i) to (vi) can be expressed as finitely many equations with coefficients $x\sigma, g\sigma \in G\sigma$ and solution in W . Therefore we obtain elements $\tilde{x}, h_1, \dots, h_{2\mu} \in G$ such that $\tilde{x} \in x \cdot V_M(\langle x\sigma \rangle)$, $h_1, \dots, h_{2\mu} \in V_M(\langle x\sigma \rangle)$ and

$$g = [[x, h_1], h_2] \cdot [[x, h_3], h_4] \cdot \dots \cdot [x, h_{2\mu-3}], h_{2\mu-2}] \cdot [x^{\tilde{x}}, h_{2\mu-1}], h_{2\mu}].$$

Now $N = V_M(\langle x\sigma \rangle)$ implies $\tilde{x} \in xN \subseteq S_k$ and

$$g \in [[x^{S_k}, N], N] \leq [[S_{k+1}^{S_k}, S_k], S_k] \leq S_{k+1}.$$

Next, consider the case $o(x\theta) = \infty$. Here, $x\sigma = (x\theta, 1)$. We decompose every $w = (1, s)$ from the base group of

$$(G_2 \lambda G_2) \text{ Wr } G/N \leq W \text{ into } w^{(1)} = (1, s^{(1)}) \text{ and } w^{(2)} = (1, s^{(2)}),$$

where

$$s^{(i)}(h) = \begin{cases} s(h) & \text{if } h = (x\theta)^{2k+i}t \text{ for some } k \in \mathbb{Z} \text{ and some } t \in T, \\ 1 & \text{else.} \end{cases}$$

Then similar considerations as in the case $o(x\theta) = \mu < \infty$ yield $g \in S_{k+1}$.

This works for every $g \in N$. Hence, $N \leq S_{k+1}$. As S is a proper subnormal subgroup of G , we have $N \leq S \neq M$. \square

Concerning Theorem 4.11 (a) it should be noted, that a corresponding result for chief factors with an element of infinite order cannot be proved by the above method, since we cannot express with finitely many equations and inequations, that an element has infinite order.

Part (b) shows, that under the assumptions of Theorem 4.11 every countable, e.c. $L\mathfrak{X}$ -group G contains exactly three kinds of normal subgroups: Firstly the normal closures $\langle g^G \rangle$, $g \in G$, which correspond to the groups M in the chief factors M/N by Theorem 4.7; secondly the verbal subgroups $V_{\langle \sigma^G \rangle}(\langle g^G \rangle)$, which are exactly the groups N in the chief factors M/N ; and finally the normal subgroups K , which do not occur in any chief factor. The latter give the pairs $(K, K) \in \bar{\Sigma}_\sigma \setminus \Sigma_\sigma$, where Σ_σ denotes the unique chief series of G (cf. Lemma 4.6), and if we index the chief factors M/N with the rationals $q \in \mathbb{Q}$ in such a way, that $q_1 < q_2 \Leftrightarrow M_{q_1} < N_{q_2}$, then we can find for every K an irrational number r such that $q_1 < r < q_2 \Leftrightarrow M_{q_1} < K < N_{q_2}$.

With regard to Theorem 4.11. (d) please note, that $|\pi_{\mathfrak{X}}| = 1$ if and only if every $L\mathfrak{X}$ -group is a p -group.

Part (f) can be used to obtain a bound on the defect of the subnormal subgroups, if we know something about the chief factors.

In order to prove the results of this section for uncountable, e.c. $L\mathfrak{X}$ -groups one could of course try to generalize Construction 4.1. Can we find for any homomorphism $\theta: G \rightarrow H$ an embedding $\sigma: G \hookrightarrow \bigcup_{S \in \Sigma} [(S\lambda S) \text{ Wr } H] \leq (G\lambda G) \text{ Wr } H$, where Σ denotes a local system of f.g. subgroups of G ? The main difficulty when trying this arises from the fact, that Σ will in general not be totally ordered by inclusion. Although we can choose Σ in such a way that it contains a minimal

member S_0 , it is not clear whether we can continue a standard embedding $\sigma_{S_0}: S_0 \hookrightarrow (S_0\lambda 1) \text{ Wr } H$ to embeddings $\sigma_S: S \hookrightarrow (S\lambda S) \text{ Wr } H$, $S \in \Sigma$, satisfying $\sigma_S \upharpoonright T = \sigma_T$ for all $T \in \Sigma$ with $T \leq S$.

In a subsequent paper we will develop other methods to prove some of the results of this section for uncountable, e.e. $L\mathfrak{X}$ -groups, where $\mathfrak{X} \subseteq L\mathfrak{Y}$.

5. The automorphism groups of the countable groups.

An automorphism α of a group G is called *locally inner*, if there exists for every finite subset F of G an element $g \in G$ satisfying $f^\alpha = fg$ for all $f \in F$. By O. H. Kegel and B. A. F. Wehrfritz [13], Theorem 6.1 every automorphism of an e.e. $L\mathfrak{Y}$ -group is locally inner.

Assume $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$. Then every automorphism α of an e.e. $L\mathfrak{X}$ -group G with $o(\alpha)$ a $\pi_{\mathfrak{X}}$ -number must be locally inner (consider the $L\mathfrak{X}$ -supergroup $\langle G, \alpha \rangle$ in the holomorph of $G!$). Conversely, if G is periodic, the order of every locally inner automorphism of G must be a $\pi_{\mathfrak{X}}$ -number or infinite. We will show now, that every countable, e.e. $L\mathfrak{X}$ -group has the maximal number of locally inner automorphisms.

THEOREM 5.1. *Let $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$, and let G be a countable, e.e. $L\mathfrak{X}$ -group. Then G has $2^{8\cdot}$ locally inner automorphisms of order m for every $\pi_{\mathfrak{X}}$ -number m and for $m = \infty$. Moreover, for any $\pi_{\mathfrak{X}}$ -number m there exist locally inner automorphisms α_1 and α_2 of order m with $o(\alpha_1\alpha_2) = \infty$.*

PROOF. Consider the case $\pi_{\mathfrak{X}} \neq \emptyset$ first. Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence of $\pi_{\mathfrak{X}}$ -numbers with $\lambda_n | \lambda_{n+1}$. Let $\{G_n\}_{n \in \mathbb{N}}$ be an ascending chain of f.g. subgroups of G with $G = \bigcup_{n \in \mathbb{N}} G_n$. We will construct an ascending chain $\{H_n\}_{n \in \mathbb{N}}$ of f.g. subgroups of G and elements $g_n, g'_n \in H_n$ such that the following holds (with $H_0 = 1$) for every $n \in \mathbb{N}$:

- (a) $G_n \leq H_n$;
- (b) $g_n \neq g'_n$, $o(g_n) = o(g'_n) = \lambda_n$, and $o(x_1 \dots x_n) = \lambda_n$ for all $x_i \in \{g_i, g'_i\}$;
- (c) $g_n, g'_n \in C_G(H_{n-1})$;
- (d) $n \leq o(g_1 g'_1 \dots g_n g'_n) < \infty$ and $\langle g_1 g'_1 \dots g_n g'_n \rangle \cap Z(H_n) = 1$;

- (e) $\{(x_1 \dots x_n)^k : 1 \leq k < \lambda_n, x_i \in \{g_i, g'_i\}\} \cap \mathbf{Z}(H_n) = \emptyset$, and $g_n^{-1} g'_n \notin \mathbf{Z}(H_n)$.

Assume by induction, that H_{n-1} has been constructed. Choose a $\pi_{\mathbf{x}}$ -number $\mu_n \geq n$ and $V_n = C_{\mu_n} \text{ wr } C_{\lambda_n}$, where $C_{\mu_n} = \langle d \rangle$ and $C_{\lambda_n} = \langle c \rangle$. Define $f: C_{\lambda_n} \rightarrow C_{\mu_n}$ by

$$f(c^i) = \begin{cases} d & \text{for } i = 0, \\ 1 & \text{for } 1 \leq i \leq \lambda_n - 1. \end{cases}$$

Then $v_1 = (c^{-1}, 1)$ and $v_2 = (c, 1)^{(1, \lambda)}$ are elements in V_n with $o(v_1) = o(v_2) = \lambda_n$ and $o(v_1 v_2) = \mu_n \geq n$. Considering the $L\mathcal{X}$ -supergroup $G \times V_n$ of G we therefore obtain elements $g_n, g'_n \in G$ satisfying (b), (c) and $n \leq \mu_n \leq o(g_1 g'_1 \dots g_n g'_n) < \infty$. Next, identify G in the obvious way with the 1-component of $G \text{ wr } C_{\lambda_n}$. No element of the top group commutes with an element of $G \setminus 1$. Hence, there must already exist $h_n \in G$ satisfying $[g_n^{-1} g'_n, h_n] \neq 1$ and $[x, h_n] \neq 1$ for all $x \in \langle g_1 g'_1 \dots g_n g'_n \rangle \setminus 1$ and all $x \in \{(x_1 \dots x_n)^k : 1 \leq k < \lambda_n, x_i \in \{g_i, g'_i\}\}$. Now choose $H_n = \langle G_n, H_{n-1}, g_n, g'_n, h_n \rangle$.

We assign to every function $\delta: \mathbf{N} \rightarrow \{0, 1\}$ the map $\alpha_\delta: G \rightarrow G$, which is defined as follows:

For every $g \in H_n$ let $g\alpha_\delta = g^{g_{\delta,1} \dots g_{\delta,n}}$, where

$$g_{\delta,i} = \begin{cases} g_i & \text{if } \delta(i) = 0, \\ g'_i & \text{if } \delta(i) = 1. \end{cases}$$

Using the properties (a) to (e) it can be shown easily, that every α_δ , is a locally inner automorphism of G , and that the assignment $\delta \rightarrow \alpha_\delta$ is one-to-one. If we choose $\{\lambda_n\}_{n \in \mathbf{N}}$ with $\lambda_n \geq n$, then every α_δ will have infinite order, whereas the choice $\lambda_n = m$ for all $n \in \mathbf{N}$ will ensure that every α_δ has order m . In the latter case $\alpha_0 \alpha_1$ is an automorphism of infinite order (here 0 (resp. 1) denotes the map $\delta \equiv 0$ (resp. $\delta \equiv 1$)).

Finally, consider the case $\pi_{\mathbf{x}} = \emptyset$. Here, we can construct similarly the chain $\{H_n\}_{n \in \mathbf{N}}$ and the elements $g_n, g'_n \in H_n$ such that the following holds for every $n \in \mathbf{N}$:

- (a') $G_n \leq H_n$;
- (b') $g_n \neq g'_n$ and $x_1 \dots x_n \neq 1$ for all $x_i \in \{g_i, g'_i\}$;
- (c') $g_n, g'_n \in \mathbf{C}_G(H_{n-1})$;

$$(d') \{ (x_1 \dots x_n)^k : 1 \leq k < n, x_i \in \{g_i, g'_i\} \} \cap \mathbf{Z}(H_n) = \emptyset \text{ and } g_n^{-1} g'_n \notin \mathbf{Z}(H_n).$$

(cf. F. Leinen [15], p. 101 for details of construction.) Now the result will follow in the same way as in the case $\pi_{\mathfrak{X}} \neq \emptyset$. \square

It should be noted, that $|\text{Aut}(G)| = 2^{|\mathcal{A}|}$ cannot be proved for uncountable, e.c. $L\mathfrak{X}$ -groups G , since S. Thomas has shown in [23], that \diamond implies the existence of e.c. $L\mathfrak{F}_\pi$ - and e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{S})$ -groups, all of whose automorphisms are inner.

COROLLARY 5.2. *Let \mathfrak{X} be a class of periodic groups with $\mathfrak{X} = S\mathfrak{X} = P\mathfrak{X}$. Then there exist countable, e.c. $L\mathfrak{X}$ -groups G, H, U and embeddings $\varrho_1: U \hookrightarrow G, \varrho_2: U \hookrightarrow H$ such that no $L\mathfrak{X}$ -group can be generated by isomorphic copies $G\eta_1$ and $H\eta_2$ of G resp. H , where $\varrho_1 \circ \eta_1 = \varrho_2 \circ \eta_2$. (*)*

PROOF. Choose any countable, e.c. $L\mathfrak{X}$ -group U . Theorem 5.1 yields automorphisms α_1 and α_2 of U with $o(\alpha_1 \alpha_2) = \infty$ and such that the subgroups $\langle U, \alpha_1 \rangle$ and $\langle U, \alpha_2 \rangle$ of the holomorph of U are countable $L\mathfrak{X}$ -groups. By J. Hirschfeld and W. H. Wheeler [8], Proposition I.1.3 there exist embeddings $\varrho_1: \langle U, \alpha_1 \rangle \hookrightarrow G$ and $\varrho_2: \langle U, \alpha_2 \rangle \hookrightarrow H$ into countable, e.c. $L\mathfrak{X}$ -groups G and H . Now, in any group $W = \langle G\eta_1, H\eta_2 \rangle$ with $\varrho_1 \eta_1 \upharpoonright U = \varrho_2 \eta_2 \upharpoonright U$ the element $(\alpha_1 \varrho_1 \eta_1) \cdot (\alpha_2 \varrho_2 \eta_2)$ induces an automorphism of infinite order on $U_{\varrho_1 \eta_1} = U_{\varrho_2 \eta_2} \leq W$. Hence, W cannot be periodic. \square

If we assume in addition to the situation in Corollary 5.2, that there exists a unique countable, e.c. $L\mathfrak{X}$ -group, then R. Grossberg and S. Shelah [1], Theorem 23 yields that $2^{\aleph_0} < 2^{\aleph_1}$ implies, that no e.c. $L\mathfrak{X}$ -group of cardinality \aleph_1 can contain an isomorphic copy of every e.c. $L\mathfrak{X}$ -group of cardinality \aleph_1 . This applies in particular to $\mathfrak{X} = L\mathfrak{F}$ and $\mathfrak{X} = L\mathfrak{F}_p$.

Now assume (1), (2) and (3), and let G be a countable, e.c. $L\mathfrak{X}$ -group. Then there exists a unique chief series $\Sigma = \{ (M_i, N_i) : i \in I \}$ in G by Theorem 4.7, and every automorphism of G must pass the pairs of Σ onto one another. Hence,

$$\text{Inv}(\Sigma) = \{ \alpha : (M\alpha, N\alpha) = (M, N) \text{ for every } (M, N) \in \Sigma \} \text{ and}$$

(*) *Added in proof:* $L\mathfrak{X}$ is not a variety by [3], Lemma 7; the negation of Corollary 5.2 holds for varieties by B. JONSSON, *Amalgamation of pure embeddings*, Algebra Univ., **19** (1984), pp. 266-268.

$\text{Stab}(\Sigma) = \{\alpha: (mN)\alpha = mN \text{ for every } (M, N) \in \Sigma \text{ and every } m \in M\}$ are two normal subgroups of $\text{Aut}(G)$ with $\text{Stab}(\Sigma) \leq \text{Inv}(\Sigma)$. The factor group $\text{Inv}(\Sigma)/\text{Stab}(\Sigma)$ can be embedded canonically into the cartesian product $\prod_{i \in I} \text{Aut}(M_i/N_i)$, while an embedding of $\text{Aut}(G)/\text{Inv}(\Sigma)$ into the group of all order-preserving permutations of the index set I is defined by $\alpha \cdot \text{Inv}(\Sigma) \mapsto \pi_\alpha$, where $(M_{\pi_\alpha(i)}, N_{\pi_\alpha(i)}) = (M_i\alpha, N_i\alpha)$ for all $i \in I$. Moreover, we note

THEOREM 5.3. *Let \mathfrak{X} satisfy (1), (2), (3), and let G be a countable, e.c. $L\mathfrak{X}$ -group with chief series Σ . Then every automorphism $\alpha \in \text{Stab}(\Sigma)$ of finite order is locally inner.*

PROOF. Because of $\mathfrak{X} = S\mathfrak{X} = Q\mathfrak{X}$ every chief factor of G is an $L\mathfrak{X}$ -group. By P. Hall and B. Hartley [6], Lemma 4 the group $\text{Stab}(\Sigma)$ must therefore have a series $\{(K_j, L_j): j \in J\}$ with $L\mathfrak{X}$ -factors. Then $\{(\langle \alpha \rangle \cap K_j, \langle \alpha \rangle \cap L_j): j \in J\}$ is a series in $\langle \alpha \rangle$ with \mathfrak{X} -factors, and $\langle \alpha \rangle$ must be a finite \mathfrak{X} -group. But now $o(\alpha)$ must be a $\pi_{\mathfrak{X}}$ -number, and α is locally inner. \square

Because of Theorem 5.3 it could be conjectured, that every automorphism of $\text{Stab}(\Sigma)$ is locally inner. But this cannot be proved with the above method, since an extension of the e.c. $L\mathfrak{X}$ -group G by C_∞ will in general not be an $L\mathfrak{X}$ -group (choose for example $\mathfrak{X} = L\mathfrak{F}_\pi$). However, in the case $\mathfrak{X} = L\mathfrak{F}_p$ it will be shown in a subsequent paper, that $\text{Stab}(\Sigma)$ coincides with the set of all locally inner automorphisms of E_p .

6. The number of countable, e.c. $L\mathfrak{S}_\pi$ -groups.

In contrast to the situation for $\mathfrak{X} = L\mathfrak{F}$ and $\mathfrak{X} = L\mathfrak{F}_p$ the following theorem holds.

THEOREM 6.1. *If π is an infinite set of primes, then there exist 2^{\aleph_0} non-isomorphic, countable e.c. $L\mathfrak{S}_\pi$ -groups.*

PROOF. Since every countable $L\mathfrak{S}_\pi$ -group can be embedded into a countable, e.c. $L\mathfrak{S}_\pi$ -group by J. Hirschfeld and W. H. Wheeler [8], Proposition I.1.3, it suffices to establish the existence of 2^{\aleph_0} non-isomorphic, f.g. \mathfrak{S}_π -groups. Define $G_\sigma = \left(\left(\prod_{p \in \sigma} C_p \right) \text{Wr } C_\infty \right) \text{Wr } C_\infty$ for every $\sigma \subseteq \pi$. Then B. H. Neumann and H. Neumann have shown

in [18], p. 470, that G_σ contains a two-generator subgroup H_σ with $\prod_{p \in \sigma} C_p \leq H_\sigma$. But H_σ is an \mathfrak{S}_σ -group containing an element of order p for every $p \in \sigma$. Hence, $H_\sigma \not\cong H_\tau$ for all $\sigma, \tau \subseteq \pi$ with $\sigma \neq \tau$. \square

It should be noted that the idea of the proof cannot be used, if \mathfrak{X} is a class of $L\mathfrak{X}$ -groups, since there exist only countably many finite groups.

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