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A class of finite q-series - II

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A Class of Finite q-Series - II.

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SUMMARY - Some elementary results are used here to prove an interesting unification (and generalization) of several finite summation formulas associated with various special hypergeometric functions of two variables. Further generalizations involving series with essentially arbitrary terms and their q-extensions are also presented. The main results (2.1), (3.1) and (3.2), and also their multivariable generalizations (3.4) and (3.9), are believed to be new.

1. Introduction.

In terms of the Pochhammer symbol $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$, let $F^{p:r;u}_{k:s;v}$ denote the generalized (Kampé de Fériet's) double hypergeometric function defined by (*cf.* [3]; see also [1], p. 150; [8], p. 63, and [9], p. 423)

$$(1.1) \qquad F^{p:r;u}_{k:s;v}\left[\begin{matrix}(a_p): & (c_r); & (\alpha_u); & \\ & & & x, y \\ (b_k): & (d_s); & (\beta_v); & \end{matrix}\right] = \sum_{l,m=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{l+m}\prod_{j=1}^{r}(c_j)_l\prod_{j=1}^{u}(\alpha_j)_m}{\prod_{j=1}^{k}(b_j)_{l+m}\prod_{j=1}^{s}(d_j)_l\prod_{j=1}^{v}(\beta_j)_m}\frac{x^l}{l!}\frac{y^m}{m!},$$

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where, for convergence of the double hypergeometric series,

(i) $p+r<k+s+1,\ p+u<k+v+1,$

$|x|<\infty$, and $|y|<\infty$,

or

(ii) $p+r=k+s+1,\ p+u=k+v+1,$

and

$$(1.2)\qquad \begin{cases} |x|^{1/(p-k)}+|y|^{1/(p-k)}<1\,, & \text{if } p>k\,,\\ \max\{|x|,\ |y|\}<1\,, & \text{if } p\leqq k\,,\end{cases}$$

unless, of course, the series terminates; here, and in what follows, (a_p) abbreviates the array of p parameters $a_1, \ldots, a_p$, with similar interpretations for (b_k), *et cetera.*

With a view to extending an earlier result of Munot ([4], p. 694, Equation (3.2)), involving a finite sum of certain double hypergeometric functions, Shah [5] proved two analogous summation formulas for some very special Kampé de Fériet functions. These finite summation formulas of Shah may be recalled here in the following (essentially equivalent) forms [1] (*cf.* [5], p. 173, Equations (2.1) and (3.1)):

$$(1.3)\qquad \sum_{n=0}^{N}\frac{(\lambda)_n(\mu)_{N-n}}{n!(N-n)!}F^{2:1;1}_{1:1;1}\left[\begin{matrix}-\varrho+\delta,\delta: & \beta; & -N+n; \\ -\varrho+\delta-\sigma: & \gamma; & \mu;\end{matrix}\ x, y\right]=$$

$$=\frac{(\lambda+\mu)_N}{N!}F^{2:1;1}_{1:1;1}\left[\begin{matrix}-\varrho+\delta,\delta: & \beta; & -N; \\ -\varrho+\delta-\sigma: & \gamma; & \lambda+\mu;\end{matrix}\ x, y\right],$$

$$(1.4)\qquad \sum_{n=0}^{N}\frac{(\lambda)_n(\mu)_{N-n}}{n!(N-n)!}F^{1:2;2}_{1:1;1}$$

$$\left[\begin{matrix}-\varrho+\delta: & \beta,\delta-\alpha-\beta-N+1; & -N+n,\alpha+\beta+N+1; \\ -\varrho+\delta-\sigma: & \gamma; & \mu;\end{matrix}\ \xi,\eta\right]=$$

(1) The summation formula (1.4) appears in Shah's paper (*cf.* [5], p. 171) with an obvious error. Precisely the same formulas as (1.3) and (1.4) happen to be the *main* results of an identical paper (with a different title: *A note on Kampé de Fériet functions*) by Shah [*Math. Notae*, **28** (1981), pp. 37-42].

$$= \frac{(\lambda+\mu)_N}{N!} F^{1:2;2}_{1:1;1} \left[\begin{matrix} -\varrho+\delta: & \beta, \delta-\alpha-\beta-N+1; & -N, \alpha+\beta+N+1; \\ -\varrho+\delta-\sigma: & \gamma; & \lambda+\mu; \end{matrix} \; \xi, \eta\right],$$

$$\{\xi \equiv x,\ \eta \equiv \tfrac{1}{2}(1-x)\}$$

where (according to Shah [5]) σ and N are *both* non-negative integers.

Shah's proofs of (1.3) and (1.4), as also Munot's similar proof of the special case of (1.3) when $\sigma = 0$, are long and involved. The object of the present sequel to our paper [7] is to derive much more general results than (1.3) and (1.4) from rather elementary considerations. Our summation formulas (2.1), (3.1), (3.2) and (3.4), and the multivariable q-extension (3.9), are believed to be new.

2. – Finite sum of generalized Kampé de Fériet functions.

We begin by establishing the following general summation formula, involving Kampé de Fériet's functions, which indeed unifies and extends the known results (1.3) and (1.4):

$$\text{(2.1)} \quad \sum_{n=0}^{N} \frac{(\lambda)_n(\mu)_{N-n}}{n!(N-n)!} F^{p:r;u+1}_{k:s;v+1} \left[\begin{matrix} (a_p): & (c_r); & -N+n, (\alpha_u); \\ (b_k): & (d_s); & \mu, (\beta_v); \end{matrix} \; x, y\right] =$$

$$= \frac{(\lambda+\mu)_N}{N!} F^{p:r;u+1}_{k:s;v+1} \left[\begin{matrix} (a_p): & (c_r); & -N, (\alpha_u); \\ (b_k): & (d_s); & \lambda+\mu, (\beta_v); \end{matrix} \; x, y\right],$$

where N is a non-negative integer, and the various parameters and variables are so constrained that each member of (2.1) exists.

PROOF. Denoting, for convenience, the left-hand member of (2.1) by $S(x, y)$, let

$$\text{(2.2)} \quad A_n = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{k}(b_j)_n}, \quad B_n = \frac{\prod_{j=1}^{r}(c_j)_n}{\prod_{j=1}^{s}(d_j)_n}, \quad C_n = \frac{\prod_{j=1}^{u}(\alpha_j)_n}{\prod_{j=1}^{v}(\beta_j)_n}, \quad n \geqq 0 .$$

If we apply the definition (1.1) and use the abbreviations in (2.2), we find from (2.1) that

$$S(x, y) = \sum_{n=0}^{N} \sum_{l=0}^{\infty} \sum_{m=0}^{N-n} A_{l+m} B_l C_m \frac{(\lambda)_n (\mu)_{N-n}}{n!(N-m-n)!(\mu)_m} \frac{x^l}{l!} \frac{(-y)^m}{m!} =$$

$$= \sum_{l, m \geqq 0} A_{l+m} B_l C_m \frac{x^l}{l!} \frac{(-y)^m}{m!} \sum_{n=0}^{N-m} \frac{(\lambda)_n (\mu+m)_{N-m-n}}{n!(N-m-n)!},$$

where we have employed such elementary identities as

$$(2.3) \qquad (\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n \quad \text{and} \quad (N-n)! = \frac{(-1)^n N!}{(-N)_n}, \quad 0 \leqq n \leqq N.$$

Since

$$(2.4) \qquad \sum_{n=0}^{N} \frac{(\lambda)_n (\mu)_{N-n}}{n!(N-n)!} = \frac{(\lambda+\mu)_N}{N!}, \quad N = 0, 1, 2, \ldots,$$

we readily have

$$S(x, y) = \sum_{l, m \geqq 0} A_{l+m} B_l C_m \frac{x^l}{l!} \frac{(-y)^m}{m!} \frac{(\lambda+\mu+m)_{N-m}}{(N-m)!} =$$

$$= \frac{(\lambda+\mu)_N}{N!} \sum_{l, m \geqq 0} A_{l+m} B_l C_m \frac{(-N)_m}{(\lambda+\mu)_m} \frac{x^l}{l!} \frac{y^m}{m!}$$

which is precisely the right-hand member of (2.1).

This evidently completes the proof of the summation formula (2.1) under the constraints stated already.

The summation formula (2.1) corresponds to Shah's result (1.3) when

$$(2.5) \qquad p-1 = k = 1, \quad r = s = 1 \quad \text{and} \quad u+1 = v = 1.$$

Moreover, in its special case when

$$(2.6) \qquad p = k = 1, \quad r-1 = s = 1 \quad \text{and} \quad u = v = 1,$$

if we further set $y = \frac{1}{2}(1-x)$, our summation formula (2.1) would yield a slightly improved version of Shah's result (1.4).

3. – Further generalizations and q-extensions.

By examining our proof of the summation formula (2.1) rather closely, we are led immediately to an obvious further generalization of (2.1) in the form:

$$(3.1)\qquad \sum_{n=0}^{N} \frac{(\lambda)_n(\mu)_{N-n}}{n!(N-n)!} \sum_{l=0}^{\infty} \sum_{m=0}^{N-n} A_{l+m} B_l C_m \frac{(-N+n)_m}{(\mu)_m} \frac{x^l}{l!} \frac{y^m}{m!} = \frac{(\lambda+\mu)_N}{N!} \sum_{l=0}^{\infty} \sum_{m=0}^{N} A_{l+m} B_l C_m \frac{(-N)_m}{(\lambda+\mu)_m} \frac{x^l}{l!} \frac{y^m}{m!},$$

where $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ are bounded complex sequences.

More generally, for every bounded double sequence $\{\Omega(l, m)\}$, we have

$$(3.2)\qquad \sum_{n=0}^{N} \frac{(\lambda)_n(\mu)_{N-n}}{n!(N-n)!} \sum_{l=0}^{\infty} \sum_{m=0}^{N-n} \Omega(l, m) \frac{(-N+n)_m}{(\mu)_m} \frac{x^l}{l!} \frac{y^m}{m!} = \frac{(\lambda+\mu)_N}{N!} \sum_{l=0}^{\infty} \sum_{m=0}^{N} \Omega(l, m) \frac{(-N)_m}{(\lambda+\mu)_m} \frac{x^l}{l!} \frac{y^m}{m!},$$

which would evidently reduce to (3.1) in the special case when

$$(3.3)\qquad \Omega(l, m) = A_{l+m} B_l C_m\,, \qquad l, m = 0, 1, 2, \ldots\,.$$

Formula (3.1) yields the hypergeometric form (2.1) when the arbitrary coefficients A_n, B_n and C_n $(n = 0, 1, 2, \ldots)$ are chosen as in (2.2).

It is not difficult to give the following multivariable extension of (3.2):

$$(3.4)\qquad \sum_{n=0}^{N} \frac{(\lambda)_n(\mu)_{N-n}}{n!(N-n)!} \sum_{l_1,\ldots,l_k=0}^{\infty} \sum_{m=0}^{N-n} \Lambda(l_1, \ldots, l_k; m) \frac{(-N+n)_m}{(\mu)_m} \cdot \frac{x_1^{l_1}}{l_1!} \cdots \frac{x_k^{l_k}}{l_k!} \frac{y^m}{m!} = \frac{(\lambda+\mu)_N}{N!} \sum_{l_1,\ldots,l_k=0}^{\infty} \sum_{m=0}^{N} \Lambda(l_1, \ldots, l_k; m) \frac{(-N)_m}{(\lambda+\mu)_m} \frac{x_1^{l_1}}{l_1!} \cdots \frac{x_k^{l_k}}{l_k!} \frac{y^m}{m!},$$

which, for $k = 1$, corresponds to (3.2), $\{\Lambda(l_1, \ldots, l_k; m)\}$ being a suitably bounded multiple sequence.

In order to present the q-extensions of the various finite summation formulas considered in this paper, we begin by recalling the definition (*cf.* [2]; see also [6], Chapter 3)

$$(3.5) \qquad (\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right)$$

for arbitrary q, λ and μ, $|q| < 1$, so that

$$(3.6) \qquad (\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q) \dots (1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, \dots\}, \end{cases}$$

and

$$(3.7) \qquad \lim_{q \to 1} \left(\frac{(q^\lambda; q)_n}{(q^\mu; q)_n} \right) = \frac{(\lambda)_n}{(\mu)_n}, \qquad n = 0, 1, 2, \dots,$$

for arbitrary λ and μ, $\mu \neq 0, -1, -2, \dots$.

We shall also need a q-analogue of the identity (2.4) in the form:

$$(3.8) \qquad \sum_{n=0}^{N} \frac{(\lambda; q)_n (\mu; q)_{N-n}}{(q; q)_n (q; q)_{N-n}} \mu^n = \frac{(\lambda\mu; q)_N}{\lambda^N (q; q)_N}, \qquad N = 0, 1, 2, \dots,$$

which is an easy consequence of a q-hypergeometric summation theorem ([6], p. 247, Equation (IV.1)).

Assuming the coefficients $\Lambda(l_1, \dots, l_k; m)$ $[l_j \geqq 0,\ j = 1, \dots, k;\ m \geqq 0]$ to be suitably bounded complex numbers, it is not difficult to prove, using (3.8) along the lines detailed in the preceding section, the following q-extension of the general result (3.4):

$$(3.9) \qquad \sum_{n=0}^{N} \frac{(\lambda; q)_n (\mu; q)_{N-n}}{(q; q)_n (q; q)_{N-n}} \mu^n \sum_{l_1, \dots, l_k = 0}^{\infty} \sum_{m=0}^{N-n} \Lambda(l_1, \dots, l_k; m) \cdot \\ \cdot \frac{(q^{-N+n}; q)_m}{(\mu; q)_m} \frac{x_1^{l_1}}{(q; q)_{l_1}} \cdots \frac{x_k^{l_k}}{(q; q)_{l_k}} \frac{y^m}{(q; q)_m} = \\ = \frac{(\lambda\mu; q)_N}{\lambda^N (q; q)_N} \sum_{l_1, \dots, l_k = 0}^{\infty} \sum_{m=0}^{N} \Lambda(l_1, \dots, l_k; m) \frac{(q^{-N}; q)_m}{(\lambda\mu; q)_m} \cdot \\ \cdot \frac{x_1^{l_1}}{(q; q)_{l_1}} \cdots \frac{x_k^{l_k}}{(q; q)_{l_k}} \frac{(\lambda y)^m}{(q; q)_m},$$

which holds true whenever both members exist.

For $k = 1$, (3.9) would immediately yield a q-extension of the summation formula (3.2). If, in the special case of (3.9) when $k = 1$, we further specialize the resulting coefficients in a manner analogous to (3.3) and (2.2) but using the definition (3.6), we can deduce appropriate q-extensions of the summation formulas (2.1) and (3.1) as particular cases of the finite q-series (3.9). Moreover, in view of the identity (3.7), the q-summation formula (3.9) can be shown to yield (3.4) in the limit when $q \to 1$.

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