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Some Commutativity Results for Rings.

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SUMMARY - It is proved that certain rings satisfying variable identities of the form $[x^m, y^n, \ldots, y^n] = 0$ (in particular $[x^m, y^n, y^n] = 0$ with n bounded) must have nil commutator ideals.

In this paper we prove results based on questions of Herstein [2, p. 357] and generalizing results of Klein, Nada and Bell [3] and Klein and Nada [4].

Let R be an associative ring and $\mathbb{Z}$ respectively $\mathbb{Z}^+$ be the set of integers respectively positive integers. For $a, b \in R$ define generalized commutators $[a, b]_k$, $k \in \mathbb{Z}^+$, as follows: $[a, b]_1 = [a, b] = ab - ba$ and for $i \in \mathbb{Z}^+$, $[a, b]_{i+1} = [[a, b]_i, b]$. R is called a k-ring if for all a, $b \in R$ there exists $m = m(a, b)$, $n = n(a, b) \in \mathbb{Z}^+$ such that $[a^m, b^n]_k = 0$. R is called a n-bounded k-ring if the above n is fixed. Let $\mathbb{Z}\{X\}$ be the free $\mathbb{Z}$-algebra generated by the noncommuting indeterminates $x_1, x_2, x_3, \ldots$ [5; pp. 2-4]. Substitute $r_i \in R$ for x_i in $f \in \mathbb{Z}\{X\}$ to get an element of R. The additive subgroup of R generated by all these elements is denoted by $f(R)$. Let $f \in \mathbb{Z}\{X\}$, and $N \in \mathbb{Z}^+$. R is called a N-f-k-ring if for all $a \in f(R)$ and $b \in R$ there exists $m = m(a, b)$, $n = n(a, b) \in \mathbb{Z}^+$ such that $m \leqslant N$ and $[a^m, b^n]_k = 0$. R is called left (right)-s-unital if $a \in Ra$ $(a \in aR)$ for all $a \in R$.

Let R_{reg} be the set of (left and right) regular elements of R, R_{nil} the set of nilpotent elements of R, R' the commutator ideal of R, $C(R)$ the center of R and $i \wedge j$ the greatest common divisor of $i, j \in \mathbb{Z}^+$.

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For $a \in R$ and $A, B \subseteq R$ let $C_R(a) = \{b \in R: ba = ab\}$ and $[A, B]$ be the additive subgroup of R generated by $\{[a, b]: a \in A, b \in B\}$. We shall prove:

THEOREM. Each of the following conditions implies $R' \subseteq R_{\text{nil}}$:

(1) R is a 2-ring and a n-bounded k-ring (in particular, R is a n-bounded 2-ring).

(2) R is a N-f-2-ring and a k-ring.

(3) R is a 2-ring and a N-f-4-ring.

(4) R is a left-or right-s-unital k-ring.

This results generalize the following sufficient conditions: For all $a, b \in R$ there exists $m = m(a, b) \in \mathbb{Z}^+$ such that $[a^m, b]_2 = 0$ [4; Theorem, p. 361]. Let $N \in \mathbb{Z}^+$. For all $a, b \in R$ there exists $m = m(a, b)$, $n = n(a, b) \in \mathbb{Z}^+$ such that $m \leqslant N$ and $[a^m, b^n]_2 = 0$ [3; Theorem 1, p. 286]. R is a k-ring with $1 \in R$ [3; Theorem 3, p. 288]. We first prove:

LEMMA. Let R be prime, torsionfree, $R = R_{\text{reg}} \cup R_{\text{nil}}$ and $C(R) = 0$.

(*a*) Let $0 \neq f \in \mathbb{Z}\{X\}$. Then there exists an ideal $I \neq 0$ of R such that $[I, R] \subseteq f(R)$.

(*b*) Let $L \neq 0$ be a Lie ideal of R and $N \in \mathbb{Z}^+$. For all $a \in R$ and $b \in R_{\text{reg}}$ suppose there exists $m = m(a, b)$, $n = n(a, b) \in \mathbb{Z}^+$ such that $m \leqslant N$ and $[a^m, b^n]_k = 0$. Then $c^2 = 0$ for all $c \in C_R(b) \cap R_{\text{nil}}$ and $b \in R_{\text{reg}}$ if $k = 4$ and $C_R(b) \cap R_{\text{nil}} = 0$ for all $b \in R_{\text{reg}}$ if $k = 2$.

(*c*) Let R be a n-bounded k-ring. Then $C_R(b^i) \subseteq C_R(b^n)$ for all $b \in R_{\text{reg}}$ and $i \in \mathbb{Z}^+$.

PROOF. (*a*) Using [5; pp. 6, 7] we get a multilinear polynomial $0 \neq g \in \mathbb{Z}\{X\}$ such that $g(R) \subseteq f(R)$. Since $g(R) \neq 0$ [5; Theorem 1.6.27, p. 47] and $[g(R), R] \subseteq g(R)$ the conclusion follows by [1; Theorem 6, p. 570].

(*b*) Let $k = 4$. Assume that there exists $b \in R_{\text{reg}}$, $c \in C_R(b)$ and $2 \leqslant l \in \mathbb{Z}^+$ such that $c^{l+1} = 0 \neq c^l$. We shall get a contradiction. For each $a \in L$ and $M \in \mathbb{Z}^+$ there exists a subset $\mathcal{M}$ of $\mathbb{Z}^+$ with M elements and $m, n \in \mathbb{Z}^+$ with $m \leqslant N$ such that $[a^m, (b + ic)^n]_4 = 0$ for all $i \in \mathcal{M}$. Using $c^{l+1} = 0$ and a Vandermonde argument analogous to [3; p. 287] we get homogeneous equations $g_j(a, b, c) = 0$, where a, b and c appear in each formal monomial exactly m, $4n - j$ and j-times. We use tacitly $b \in R_{\text{reg}}$ and $c^{l+1} = 0$.

Since

$$0 = g_1(a, b, c)c^l = 4\binom{n}{1}[[a^m, b^n]_3, b^{n-1}c]c^l = 4nb^{n-1}c[a^m, b^n]_3c^l$$

we have $c^l[a^m, b^n]_3c^l = 0$ and $c^{l-1}g_2(a, b, c)c^{l-1} = 0$, hence

$$0 = 4\binom{n}{2}c^{l-1}[[a^m, b^n]_3, b^{n-2}c^2]c^{l-1} +$$
$$+ 6\binom{n}{1}\binom{n}{1}c^{l-1}[[a^m, b^n]_2, b^{n-1}c]_2c^{l-1} = -12n^2b^{n-1}c^l[a^m, b^n]_2c^lb^{n-1},$$

therefore

$$c^l[a^m, b^n]_2c^l = 0 \quad \text{and} \quad c^{l-2}g_3(a, b, c)c^{l-1} = 0\,.$$

Analogously we get

$$c^l[a^m, b^n]c^l = 0 \quad \text{and} \quad c^{l-2}g_4(a, b, c)c^{l-2} = 0$$

and finally $c^la^mc^l = 0$.

Choose $m(a)$ in $\mathbb{Z}^+$ maximal with respect to $c^la^{m(a)}c^l = 0$. Put $M = \max\{m(a)\colon a \in L\}$. Choose $d \in L$ such that $m(d) = M$. For each $a \in L$ there exists $M \geqslant m \in \mathbb{Z}^+$ such that $c^l(ia + d)^mc^l = 0$ for infinitely many $i \in \mathbb{Z}^+$. Using a Vandermonde argument we get $c^la^mc^l = 0 = c^ld^mc^l$, hence $m = M$. We have proved that $c^la^Mc^l = 0$ for all $a \in L$. There exists an ideal $I \neq 0$ of R such that $[I, I] \subseteq L$ [1; Theorem 6, p. 570]. Using (a) for $R = I$ and $f = [x_1, x_2]^M$ we get an ideal $J \neq 0$ of I such that $[J, J] \subseteq f(I)$. Then $K = IJI \neq 0$ is an ideal of R and $0 = c^l[K, cK]c^l = c^lKcKc^l$, hence $c^l = 0$, a contradiction.

Let $k = 2$. The condition for $k = 4$ is still satisfied, hence $c^2 = 0$ for all $b \in R_{\text{reg}}$ and $c \in C_R(b) \cap R_{\text{nil}}$. As above we get $c = 0$ using $0 = g_2(a, b, c) = 2n^2b^{n-1}ca^mcb^{n-1}$.

(c) Let $b \in R_{\text{reg}}$, $i, j \in \mathbb{Z}^+$ und $l = i \wedge n$. We show (i)-(iv) step by step.

(i) $C_R(b^i) \cap C_R(b^j) \subseteq C_R(b^{i\wedge j})$.

We can assume that $i < j$. For $a \in C_R(b^i) \cap C_R(b^j)$ we have

$$0 = [a, b^j] = [a, b^i]b^{j-i} + b^i[a, b^{j-i}] = b^i[a, b^{j-i}]\,,$$

hence $[a, b^i] = 0 = [a, b^{j-i}]$. By induction over $i + j$ we get the conclusion.

(ii) Let $a \in C_R(b^i)$ and $a^2 = 0$. Then $a \in C_R(b^l)$.

Let $m \in \mathbb{Z}^+$ be such that $0 = [(a + b^i)^m, b^n]_k = mb^{i(m-1)} [a, b^n]_k$. Then $[a, b^n]_k = 0$. For $c = [a, b^n]_{k-1}$ we have $[c, b^n] = 0 = [c, b^i]$, hence $[c, b^l] = 0$ by (i). For $c = [a, b^l]$ we have $[c, b^n]_{k-1} = 0 = [c, b^i]$, hence $[a, b^l]_k = 0$ by induction over k. For $c = [a, b^l]_{k-2}$ and $j = i/l$ we have $0 = [c, b^i] = jb^{i-l}[c, b^l]$, hence $[a, b^l]_{k-1} = 0$, therefore $a \in C_R(b^l)$ by induction over k.

By induction over the index of nilpotence of a we get

(iii) Let $a \in C_R(b^i) \cap R_{\text{nil}}$. Then $a \in C_R(b^l)$.

(iv) $C_R(b^i) \subseteq C_R(b^l) \subseteq C_R(b^n)$.

If $C_R(b^i) \subseteq R_{\text{reg}}$, then $C_R(b^i)$ is commutative by [3; Lemma, p. 286], hence (iv). Otherwise let $a \in C_R(b^i)$, $a^2 = 0$ and $c \in C_R(b^i) \cap R_{\text{reg}}$. Then $ac \in C_R(b^i) \cap R_{\text{nil}}$, hence $0 = [ac, b^l] = a[c, b^l]$ by (iii), therefore $[c, b^l] \in C_R(b^i) \cap R_{\text{nil}}$, hence $[c, b^l]_2 = 0$ by (iii), finally $c \in C_R(b^l)$ as above.

Proof of Theorem. (1)-(3) Let us assume that $R' \not\subseteq R_{\text{nil}}$. We shall get a contradiction. By [2] we can assume, that R is prime, torsionfree, $R = R_{\text{reg}} \cup R_{\text{nil}}$, $C(R) = 0$, R is a k-ring but not a k-1-ring and $k > 1$.

(1) We show (i)-(iii) step by step.

(i) Let $a \in R$, $b \in R_{\text{reg}}$ and m, $i \in \mathbb{Z}^+$. Then $[a^m, b^i]_2 = 0$ implies $[a^m, b^n]_2 = 0$.

By (c) we have $0 = [[a^m, b^i], b^n] = [[a^m, b^n], b^i]$, hence $[a^m, b^n]_2 = 0$.

(ii) For a $b \in R_{\text{reg}}$ there exists $m = m(a, b) \in \mathbb{Z}^+$ such that $n|m$ and $[a^i, b^j]_2 = [b^j, a^i]_2 = [a^i, b^j]^2 = 0$ for all i, $j \in m\mathbb{Z}^+$.

By (i) there exists r, $s \in \mathbb{Z}^+$ such that $[a^{nr}, b^n]_2 = 0 = [b^{ns}, a^n]_2$. For $u = a^{nr}$ and $v = b^{ns}$ we have $[u, v]_2 = 0 = [v, u]_2$. By (i) there exists $1 < t \in \mathbb{Z}^+$ such that $0 = [u^t, b^n]_2$. Hence $0 = [u^t, v]_2 = t(t - 1)u^{t-2}[u, v]^2$, therefore $[u, v]^2 = 0$. We have $[u^i, v^j]_2 = [v^j, u^i]_2 = [u^i, v^j]^2 = 0$ for all i, $j \in \mathbb{Z}^+$. Using $m = nrs$ we get (ii).

(iii) Let a, $b \in R_{\text{reg}}$, $c \in C_R(a)$ and $c^2 = 0$. Then $[c, b^n]_3 = 0$.

Let $m = m(a, b)$ as in (ii). By (i) there exists $1 < l \in \mathbb{Z}^+$ such that $[(a^m + c)^l, b^m]_2 = 0$. Analogous to [2; p. 355] we get (iii).

By an argument as in [2; pp. 355, 356] we get a Lie ideal $L \neq 0$ of R such that $[a, b^n]_3 = 0$ for all $a \in L$ and $b \in R_{\text{reg}}$. By (*b*) we have $c^2 = 0$ for all $b \in R_{\text{reg}}$ and $c \in C_R(b) \cap R_{\text{nil}}$. We conclude the proof as in [4; p. 361].

(2) Since R is not a k-1-ring there exists $a, b \in R$ such that $[a^i, b^j]_{k-1} \neq 0$ for all $i, j \in \mathbb{Z}^+$. There exists $m, n, 1 < l \in \mathbb{Z}^+$ such that $[a^m, b^n]_k = 0 = [a^{ml}, b^n]_k$. Using the formula $[uv, w] = [u, w]v + u[v, w]$ we get $0 = [a^{ml}, b^n]_{(k-1)l} = [a^m, b^n]_{k-1}^l$. Hence there exists $b \in R_{\text{reg}}$ and $c \in C_R(b)$ such that $c \neq 0 = c^2$ in contradiction to (*a*) and (*b*).

(3) We have $c^2 = 0$ for all $b \in R_{\text{reg}}$ and $c \in C_R(b) \cap R_{\text{nil}}$ by (*a*) and (*b*) and can conclude the proof as in [4; p. 361].

(4) Let R be left-s-unital. Assume that $R' \not\subseteq R_{\text{nil}}$. Choose a finite subset A of R and $e \in R$ such that $S' \not\subseteq S_{\text{nil}}$ for the subring S of R generated by A and $ea = a$ for all $a \in A$ [6]. Let T be the subring of R generated by $A \cup \{e\}$ and I the ideal of T generated by $\{ae - a: a \in A \cup \{e\}\}$. Since $IA = 0$ T/I has no nil commutator ideal. But T/I ist a k-ring with 1 in contradiction to [2; Theorem 3, p. 288].

Remark. For $a, b \in R$ define $a \circ b = ab + ba$. Let $a, b \in R$, m, $n_i \in \mathbb{Z}^+$ and $*_i \in \{[\ ,\ ], \circ\}$ such that $\left(\left((a^m *_1 b^{n_1}) *_2 b^{n_2}\right) \dots\right) *_k b^{n_k} = 0$. Then using the formula $[u, v^2] = [u, v] \circ v = [u \circ v, v]$ we get $[a^m, b^n]_k = 0$ for $n = 2\Pi n_i$. Thus the use of $\circ$ as well as $[\ ,\ ]$ provides no real generalisation.

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