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## A Criterion for a Rational Projectively Normal Variety to be Almost-Factorial.

REMO GATTAZZO (\*)

SUNTO - Si dimostra che una varietà razionale proiettivamente normale  $\mathcal{F} \subset \mathbb{P}^n$ ,  $n = \dim \mathcal{F}$ , è semifattoriale se e solo se ammette una parametrizzazione  $\sigma_p: \mathbb{P}^n \rightarrow \mathcal{F}$  che gode della seguente proprietà: sono sottoinsieme intersezione completa di  $\mathcal{F}$  tutte le componenti di  $\mathcal{F} \cap S$ , dove  $S$  è una ipersuperficie di  $\mathbb{P}^n$  legata alla parametrizzazione  $\sigma_p$ . Vengono date applicazioni ed esempi.

### 0. Introduction.

In the forthcoming paper « Factorial singularities on rational quartic surfaces of  $\mathbb{P}^3$  », written in collaboration with P. C. Craighero, the properties of such surfaces in connection with their parametric representation on a plane  $\mathbb{P}^2$  have been deeply investigated. In such a research, the curves on the surfaces coming from particular points of the plane, that is the exceptional curves, play a leading role. This fact has suggested the author the investigation between the relation on almost-factoriality of rational surfaces and one of its parametric representation.

This paper presents the answer to the matter. It holds that, if  $\mathcal{F}$  is a rational projectively normal variety,  $\mathcal{F}$  is almost-factorial iff are set-theoretic complete intersection on  $\mathcal{F}$  only a finite subset of sub-

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varieties of codimension 1 of  $\mathcal{F}$  which are *referred* to the parametrization (see def. 5); however, among these there are also the exceptional subvarieties (see Prop. 1). An immediate affine version of this result is given and some applications to classes of surfaces of  $A^3$  or of  $P^3$  are enunciated.

The result is an extension of the well-known criterion of D. Galarati for the almost-factoriality which holds for monoid hypersurfaces in  $P^N$ . According to it, one can, for example, tackle the question of the classification of the almost-factorial rational surfaces of degree four with only double points.

On the other hand, as it is proved in [1] Prop. 2.11 p. 260, the almost-factoriality is a non local property which is unaffected by isomorphisms in the class of the projectively normal varieties. Since  $P^n$  is factorial (so almost-factorial), for every  $n > 0$  one can find very large classes of almost-factorial varieties by means of isomorphisms: for example all the  $m$ -ple embeddings of  $P^n$  in  $P^N$ ,  $N = \binom{n+m}{n} - 1$ ,  $m > 1$ . According to the new result, one can build more models of rational almost-factorial varieties which are not necessarily isomorphic to some  $P^n$  or to some monoid.

At the end of the paper a rational quartic surface with only double points which results to be 12-almostfactorial is examined as a detailed example, and some classes of rational almost-factorial surfaces of  $A^3$  are pointed out as well.

**1.** Let  $k$  be an algebraically closed field of any characteristic.  $P^N$  and  $A^N$  denote respectively the projective and the affine space of dimension  $N$  over  $k$ . Variety on  $P^N$  (or on  $A^N$ ), will mean always an algebraic irreducible and reduced closed subset on  $P^N$  (or on  $A^N$ ),  $N \geq 3$ .

A prime divisor (or more shortly) a *prime* on a variety  $\mathcal{F}$ , non singular in codimension 1, will be an irreducible and reduced subvariety on  $\mathcal{F}$  of codimension 1. Curves surfaces, hypersurfaces on  $P^N$  (or on  $A^N$ ) will be varieties of dimension 1, 2,  $N - 1$  respectively.

Let  $\mathcal{F}$  be a variety on  $P^N$ :  $J(\mathcal{F})$ ,  $k[\mathcal{F}]$ ,  $k(\mathcal{F})$  denote respectively the (prime and homogeneous) ideal of  $\mathcal{F}$  in  $k[X_0, \dots, X_N]$ , ( $= k[X]$ ), the quotient ring  $k[X]/J(\mathcal{F})$ , the field of rational functions on  $\mathcal{F}$ , ( $=$  field of quotients of elements in  $k[\mathcal{F}]$  of same degree).

**DEFINITIONS 1.** A Variety  $\mathcal{F} \subset P^N$  is called *projectively normal* if  $k[\mathcal{F}]$  is integrally closed.

2. A prime  $C$  on a variety  $\mathcal{F} \subset \mathbf{P}^N$ , is *set-theoretic complete intersection* (S.T.C.I.) on  $\mathcal{F}$  with multiplicity  $\lambda$ , if exists a hypersurface  $\mathcal{G} \subset \mathbf{P}^N$  such that  $\mathcal{F} \cdot \mathcal{G} = \lambda C$ , i.e.  $\lambda C$  is the complete intersection of  $\mathcal{F}$  and  $\mathcal{G}$ .

3. A Variety  $\mathcal{F}$  projectively normal is *almost-factorial* if every prime  $C \subset \mathcal{F}$  is S.T.C.I. on  $\mathcal{F}$ . In particular  $\mathcal{F}$  is  $\varrho$ -almostfactorial if every prime  $C$  on  $\mathcal{F}$  is S.T.C.I. on  $\mathcal{F}$  with multiplicity  $\lambda < \varrho$ .

Let  $\mathcal{F}$  be a projective variety of  $\mathbf{P}^N$ , with  $n = \dim \mathcal{F}$ . We recall some well known facts (see [4], pp. 107-124) about birational correspondences between projective spaces and in particular between  $\mathbf{P}^n$  and  $\mathcal{F}$ .

DEFINITION 4. A *parametrization*  $p$  on  $\mathbf{P}^n$  of a projective variety  $\mathcal{F}$  is a set

$$H_0, \dots, H_N \in k[U_0, \dots, U_n], \quad (= k[U])$$

of homogeneous polynomials (forms) of the same degree such that

a) substituting

$$(1) \quad X_i \rightarrow H_i \quad i = 0, \dots, N,$$

for every  $P \in J(\mathcal{F})$  it follows  $P(H_0, \dots, H_N) = 0$ ;

b) if  $g$  denotes the image of  $G$  in the canonical projection  $k[X] \rightarrow k[\mathcal{F}] = k[x_0, \dots, x_n]$ , for every  $G \in k[X]$ , there exists a set

$$F_0, \dots, F_n \in k[X]$$

of forms of the same degree such that

$$(2) \quad M U_j = f_j(H_0, \dots, H_N) \quad j = 0, \dots, n,$$

for a suitable  $M \in k[U]$ ,  $M \neq 0$ .

We note that in b) the forms  $F_0, \dots, F_n$  are chosen in  $k[X]$  only mod  $J(\mathcal{F})$ ; thus the polynomial  $M$  in (2) can change with the set  $F_0, \dots, F_n$ ; however it must be homogeneous from (2) itself. Moreover (1) and (2) together imply there exists a  $k$ -homomorphism between the fields  $k(\mathcal{F})$  and  $k(U)$  of rational functions on  $\mathcal{F}$  and  $\mathbf{P}^n$  respectively. By this, there must exist a suitable (homogeneous) polynomial  $Q \in$

$\in k[X]$ ,  $Q \notin J(\mathcal{F})$  such that

$$(3) \quad qx_j = H_j(f_0, \dots, f_n) \quad j = 0, \dots, N$$

(see also [4] p. 116). We remark also that for every point  $A \in \mathbf{P}^n - \{M = 0\}$  is  $M(A) \neq 0$ , if exists  $j$ ,  $0 \leq j \leq N$ , for which  $U_j \neq 0$  and, by (2)

$$0 \neq M(A) U_j(A) = f_j(H_0(A), \dots, H_n(A)) \Rightarrow H_i(A) \neq 0$$

for at least one  $i$ ,  $0 \leq i \leq n$ . This proves that the parametrization  $p$  rises, by means of (1), to a map

$$\sigma_p: \mathbf{P}^n \rightarrow \mathcal{F} \quad (\subset \mathbf{P}^N)$$

which is regular in  $\mathbf{P}^n - \{M = 0\}$  and  $\sigma_p$  is invertible by (3).

As above one sees that  $\sigma_p^{-1}: \mathcal{F} \rightarrow \mathbf{P}^n$  is surely regular in  $\mathcal{F} - \mathcal{F} \cap \{Q = 0\}$ , but in general nothing can be said for the points of the set  $\mathcal{F} \cap \{Q = 0\}$ . If we consider another parametrization  $q$  on  $\mathbf{P}^n$  of  $\mathcal{F}$ , being  $\mathcal{F}$  irreducible, we have that the map  $\sigma_q: \mathbf{P}^n \rightarrow \mathcal{F}$  risen by  $q$  coincides with  $\sigma_p$  in an open suitable subset of  $\mathbf{P}^n$ . By this they give a birational map  $\sigma: \mathbf{P}^n \rightarrow \mathcal{F} (\subset \mathbf{P}^N)$  which may be biregular too. For example it is what happens in the  $m$ -ple embedding  $\mathbf{P}^n \rightarrow \mathcal{F} \subset \mathbf{P}^N$  with  $N = \binom{n+m}{n} - 1$ .

**2.** In the following we suppose  $\mathcal{F} \subset \mathbf{P}^N$  to be a rational variety (i.e. every variety  $\mathcal{F}$  for which exists a birational map  $\mathbf{P}^n \rightarrow \mathcal{F}$ ,  $n = \dim \mathcal{F}$ ) and let  $p$  a parametrization of  $\mathcal{F}$  on  $\mathbf{P}^n$ .

We are precisely concerned with the particular map  $\sigma_p: \mathbf{P}^n \rightarrow \mathcal{F}$  given by  $p$  and the sets  $\mathbf{P}^n - \{M = 0\}$  and  $\mathcal{F} - \mathcal{F} \cap \{Q = 0\}$  which depend from  $p$  according to the previous notations, and the said situation.

**DEFINITIONS 5.** Given a parametrization  $p$  on  $\mathbf{P}^n$  of a projective variety  $\mathcal{F}$ , we call *referred to  $p$*  all the subvarieties of codimension 1 in  $\mathcal{F}$  which belong to  $\{Q = 0\}$ .

**6.** Rational variety of  $\mathbf{A}^N$  will be a variety of  $\mathbf{A}^N$  whose projective closure is a rational variety in  $\mathbf{P}^N$ .

The properties of the map  $\sigma$  in the present hypothesis are well known; we recall someone of them in the

PROPOSITION 1.a) *To every subvariety  $\mathcal{U} \subset \mathcal{F}$  not referred to  $p$ , it corresponds a subvariety  $\sigma_p^{-1}(\mathcal{U})$  such that  $\dim \sigma_p^{-1}(\mathcal{U}) = \dim \mathcal{U}$  and  $\sigma_p^{-1}(V) \not\subseteq \{M = 0\}$  (see [4], Satz VI, p. 120)*

b) *The restriction of  $\sigma_p^{-1}$  to the set of non singular points of  $\mathcal{U} - \mathcal{U} \cap \{Q = 0\}$  is bijective, moreover to every non singular point corresponds a non singular point (see [4] Korollar, p. 121).*

Let  $\mathcal{F}'$  and  $\mathcal{F}$  be two varieties of dimension  $n$  and  $\tau$  a birational map  $\tau: \mathcal{F}' \rightarrow \mathcal{F}$ . We recall that a prime  $\mathcal{U} \subset \mathcal{F}$  is said *exceptional* for  $\tau$  if  $\dim \tau^{-1}(\mathcal{U}) < n - 1$ . The rational variety  $\mathcal{F}$  can have only a finite number of exceptional primes for the birational map  $\mathbb{P}^n \rightarrow \mathcal{F}$ : indeed they can belong among the maximal components of  $\mathcal{F} \cap \{Q = 0\}$  for every parametrization  $p$  on  $\mathbb{P}^n$  of  $\mathcal{F}$ ; they are then referred to every parametrization  $p$ . On the other hand, if the birational map  $\mathbb{P}^n \rightarrow \mathcal{F}$  is biregular, no prime of  $\mathcal{F}$  is exceptional for it.

LEMMA 1. *Let  $\mathcal{F}$  be a rational variety,  $\mathcal{F} \subset \mathbb{P}^n$ , and  $p$  be a parametrization on  $\mathbb{P}^n$  of  $\mathcal{F}$ . For each prime  $\mathcal{U} \subset \mathcal{F}$  not referred to  $p$ , it exists at least an irreducible form  $\Psi \in k[U]$  such that*

$$\Psi(F_0, \dots, F_n) \in J(\mathcal{U}).$$

PROOF. Let  $C_1, \dots, C_s \in k[X]$  such that  $J(\mathcal{U}) = (C_1, \dots, C_s)$  and let be

$$D_i = C_i(H_0, \dots, H_N) \quad i = 1, \dots, s.$$

Let us denote always with  $\sigma_p: \mathbb{P}^n \rightarrow \mathcal{F}$  the rational mapping rised by  $p$  and  $Q \in [X]$ ,  $M \in k[U]$  the forms in the (3) and (2) respectively. First we have

$$\sigma_p^{-1}(\mathcal{U}) = \{D_1 = \dots = D_s = 0\}.$$

Indeed obviously  $\sigma_p^{-1}(\mathcal{U}) \subseteq \{D_1 = \dots = D_s = 0\}$ ; on the other hand, by Prop. 1.a),  $\sigma_p^{-1}(\mathcal{U})$  is irreducible and of codimension 1 in  $\mathbb{P}^n$ , so it must be a hypersurface of  $\mathbb{P}^n$ . From this  $\sigma_p^{-1}(\mathcal{U}) \supseteq \{D_1 = \dots = D_s = 0\}$  and  $J(\sigma_p^{-1}(\mathcal{U}))$  will be a principal ideal generated by  $\Psi = \text{G.C.D. } \{D_1, \dots, D_s\}$  and  $\Psi$  will be irreducible and  $\Psi$  does not divide  $M$ . Later,

being the ring  $k[U]$  U.F.D. and  $\Psi = \text{G.C.D}\{D_1, \dots, D_s\}$ , there exist  $A_i, B_i \in k[U]$ ,  $i = 1, \dots, s$ , such that

$$D_i = \Psi A_i \quad \text{for } i = 1, \dots, s; \quad 1 = \sum_{i=1}^s A_i B_i$$

by which  $\Psi = \Psi \sum_{i=1}^s A_i B_i = \sum_{i=1}^s D_i B_i = \sum_{i=1}^s C_i(H_0, \dots, H_N) B_i$ :

Let us consider now the form  $\Psi(F_0, \dots, F_n) \in k[X]$  obtained by substituting  $F_i$  to place of  $U_i$ ,  $i = 0, \dots, n$ . It results

$$\begin{aligned} (\#) \quad \Psi(F_0, \dots, F_n) &= \\ &= \sum_{i=1}^s C_i(H_0(F_0, \dots, F_n), \dots, H_N(F_0, \dots, F_n)) B_i(F_0, \dots, F_n). \end{aligned}$$

Now we want to calculate its image  $\Psi(f_0, \dots, f_n)$  in  $k[\mathcal{F}]$ . From (3) it is

$$C_i(H_0(f_0, \dots, f_n), \dots, H_N(f_0, \dots, f_n)) = C_i(qx_0, \dots, qx_N) = q^{\deg C_i} C_i(x_0, \dots, x_N),$$

by this, from (#) one gets

$$\Psi(f_0, \dots, f_n) = \sum_{i=1}^s q^{\deg C_i} C_i(x_0, \dots, x_N) B_i(f_0, \dots, f_n)$$

which belongs to the image of  $J(\mathcal{U})$  in  $k[\mathcal{F}]$ , whence

$$\Psi(F_0, \dots, F_n) \in J(\mathcal{U}).$$

**PROPOSITION 2.** *Let  $\mathcal{F} \subset \mathbf{P}^N$  be a rational projectively normal variety of  $\dim \mathcal{F} = n$  and  $p$  a parametrization of  $\mathcal{F}$  on  $\mathbf{P}^n$ . The following are equivalent:*

- a)  $\mathcal{F}$  is almost-factorial;
- b) every prime on  $\mathcal{F}$  referred to  $p$  is set-theoretic complete intersection on  $\mathcal{F}$ .

*More precisely if  $C_1, \dots, C_t$  are the primes on  $\mathcal{F}$  referred to  $p$  and  $\lambda_i C_i$  is the complete intersection  $\mathcal{F}$  with a suitable hypersurface  $\mathcal{S}_i \subset \mathbf{P}^N$ ,  $i = 1, \dots, t$ , then for every prime  $C \subset \mathcal{F}$  it exists a hypersurface  $\mathcal{S} \subset \mathbf{P}^N$*

such that

$$\mathcal{F} \cdot \mathcal{C} = \lambda \mathcal{C} \quad \text{where} \quad \lambda = \text{L.C.M.}\{\lambda_1, \dots, \lambda_t\},$$

that is  $\mathcal{F}$  is  $\lambda$ -almostfactorial.

PROOF.  $a) \Rightarrow b)$  is obvious. So we have only to prove  $b) \Rightarrow a)$   
 Let  $\mathcal{C}_1, \dots, \mathcal{C}_t$  be all the primes of  $\mathcal{F}$  referred to  $p$ . By hypothesis  $b)$  there exist hypersurfaces  $\mathcal{L}_i = \{L_i = 0\} \subset P^N$  and  $\lambda_i > 0, i = 1, \dots, t$ , such that

$$(4) \quad \mathcal{F} \cdot \mathcal{L}_i = \lambda_i \mathcal{C}_i \quad i = 1, \dots, t.$$

Let  $\lambda = \text{L.C.M.}\{\lambda_1, \dots, \lambda_t\}$  and let  $n_1, \dots, n_t$  be positive integers such that

$$(5) \quad \lambda = n_i \lambda_i \quad i = 1, \dots, t.$$

For every prime  $\mathcal{D} \subset \mathcal{F}$  we denote with  $C(\mathcal{D})$  the affine cone of  $\mathcal{D}$  and let  $C(\mathcal{F})$  be the affine cone of  $\mathcal{F}$ , both in  $\mathbf{A}^{N+1}$ . Obviously  $C(\mathcal{D})$  has codimension 1 in  $C(\mathcal{F})$  for every prime  $\mathcal{D} \subset \mathcal{F}$ . Moreover the ring  $k[\mathcal{F}]$  can be considered as the ring of the regular functions on  $C(\mathcal{F})$ . Let  $K$  be the quotient field of  $k[\mathcal{F}]$ . For every prime  $\mathcal{C} \subset \mathcal{F}$  not referred to  $p$  it exists, by Lemma 1, an irreducible polynomial  $\Psi \in k[U]$  such that

$$\Psi(F_0, \dots, F_n) \in J(\mathcal{U}).$$

Let be  $H = \Psi(F_0, \dots, F_n) \in k[X]$  and let  $h$  be its projection in  $k[\mathcal{F}]$ . We have

$$(6) \quad \{H = 0\} \cdot \mathcal{F} = \text{div}(h) = \mu \mathcal{C} + \nu_1 \mathcal{D}_1 + \dots + \nu_r \mathcal{D}_r,$$

$$\mu > 0, \nu_i > 0, i = 1, \dots, r,$$

where  $\mu = 1$  by Prop. 1.b) because  $\mathcal{C}$  is not referred to  $p$ , and  $\mathcal{D}_1, \dots, \mathcal{D}_r$  are distinct primes on  $\mathcal{F}$ , different from  $\mathcal{C}$ , which are necessarily referred to  $p$ . Indeed, if  $\mathcal{D}_i$  is one of  $\mathcal{D}_1, \dots, \mathcal{D}_r$ , it is or exceptional for  $p$  (and then referred to  $p$ ), or, if it would not be referred to  $p$ , the ideal  $J(\sigma_p^{-1}(\mathcal{D}_i))$ , by Lemma 1 is a principal ideal which contains  $\Psi$  itself. Since  $\Psi$  is irreducible, then  $J(\sigma_p^{-1}(\mathcal{D}_i)) = (\Psi) = J(\sigma_p^{-1}(\mathcal{C}))$ . On the other hand  $\{\Psi = 0\} \not\subset \{M = 0\}$  and  $\sigma_p$  is regular in  $\mathbf{P}^n - \{M = 0\}$  so we would have  $\mathcal{D}_i = \mathcal{C}$ . So  $\mathcal{D}_i, i = 1, \dots, r$ , is in any case referred to  $p$ . Of course  $r \leq t$ , being  $t$  the number of all primes of  $\mathcal{F}$  referred



to  $p$  and among them there are  $\mathfrak{D}_1, \dots, \mathfrak{D}_r$ . We can suppose that the primes of  $\mathcal{F}$  referred to  $p$  in (6) to be  $\mathfrak{D}_1 = \mathfrak{C}_1, \dots, \mathfrak{D}_r = \mathfrak{C}_r$ .

Let us consider now the polynomials  $L_i^{v_i} \in k[X]$ ,  $i = 1, \dots, r$  and we denote with  $p_i$  their images in  $k[\mathcal{F}]$ . Since is

$$g = h^\lambda / (p_1 \dots p_r) \in K.$$

it results, by (4), (5) and (6),

$$(7) \quad \operatorname{div}(g) = \lambda \mathfrak{C} + \lambda v_1 \mathfrak{C}_1 + \\ + \dots + \lambda v_r \mathfrak{C}_r - [v_1 \lambda_1 n_1 \mathfrak{C}_1 + \dots + v_r \lambda_r n_r \mathfrak{C}_r] = \lambda \mathfrak{C}.$$

Note (7) means that for every valuation  $v_\varepsilon$  of the field  $K$  centered in the subvariety  $\varepsilon$  of codimension 1 in  $C(\mathcal{F})$  is

$$v_\varepsilon(g) = 0 \text{ if } \varepsilon \neq C(\mathfrak{C}) \quad \text{and} \quad v_\varepsilon(g) = \lambda \text{ if } \varepsilon = C(\mathfrak{C}).$$

By this  $g$  is an element which belongs to the integral closure of  $k[\mathcal{F}]$ , by the structure theorem of noetherian integrally closed domains. On the other hand, being  $k[\mathcal{F}]$  normal because  $\mathcal{F}$  is projectively normal (see d.f. 1),  $g \in k[\mathcal{F}]$ . It exists then at least a homogeneous  $G \in k[X]$  such that its projection in  $k[\mathcal{F}]$  is  $g$ . Moreover we get

$$(8) \quad \mathcal{F} \cdot \mathfrak{G} = \lambda \mathfrak{C}.$$

We note that the integer  $\lambda$  in (8) does not depend on  $\mathfrak{C}$  but only on all the primes referred to  $p$ . So  $\mathcal{F}$  is  $\lambda$ -almostfactorial.

REMARK 1. Prop. 2 is an extension of a well known criterion of D. Gallarati on the monoid hypersurfaces  $\mathcal{M} \subset \mathbb{P}^N$  (see [3], cap. III, 17, p. 38, and also [7], Prop. 1):

*Every prime of  $\mathcal{M}$  is set-theoretic complete intersection of  $\mathcal{M}$  iff all the primes of the cone of the straight lines passing through the vertex of  $\mathcal{M}$  are set-theoretic complete intersection.*

Indeed the projection from the vertex  $V$  of  $\mathcal{M}$  onto a hyperplane not passing through  $V$ , gives a parametrization of  $\mathcal{M}$  on that hyperplane. The primes of  $\mathcal{M}$  referred to this parametrization are just the primes of the cone of straight lines of  $\mathcal{M}$  passing through  $V$ . They are all exceptional too.

REMARK 2. Prop. 2 also shows that the image  $\mathcal{F}$  of a  $m$ -ple embedding of  $\mathbb{P}^n$  in  $\mathbb{P}^N$ ,  $N = \binom{n+m}{n} - 1$ , is  $m$ -almostfactorial (fact well known). Indeed in the usually parametrization  $p$  of  $\mathcal{F}$  on  $\mathbb{P}^n$  (see [4], pag. 124 for example) is referred to  $p$  a prime  $\mathcal{C} \subset \mathcal{F}$  which can be supposed to be the image of a hyperplane of  $\mathbb{P}^n$ . It results that  $m\mathcal{C}$  is just the complete intersection of  $\mathcal{F}$  with a suitable hyperplane in  $\mathbb{P}^N$ . Since  $\mathcal{F}$  is projectively normal one gets that  $\mathcal{F}$  is  $m$ -almostfactorial.

We can formulate an affine version of Prop. 2 in the following

PROPOSITION 3. *Let  $\mathcal{E} \subset \mathbb{A}^N$  be a rational normal variety. The following facts are equivalent:*

- a)  $\mathcal{E}$  is almost-factorial;
- b) every prime  $\mathcal{C}_a$  on  $\mathcal{E}$  referred to a parametrization  $p$  of the projective closure  $\bar{\mathcal{E}}$  is S.T.C.I. on  $\mathcal{E}$ .

PROOF. Let  $\mathcal{F} = \bar{\mathcal{E}}$  be the projective closure of  $\mathcal{E}$  and  $\mathcal{C}_a \subset \mathcal{E}$  be a prime whose projective closure  $\mathcal{C} = \bar{\mathcal{C}}_a$  is not referred to the parametrization of  $\mathcal{F}$ . To obtain relation (7) we argue as in Prop. 2. (7) induces on  $\mathcal{E}$

$$(7') \quad \operatorname{div}(g_a) = \lambda \mathcal{C}_a,$$

where  $g_a$  now belongs to the quotient field of  $\mathcal{E}$ . Since  $\mathcal{E}$  is normal, the ring  $k[\mathcal{E}]$  coincides with its integral closure. The arguments as at the end of the proof of Prop. 2 prove that  $g_a \in k[\mathcal{E}]$ . So it exists a suitable polynomial  $G$  for which it results

$$(8') \quad \mathcal{E} \cdot \{G = 0\} = \lambda \mathcal{C}_a.$$

This proves  $b) \Rightarrow a)$ , while  $a) \Rightarrow b)$  is obvious.

### 3. Applications and examples.

It is well known that a hypersurface  $\mathcal{F} \subset \mathbb{P}^N$  is projectively normal iff is non singular in codimension 1 (see [6] Prop. 1 p. 389 and Prop 2 p. 391) arguing, in the projective case, on the affine cone of  $\mathcal{F}$ . In the case of surfaces of  $\mathbb{P}^3$  we can apply Prop. 2 to state the

**COROLLARY 1.** *A rational non singular surface  $\mathcal{F} \in \mathbf{P}^3$  is almost-factorial only if it is a plane. (So it is factorial).*

**PROOF.** Let  $d$  be the degree of  $\mathcal{F}$ . Since  $\mathcal{F}$  is non singular and rational, the geometric genus

$$p_g(\mathcal{F}) = p_g(\mathbf{P}^2) = (d-1)(d-2)(d-3)/6 = 0.$$

So, if  $\mathcal{F}$  is not a plane,  $\mathcal{F}$  is a non singular quadric or cubic. But these surfaces are not almost-factorial (see [3], or [7]).

**COROLLARY 2.** *A rational surface  $\mathcal{F} \subset \mathbf{P}^3$  of degree  $d > 1$  is almost-factorial if only if it has a positive (finite) number of singular points, and has a parametrization  $p$  on  $\mathbf{P}^2$  such that very curve on  $\mathcal{F}$  which is referred to  $p$  is S.T.C.I. of  $\mathcal{F}$ .*

**PROOF.** It follows from Prop. 2 and from what we have recalled about the condition for a hypersurface of  $\mathbf{P}^N$  to be projectively normal.

**EXAMPLE 1.** *A quartic surface in  $\mathbf{P}^3$  with only two double singular points.* Let us denote with  $\{T, X, Y, Z\}$  the coordinates in  $\mathbf{P}^3$ . Let  $\mathcal{F}$  be:

$$\mathcal{F} = \{T^2X^2 + TY^3 - Z^4 = 0\}.$$

The surface  $\mathcal{F}$  is singular only in the double points  $(1, 0, 0, 0)$  and  $(0, 1, 0, 0)$ .

A parametrization  $p$  of  $\mathcal{F}$  is, for example, given by

$$\begin{aligned} H_0 &= W^3(W^2 - U^2)^2, & H_1 &= V^4U, & H_2 &= V^4W(W^2 - U^2), \\ H_3 &= V^3W^2(W^2 - U^2) \in k[W, U, V] \end{aligned}$$

because we have, first

$$H_0^2H_1^2 + H_0H_2^3 - H_3^4 = 0,$$

secondly, let us choose  $F_0 = Z^2$ ,  $F_1 = TX$ ,  $F_2 = YZ \in k[T, X, Y, Z]$ . We can consider then the rational map

$$\sigma: (W, U, V) \rightarrow (T = H_0, X = H_1, Y = H_2, Z = H_3) \subseteq \mathcal{F} \subset \mathbf{P}^3.$$

The restriction on  $\mathcal{F}$ , of the map

$$\pi: (T, X, Y, Z) \rightarrow (W = F_0, U = F_1, V = F_2) \subset \mathbf{P}^2$$

is just  $\sigma^{-1}$ . Indeed for every  $F \in k[T, X, Y, Z]$  let  $f$  be the canonical projection of  $F$  in

$$k[T, X, Y, Z]/J(\mathcal{F}) = k[\mathcal{F}] = k[t, x, y, z];$$

one gets the relations

$$\begin{aligned} H_0(f_0, f_1, f_2) &= \dots = (ty^6 z^6) t, & H_1(f_0, f_1, f_2) &= \dots = (ty^6 z^6) x, \\ H_2(f_0, f_1, f_2) &= \dots = (ty^6 z^6) y, & H_3(f_0, f_1, f_2) &= \dots = (ty^6 z^6) z \end{aligned}$$

from which it results  $Q = TY^6Z^6$ . Moreover the product-map

$$\begin{aligned} (t, x, y, z) \rightarrow (W = f_0, U = f_1, V = f_2) \rightarrow \\ \rightarrow (t = H_0, x = H_1, y = H_2, z = H_3) \end{aligned}$$

is the identity on  $\mathcal{F}$ , while the identities

$$\begin{aligned} F_0(H_0, \dots, H_3) &= H_3^2 = [V^6 W^3(W^2 - U^2)^2] W, \\ F_1(H_0, \dots, H_3) &= H_0 H_3 = [V^6 W^3(W^2 - U^2)^2] U, \\ F_2(H_0, \dots, H_3) &= H_1 H_2 = [V^6 W^3(W^2 - U^2)^2] V, \end{aligned}$$

show that one must assume  $M = V^6 W^3(W^2 - U^2)^2$  and they prove that the product-map

$$\begin{aligned} (W, U, V) \rightarrow (t = H_0, x = H_1, y = H_2, z = H_3) \rightarrow \\ \rightarrow (W = f_0, U = f_1, V = f_2) \end{aligned}$$

is the identity on  $\mathbf{P}^2$ . The components of  $\mathcal{F} \cap \{Q = 0\}$  are

$$\begin{aligned} \mathcal{R} = \{T = Z = 0\}, & \quad \mathcal{C}_3 = \{Z = TX^2 + Y^3 = 0\}, \\ \mathcal{C}_2 = \{Y = TX - Z^2 = 0\}, & \quad \mathcal{C}'_2 = \{Y = TX + Z^2 = 0\} \end{aligned}$$

and they are the curves on  $\mathcal{F}$  referred to  $p$ .

Now we have to prove that every such curves is S.T.C.I. of  $\mathcal{F}$ . Clearly is

$$\begin{aligned}\mathcal{F} \cdot \{T = 0\} &= 4\mathcal{R}, & \mathcal{F} \cdot \{TX^2 + Y^3 = 0\} &= 4\mathcal{C}_3, \\ \mathcal{F} \cdot \{TX - Z^2 = 0\} &= 3\mathcal{C}_2 + 2\mathcal{R}, & \mathcal{F} \cdot \{TX + Z^2 = 0\} &= 3\mathcal{C}'_2 + 2\mathcal{R}.\end{aligned}$$

Let us consider on  $\mathcal{F}$  the divisor defined by the quotient

$$[(tx - z^2)^2]/t.$$

It is just the divisor of  $2tx^2 - 2xz^2 + y^3$  because in  $k[\mathcal{F}]$  equality  $z^4 = t(tx^2 + y^3)$  holds. From this, it follows

$$\mathcal{F} \cdot \{2TX^2 - 2XZ^2 + Y^3 = 0\} = 6\mathcal{C}_2.$$

By the same arguments one gets

$$\mathcal{F} \cdot \{2TX^2 + 2XZ^2 + Y^3 = 0\} = 6\mathcal{C}'_2.$$

Since every curve of  $\mathcal{F}$  referred to  $p$  is S.T.C.I. on  $\mathcal{F}$  we can apply to  $\mathcal{F}$  Prop. 2:  $\mathcal{F}$  is then *almost-factorial*; more precisely  $\mathcal{F}$  is *12-almostfactorial*.

As example let us consider the rational curve

$$\mathcal{C}_5 = \{t = (1 - s)^2, x = 4s^4(1 - 2s), y = 4s^3(1 - s), z = 2s^2(1 - s)\}.$$

$\mathcal{C}_5$  belongs to  $\mathcal{F}$ . The curve  $\mathcal{C}_5$  belongs even to the surfaces  $\{Z^3 + TXZ - TY^2 = 0\}$ ,  $\{Y^2Z + TXY + 2TXZ - TY^2 = 0\}$  etc., and even to the quadric  $Q = \{TX + YZ - Z^2 = 0\}$ . The images on  $\mathbb{P}^2$ , by means of  $\sigma^{-1}$ , of the intersections with  $\mathcal{F}$  of such surfaces are curves of which a common component, not component of  $\{M = 0\} = \{V^6 W^3 (W^2 - U^2)^2 = 0\}$ , is the line  $\{U + V - W = 0\}$ . By this we can suppose  $\Psi(W, U, V) = U + V - W$ , so  $\Psi(F_0, F_1, F_2) = TX + YZ - Z^2$ .

On the other hand it is

$$\mathcal{F} \cdot Q = \mathcal{C}_5 + \mathcal{C}_2 + \mathcal{R}.$$

Now  $\mathcal{C}_2$  and  $\mathcal{R}$  are S.T.C.I. on  $\mathcal{F}$ , so it is also  $\mathcal{C}_5$  with multiplicity at most 12. It is enough to consider on  $\mathcal{F}$  the divisor defined by the

quotient

$$[tx - z^2 + yz]^{12}/[(tx - z^2)^4]t.$$

Indeed in  $k[\mathcal{F}]$  the identities

$$ty^3 = - (tx - z^2)(tx + z^2), \quad z^4 = t(tx^2 + y^3);$$

hold, which give, first

$$\begin{aligned} [tx - z^2 + yz]^3/(tx - z^2) &= \\ &= [(tx - z^2)]^2 + 3(tx - z^2)yz + 3y^2z^2 + y^3z^3/(tx - z^2) = \\ &= 2t^2x^2 - 2txz^2 + ty^3 + 3txyz - 3yz^3 + 3y^2z^2 - zy^3 - tx^2z - xz^3 = \\ &= tL + zN \end{aligned}$$

where we have assumed, for example,

$$L = 2x^2t - 2xz^2 + y^3 + 3xyz - x^2z, \quad N = -y^3 - 3yz^2 + 3y^2z - xz^2.$$

Secondly, the divisor of the quotient

$$(tL + zN)^4/t$$

coincides with the divisor of

$$g = t^3L^4 + 4t^2zL^3N + 6tz^2L^2N^2 + 4z^3LN^3 + (tx^2 + y^3)N^4.$$

It then exists in  $k[X]$  a polynomial  $G$ , homogeneous of degree 15, such that its image on  $k[\mathcal{F}]$  is  $g$ ; then we get  $\{G = 0\} \cdot \mathcal{F} = 12C_5$ .

**EXAMPLE 2.** *Classes of rational surfaces which an affine part isomorphic to a plane.* It is easy to determine a class of surfaces  $\mathcal{F}_n \subset \mathbb{A}^3$  of degree  $n$ , for every  $n > 0$ , which are *isomorphic* to a plane and having projective closure non singular in codimension 1 and  $n$ -almostfactorial.

P. C. Craighero has pointed this example to me in the case  $n = 4$ .

Let  $a(T)$ ,  $b(T)$ ,  $c(T)$  be arbitrary polynomials of  $k[T]$  of degree  $r$ ,  $s$ ,  $m$  respectively for which is

$$rs + 1 = m = n \quad \text{or} \quad rs = n = m + 1.$$

Let us consider the two isomorphisms of  $\mathbf{A}^3$

$$\eta: (U, V, W) \rightarrow (U' = U + a(W), V' = V, W' = W)$$

and

$$\chi: (U', V', W') \rightarrow (U'' = U', V'' = V', W'' = W' + b(U') + c(V'))$$

and their product

$$\chi \circ \eta: (U, V, W) \rightarrow$$

$$\rightarrow (U'' = U + a(W), V'' = V, W'' = W + b(U + a(W)) + c(V)).$$

The affine surfaces of  $\mathbf{A}^3$ :

$$\mathcal{F}_n = \{Z + b(X + a(Z)) + c(Y) = 0\}$$

are isomorphic to a plane (and more precisely to the plane  $\{W'' = 0\}$ ) by means of  $\chi \circ \eta$  and are of degree  $n$ , and, for example, they admit the parametrization

$$X = U - a[-b(U) - c(V)], \quad Y = V, \quad Z = -b(U) - c(V)$$

whose inverse is  $(U = X + a(Z), V = Y)$ , with  $Z + b(X + a(Z)) + c(Y) = 0$ . Such surfaces are then rational and factorial. Their projective closure  $\overline{\mathcal{F}}_n$  is non singular in codimension 1; the section of such surfaces with the plane at the infinity is a straight line which is just the complete intersection of such two surfaces. Such a line is the only curve on the surface  $\overline{\mathcal{F}}_n$  which is referred to the parametrization of  $\overline{\mathcal{F}}_n$ : by this  $\overline{\mathcal{F}}_n$  is  $n$ -almostfactorial.

If the polynomial  $b(T)$  is linear ( $s = 1$ ) such surfaces are monoids; then one can apply Gallarati's criterion to them with the same conclusions.

**EXAMPLE 3.** *Classes of trinomial rational surfaces* (of the kind  $\{X^a + Y^b = Z^c\}$ ). Let  $m, n$  be coprime positive integers, with  $m < n$ . Let  $(r_0, s_0)$  a integer solution of the diophantine equation

$$(\alpha) \quad xm - yn = 1.$$

Every other integer solution  $(r, s)$  of such equation is

$$\begin{aligned}
 (\beta_1) \qquad \qquad \qquad r &= r_0 + tm, \\
 (\beta_2) \qquad \qquad \qquad s &= s_0 + tm \quad \text{for every } t \in \mathbf{Z}.
 \end{aligned}$$

For every integer  $r$  in  $(\beta_1)$ , we consider the classes of affine surfaces of  $\mathbf{A}^3$

$$\begin{aligned}
 \mathcal{E}_{m,n,r} &= \{X^m + Y^n = Z^{r^{m-1}}\}, & \mathcal{F}_{m,n,r} &= \{X^m + Y^n = Z^{rm}\} & \text{for } r > 0, \\
 \mathcal{G}_{m,n,r} &= \{X^m + Y^n = Z^{-rm}\}, & \mathcal{H}_{m,n,r} &= \{X^m + Y^n = Z^{1-rm}\} & \text{for } r < 0.
 \end{aligned}$$

One gets a parametrization of such surfaces assuming

$$\begin{aligned}
 x &= z^r u, & y &= z^s v & \text{with } z &= (1 - v^n)/u^m & \text{for the } \mathcal{E}_{m,n,r}, \\
 x &= z^r u, & y &= z^s v & \text{with } z &= v^n/(1 - u^m) & \text{for the } \mathcal{F}_{m,n,r}, \\
 x &= z^{-r} u, & y &= z^{-s} v & \text{with } z &= (1 - u^m)/v^n & \text{for the } \mathcal{G}_{m,n,r}, \\
 x &= z^{-r} u, & y &= z^{-s} v & \text{with } z &= u^m/(1 - v^n) & \text{for the } \mathcal{H}_{m,n,r},
 \end{aligned}$$

while for the inverse map we have to assume for the  $\mathcal{E}_{m,n,r}$  and  $\mathcal{F}_{m,n,r}$ :

$$u = x/z^r, \quad v = y/z^s$$

and  $u = xz^r, v = yz^s$  for the  $\mathcal{G}_{m,n,r}$  and for the  $\mathcal{H}_{m,n,r}$  respectively. Every surface  $\mathcal{E}_{m,n,r}$  and  $\mathcal{H}_{m,n,r}$  is *m-almostfactorial*, while the surfaces  $\mathcal{F}_{m,n,r}$  and  $\mathcal{G}_{m,n,r}$  are *n-almostfactorial*, by Prop. 2 and Prop. 3.

There are other classes of affine rational surfaces which are trinomial and almost-factorial. For example, for every pair of positive integers  $(m, n)$  the affine surfaces in  $\mathbf{A}^3$ :

$$\mathcal{B}_{m,n} = \{X^m + Y^n = Z^{mn+1}\} \quad \text{and} \quad \mathcal{C}_{m,n} = \{X^m + Y^n = Z^{mn-1}\}.$$

$\mathcal{B}_{m,n}$  admits a parametrization  $x = z^n u, y = z^m v, z = u^m + v^n$  whose inverse is  $u = x/z^n, v = y/z^m, \mathcal{C}_{m,n}$  admits a parametrization  $x = z^n u, y = z^m v, z = 1/(u^m + v^n)$  whose inverse is  $u = x/z^n, v = y/z^m$ . The surfaces  $\mathcal{B}_{m,n}$  and  $\mathcal{C}_{m,n}$  are *factorial* if  $m, n$  are coprimes,  $\mathcal{B}_{m,n}$  is  $(mn + 1)$ -almostfactorial and  $\mathcal{C}_{m,n}$  is  $(mn - 1)$ -almostfactorial otherwise.



Note the surface  $C_{2,3}$  which belongs to the family  $C_{m,n}$ . By interchanging  $-z$  with  $z$ , one gets the well known surface  $\{x^2 + y^3 + z^5 = 0\}$  which is factorial even if it has a singular point in  $0 = (0, 0, 0)$  (cfr. [5] Example 5.8 p. 420).

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