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A Note on Certain Subclasses of Spiral-Like Functions.

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SUMMARY - The object of the present paper is first to prove a representation formula and a distortion theorem for functions belonging to the class $S_p(\alpha, a, b)$ of α-spiral-like functions introduced here, and then to determine the radius of starlikeness for this class. By specializing the various parameters involved, the corresponding results for the function classes studied earlier (by, for example, Z. J. Jakubowski [3], S. Owa [4], H. Silverman [5], and E. M. Silvia [6]) can be derived.

1. – Introduction.

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathcal{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Further, let

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$\mathcal{S}_p$ denote the subclass of $\mathcal{A}_p$ consisting of analytic and univalent functions $f(z)$ in the unit disk $\mathcal{U}$. Then a function $f(z)$ of $\mathcal{S}_p$ is said to be α-spiral-like if $f(z)$ satisfies (*cf.*, *e.g.*, [2, p. 52 *et seq.*]; see also [1])

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \tag{1.2}$$

for some α ($|\alpha| < \pi/2$). We denote by $\mathcal{S}_p(\alpha)$ the class of all α-spiral-like functions. We note that $\mathcal{S}_p(\alpha) \subset \mathcal{S}$, where $\mathcal{S}$ denotes the class of functions of the form (1.1) *with* $p = 1$ which are analytic and univalent in the unit disk $\mathcal{U}$. The class $\mathcal{S}_1(\alpha)$ was introduced by L. Špaček in 1933.

For α, a, b real and such that

$$|\alpha| < \frac{\pi}{2}, \quad a + b \geqq 1, \qquad b \leqq a < b + 1, \tag{1.3}$$

we say that a function $f(z)$ is in the class $\mathcal{S}_p(\alpha, a, b)$ if $f(z) \in \mathcal{S}_p$ and

$$\left|e^{i\alpha}\frac{zf'(z)}{f(z)} - (a\cos\alpha + i\sin\alpha)\right| \leqq b\cos\alpha \qquad (z \in \mathcal{U}). \tag{1.4}$$

Clearly, each $\mathcal{S}_p(\alpha, a, b)$ is a subclass of $\mathcal{S}_p(\alpha)$.

The class $\mathcal{S}_1(\alpha, a, b)$ was studied by Silvia [6], the class $\mathcal{S}_p(0, a, b)$ was studied by Owa [4], and the class $\mathcal{S}_1(0, a, b)$ was studied by Silverman [5]. Furthermore, by setting $a = \lambda + m(1-\lambda)$ and $b = M(1-\lambda)$ in (1.4), where $0 \leqq \lambda < 1$, and *either* $\frac{1}{2} < m < 1$ and $1 - m < M \leqq m$, *or* $m \geqq 1$ and $m - 1 \leqq M \leqq m$, we can see that $\mathcal{S}_1(\alpha, a, b)$ is an alternate definition of the class for which Jakubowski [3] obtained sharp bounds for the moduli of the coefficients in (1.1) with $p = 1$.

2. – A representation formula.

In the sequel, it will be convenient to set

$$c = b^2 - (1-a)^2. \tag{2.1}$$

Note that $c \geqq 0$ whenever a and b satisfy the relevant inequalities in (1.3).

Our first representation formula is contained in

THEOREM 1. *Let the function* $f(z)$ *be defined by* (1.1). *Then* $f(z)$ *is in the class* $S_p(\alpha, a, b)$ *if and only if it can be expressed in the form*

$$f(z) = z \exp\left(e^{-i\alpha} \cos\alpha \int_0^z \frac{ct^{p-1}\varphi(t)}{b + (1-a)t^p\varphi(t)}\, dt\right), \tag{2.2}$$

where $\varphi(z)$ *is an analytic function with* $|\varphi(z)| \leqq 1$ *for* $z \in \mathcal{U}$.

PROOF. Assume that $f(z)$ defined by (1.1) is in the class $S_p(\alpha, a, b)$. Then

$$g(z) = \left(e^{i\alpha}\frac{zf'(z)}{f(z)} - (a\cos\alpha + i\sin\alpha)\right)\frac{\sec\alpha}{b} \tag{2.3}$$

has modulus at most one in the unit disk $\mathcal{U}$. Hence the function

$$z^{p-1}h(z) = \frac{g(z) - g(0)}{1 - g(0)g(z)} \tag{2.4}$$

satisfies the condition of the Schwarz lemma, and we can write $h(z) = z\varphi(z)$, where $\varphi(z)$ is analytic with $|\varphi(z)| \leqq 1$ for $z \in \mathcal{U}$. Thus we find that

$$\frac{zf'(z)}{f(z)} = \frac{b + (ce^{-i\alpha}\cos\alpha + 1 - a)z^p\varphi(z)}{b + (1-a)z^p\varphi(z)}. \tag{2.5}$$

Subtracting 1 from each member of (2.5), and then dividing by z, we obtain

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{(ce^{-i\alpha}\cos\alpha)z^{p-1}\varphi(z)}{b + (1-a)z^p\varphi(z)}. \tag{2.6}$$

On integrating both sides of (2.6) from 0 to z, we have the representation formula (2.2).

For the converse, if $f(z)$ has the representation (2.2), (2.5) is readily seen to yield

$$e^{i\alpha}\frac{zf'(z)}{f(z)} - (a\cos\alpha + i\sin\alpha) = b\cos\alpha\left\{\frac{[(1-a)/b] + z^p\varphi(z)}{1 + [(1-a)/b]z^p\varphi(z)}\right\}. \tag{2.7}$$

Since $|(1-a)/b| \leqq 1$, the function

$$w(z)=\frac{[(1-a)/b]+z^p\varphi(z)}{1+[(1-a)/b]z^p\varphi(z)} \tag{2.8}$$

maps the unit disk $\mathcal{U}$ onto itself. This completes the proof of the theorem.

COROLLARY 1. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z)$ is in the class $\mathcal{S}_p(\alpha)$ if and only if it can be expressed in the form*

$$f(z)=z\exp\left(2e^{-i\alpha}\cos\alpha\int_0^z\frac{t^{p-1}\varphi(t)}{1-t^p\varphi(t)}\,dt\right), \tag{2.9}$$

where $\varphi(z)$ is an analytic function with $|\varphi(z)|\leqq 1$ for $z\in\mathcal{U}$.

PROOF. By a simple application of (1.3) and (2.1), we can put $a=b$ and show that

$$\lim_{b\to\infty}\frac{c}{b}=2 \quad \text{and} \lim_{b\to\infty}\frac{1-a}{b}=-1\,. \tag{2.10[}$$

Thus Corollary 1 follows from the assertion (2.2) of Theorem 1.

REMARK 1. For $\alpha=0$, Theorem 1 reduces to the representation formula obtained by Owa [4]. For $p=1$, we similarly have the representation formula of Silvia [6]. Furthermore, for $p=1$ and $\alpha=0$, we have the representation formula due to Silverman [5].

3. – A distortion theorem.

We need the following lemma in proving our main distortion theorem (Theorem 2 below):

LEMMA 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}_p(\alpha, a, b)$. Then*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\geqq 1-\frac{c\{b+(1-a)|z|^p\cos\alpha\}\,|z|^p\cos\alpha}{b^2-(1-a)^2|z|^{2p}} \tag{3.1}$$

and

$$(3.2) \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq 1 + \frac{c\{b-(1-a)|z|^p \cos\alpha\}|z|^p \cos\alpha}{b^2-(1-a)^2|z|^{2p}}$$

for $z \in \mathcal{U}$. *The results* (3.1) *and* (3.2) *are sharp.*

PROOF. Since $f(z)$ is in the class $S_p(\alpha, a, b)$, in view of Theorem 1, we have

$$(3.3) \qquad \frac{zf'(z)}{f(z)} = \frac{b + (ce^{-i\alpha}\cos\alpha + 1 - a)z^p\varphi(z)}{b + (1-a)z^p\varphi(z)},$$

where $\varphi(z)$ is an analytic function with $|\varphi(z)| \leqq 1$ for $z \in \mathcal{U}$. After a simple computation, we find from (3.3) that

$$(3.4) \qquad \left|\frac{zf'(z)}{f(z)} - \frac{b^2-(ce^{-i\alpha}\cos\alpha + 1 - a)(1-a)|z|^{2p}}{b^2-(1-a)^2|z|^{2p}}\right| \leqq \frac{bc|z|^p\cos\alpha}{b^2-(1-a)^2|z|^{2p}},$$

which implies (3.1) and (3.2).

Finally, by taking the function $f(z)$ defined by (2.2), we can show that the results (3.1) and (3.2) are sharp.

REMARK 2. For $p = 1$, Lemma 1 reduces to the corresponding result obtained recently by Silvia [6].

We now state our main distortion theorem as

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_p(\alpha, a, b)$. *Then*

$$(3.5) \qquad |f(z)| \geqq |z|\{b-(1-a)|z|^p\}^{[c/(1-a)p]\cos\alpha}$$

and

$$(3.6) \qquad |f(z)| \leqq |z|\{b+(1-a)|z|^p\}^{[c/(1-a)p]\cos\alpha}$$

for $a \neq 1$. *Furthermore, for* $a = 1$,

$$(3.7) \qquad |f(z)| \geqq |z|\exp\left(-\frac{b}{p}|z|^p\cos\alpha\right)$$

and

$$|f(z)| \leqq |z| \exp\left(\frac{b}{p}|z|^p \cos\alpha\right). \tag{3.8}$$

PROOF. Since $f(z)$ is in the class $S_p(\alpha, a, b)$, by using Lemma 1, we obtain

$$\log\left|\frac{f(z)}{z}\right| = \mathrm{Re}\left(\log\left\{\frac{f(z)}{z}\right\}\right) = \mathrm{Re}\left(\int_0^z\left(\frac{f'(t)}{f(t)}-\frac{1}{t}\right)dt\right) \tag{3.9}$$

$$= \int_0^{|z|} t^{-1}\,\mathrm{Re}\left(te^{i\theta}\frac{f'(te^{i\theta})}{f(te^{i\theta})}-1\right)dt \leqq \int_0^{|z|}\frac{ct^{p-1}\cos\alpha}{b+(1-a)t^p}\,dt\,.$$

Hence, for $a \neq 1$,

$$\log\left|\frac{f(z)}{z}\right| \leqq \frac{c\cos\alpha}{(1-a)p}\log\{b+(1-a)|z|^p\}\,, \tag{3.10}$$

which immediately gives the estimate (3.6).

In case $a = 1$, we find from (2.1) that $c = b^2$, and (3.9) reduces to

$$\log\left|\frac{f(z)}{z}\right| \leqq b\cos\alpha\int_0^{|z|} t^{p-1}\,dt = \frac{b}{p}|z|^p\cos\alpha\,, \tag{3.11}$$

which implies (3.8).

In order to establish the remaining estimates (3.5) and (3.7), by virtue of Lemma 1, we observe that

$$r\,\mathrm{Re}\left(\frac{\partial}{\partial r}\left\{\log\left(\frac{f(z)}{z}\right)\right\}\right) = \mathrm{Re}\left(\frac{zf'(z)}{f(z)}-1\right) \tag{3.12}$$

$$\geqq \frac{b-(c\cos\alpha+1-a)r^p}{b-(1-a)r^p}-1 = -\frac{cr^p\cos\alpha}{b-(1-a)r^p}$$

for $|z| = r < 1$. Consequently, we have

$$\log\left|\frac{f(z)}{z}\right| = \mathrm{Re}\left(\log\left\{\frac{f(z)}{z}\right\}\right) \geqq -\int_0^r\frac{ct^{p-1}\cos\alpha}{b-(1-a)t^p}\,dt\,. \tag{3.13}$$

This gives

$$\log\left|\frac{f(z)}{z}\right| \geqq \frac{c\cos\alpha}{(1-a)p}\log\{b-(1-a)r^p\}\,, \qquad a\neq 1\,, \tag{3.14}$$

which evidently yields the estimate (3.5).

From (3.13) we also have

$$\log\left|\frac{f(z)}{z}\right| \geqq -b\cos\alpha\int_0^r t^{p-1}\,dt = -\frac{b}{p}\,r^p\cos\alpha\,, \tag{3.15}$$

for $a=1$, and the assertion (3.7) of Theorem 2 follows at once.

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_p(\alpha, a, b)$. *Then* $f(z)$ *maps the unit disk* $\mathcal{U}$ *onto the domain which contains*

$$|w| < (a+b-1)^{[c/(1-a)p]\cos\alpha} \tag{3.16}$$

for $a\neq 1$, *and*

$$|w| < \exp\left(-\frac{b}{p}\cos\alpha\right) \tag{3.17}$$

in case $a=1$.

REMARK 3. For $\alpha=0$, Theorem 2 reduces to the distortion theorem proved by Owa [4]. Moreover, for $p=1$ and $\alpha=0$, Theorem 2 yields the corresponding result due to Silverman [5].

4. – Radius of starlikeness for $S_p(\alpha, a, b)$.

In this section, we determine the radius of univalence for functions belonging to the class $S_p(\alpha, a, b)$, given by

THEOREM 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_p(\alpha, a, b)$ *with*

$$a+b>1 \quad \textit{and} \quad (b-a+1)\cos\alpha>1\,. \tag{4.1}$$

Then $f(z)$ *is univalent and starlike in the disk*

$$|z| < \{b/(c\cos\alpha+1-a)\}^{1/p}\,. \tag{4.2}$$

PROOF. The inequalities in (4.1), together, yield

$$(a+b-1)\{(b-a+1)\cos\alpha-1\}>0 \tag{4.3}$$

or, equivalently,

$$b/(c\cos\alpha+1-a)<1\,. \tag{4.4}$$

By applying Lemma 1, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0 \tag{4.5}$$

for

$$|z|<\left(\frac{b}{c\cos\alpha+1-a}\right)^{1/p}<1\,. \tag{4.6}$$

This shows that the function $f(z)$ is univalent and starlike for z given by (4.2).

REMARK 4. Theorem 3 reduces, when $\alpha=0$, to the corresponding result obtained by Owa [4]. On the other hand, for $p=1$ and $\alpha=0$, it was proved earlier by Silverman [5].

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