Rendiconti
del
Seminario Matematico
della
Università di Padova

Claudia Menini
Adalberto Orsatti

Representable equivalences between categories of modules and applications

Rendiconti del Seminario Matematico della Università di Padova, tome 82 (1989), p. 203-231
<http://www.numdam.org/item?id=RSMUP_1989__82__203_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 82 (1989)

Representable Equivalences between Categories of Modules and Applications.

CLAUDIA MENINI - ADALBERTO ORSATTI (*)

Dedicato a Giovanni Zacher.

1. Introduction.

All rings considered in this paper have a nonzero identity and all modules are unital. For every ring R, Mod-R (R-Mod) denotes the category of all right (left) R-modules. The symbol M_R (${}_RM$) is used to emphasize that M is a right (left) R-module.

Categories and functors are understood to be additive. Any subcategory of a given category is full and closed under isomorphic objects.

1.1. Let A and R be two rings, $\mathcal{D}_A$ and $\mathcal{G}_R$ subcategories of Mod-A and Mod-R respectively.

Assume that a category equivalence (F, G), $F\colon \mathcal{D}_A \to \mathcal{G}_R$ and $G\colon \mathcal{G}_R \to \mathcal{D}_A$ is given. We say that the equivalence (F, G) is *representable* if there exists a bimodule ${}_AP_R$, with $P_R \in \mathcal{G}_R$, such that the following natural equivalences of functors hold:

$$F \approx (- \otimes_A P)|\mathcal{D}_A\,, \qquad G \approx \mathrm{Hom}_R(P_R, -)|\mathcal{G}_R\,.$$

(*) Indirizzo degli A.A.: C. MENINI: Dipartimento di Matematica, via Roma, I-67100 L'Aquila; A. ORSATTI: Dipartimento di Matematica Pura e Applicata, Università di Padova, via Belzoni 7, I-35100 Padova.

This paper was written while the Authors were members of the G.N.S. A.G.A. of the « Consiglio Nazionale delle Ricerche », with a partial financial support from Ministero della Pubblica Istruzione.

In this case we say that the bimodule ${}_AP_R$ induces the equivalence (F, G). Note that, if $A_A \in \mathcal{D}_A$, then A is canonically isomorphic to $\mathrm{End}(P_R)$.

For example if $\mathcal{D}_A = \text{Mod-}A$ and $\mathcal{G}_R = \text{Mod-}R$, then a classical Morita's result [M] asserts that (F, G) is representable by a faithful balanced bimodule ${}_AP_R$ which is a progenerator on both sides and conversely any such a bimodule induces an equivalence between Mod-A and Mod-R.

1.2. More recently Fuller [F] proved the following result: if $\mathcal{D}_A = \text{Mod-}A$ and if $\mathcal{G}_R$ is closed under submodules, epimorphic images and arbitrary direct sums, then (F, G) is representable by a bimodule ${}_AP_R$ such that P_R is a *quasi-progenerator* i.e. P_R is quasi-projective, finitely generated (f.g.) and generates all its submodules. Conversely any quasi-progenerator P_R with $A = \mathrm{End}(P_R)$ induces such an equivalence. If P_R is a progenerator then $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$ and R is dense in $\mathrm{End}({}_AP)$ endowed with its finite topology. For unexplained terms see Section 2.

1.3. In this paper we prove the following representation theorem. Assume that

a) $A_A \in \mathcal{D}_A$ and $\mathcal{D}_A$ is closed under submodules.

b) $\mathcal{G}_R$ is closed under arbitrary direct sums and epimorphic images.

c) A category equivalence (F, G), $F\colon \mathcal{D}_A \to \mathcal{G}_R$, $G\colon \mathcal{G}_R \to \mathcal{D}_A$ is given

and let Q_R be a fixed, but arbitrary, injective cogenerator of Mod-R.

Then there exists a bimodule ${}_AP_R$ with the following properties:

1) $P_R \in \mathcal{G}_R$, $A \simeq \mathrm{End}(P_R)$.

2) $\mathcal{G}_R = \mathrm{Gen}(P_R)$, $\mathcal{D}_A = \mathcal{D}(K_A)$ where $K_A = \mathrm{Hom}_R(P_R, Q_R)$ and $\mathcal{D}(K_A)$ is the subcategory of Mod-A cogenerated by K_A.

3) The bimodule ${}_AP_R$ induces the equivalence (F, G).

1.4. The categories $\mathcal{D}_A$ and $\mathcal{G}_R$ involved in 1.3 are the largest possible. Indeed, given any bimodule ${}_AP_R$ and setting $T = - \otimes_A P$, $H = \mathrm{Hom}_R(P_R, -)$ we have $\mathrm{Im}(T) \subseteq \mathrm{Gen}(P_R)$ and $\mathrm{Im}(H) \subseteq \mathcal{D}(K_A$.

1.5. Under the assumptions *a*), *b*), *c*) in 1.3, suppose that, in addition, $\mathcal{G}_R$ is closed under submodules. Then we prove that $\mathfrak{D}(K_A) = $ $= \text{Mod-}A$ so that P_R is a quasi-progenerator.

Thus we obtain, in this way, a non trivial generalization of Fuller's Theorem on equivalences.

1.6. Under the assumptions *a*), *b*), *c*) in 1.3 it holds, in general, that $\mathfrak{D}_A \neq \text{Mod-}A$. This will be proved in Section 4 using tilting modules of Happel and Ringel [HR$_2$]. Nevertheless we are able to give, in Section 5, a number of conditions in order that $\mathfrak{D}_A = \text{Mod-}A$. In particular this is true if P_R is quasi-projective.

The following question is still open: characterize the modules $P_R \in \text{Mod-}R$ such that, setting $A = \text{End}(P_R)$, the bimodule ${}_AP_R$ induces an equivalence between $\mathfrak{D}(K_A)$ and $\text{Gen}(P_R)$.

1.7. Using Pontryagin duality on R, Theorem in 1.3 can be translated in a representation theorem for a given duality between the category $\mathfrak{D}_A$ in 1.3 and a category ${}_R\mathcal{C}$ of compact modules which is assumed to be closed under topological products and closed submodules.

This representation theorem leads us to solve and old question of ours [MO]: there exist dualities between $\mathfrak{D}_A$ and ${}_R\mathcal{C}$ which are not « good dualities ».

Acknowledgements. The authors are deeply indebted to Prof. Masahisa Sato for pointing out to them that tilting modules provide examples of $\mathfrak{D}(K_A) \neq \text{Mod-}A$.

They are also grateful to Prof. G. D'Este and Dr. E. Gregorio for many useful suggestions.

2. Preliminaries.

2.1. Let A and R be two rings, ${}_AP_R$ any bimodule. Define the functors T and H by setting

$$T = - \underset{A}{\otimes} P\colon \ \text{Mod-}A \to \text{Mod-}R\,,$$

$$H = \text{Hom}_R(P_R, -)\colon \ \text{Mod-}R \to \text{Mod-}A\,.$$

Let $\mathrm{Gen}\,(P_R)$ be the full subcategory of Mod-R generated by P_R. Recall that a module $M \in$ Mod-R belongs to $\mathrm{Gen}\,(P_R)$ if there exists an epimorphism $P_R^{(X)} \to M \to 0$ where X is a suitable set. $\mathrm{Gen}\,(P_R)$ is closed under taking epimorphic images and arbitrary direct sums.

Denote by $\overline{\mathrm{Gen}}\,(P_R)$ the smallest subcategory of Mod-R containing $\mathrm{Gen}\,(P_R)$ and closed under taking submodules, epimorphic images and direct sums. Clearly $\mathrm{Gen}\,(P_R) = \overline{\mathrm{Gen}}\,(P_R)$ if and only if $\mathrm{Gen}\,(P_R)$ is closed under submodules.

Let $C(P_R)$ be the subcategory of Mod-R consisting of all modules $M \in$ Mod-R having *P_R-codominant dimension* $\geqslant 2$ i.e. for which there is an exact sequence of the form

$$P_R^{(X)} \to P_R^{(Y)} \to M \to 0 .$$

Clearly

$$C(P_R) \subseteq \mathrm{Gen}\,(P_R) \subseteq \overline{\mathrm{Gen}}\,(P_R) .$$

Let Q_R be a fixed, but arbitrary, injective cogenerator of Mod-R and set $K_A = \mathrm{Hom}_R\,(P_R, Q_R)$. Denote by $\mathfrak{D}(K_A)$ the full subcategory of Mod-A cogenerated by K_A and by $L(K_A)$ the subcategory of Mod-A consisting of all modules $L \in$ Mod-A having *K_A-dominant dimension* $\geqslant 2$. This means that there exists an exact sequence

$$0 \to L \to K_A^X \to K_A^Y .$$

Clearly

$$L(K_A) \subseteq \mathfrak{D}(K_A) .$$

Finally, for every $M \in$ Mod-R, set

$$t_P(M) = \sum \{\mathrm{Im}\,(f)\colon f \in \mathrm{Hom}_R\,(P_R, M)\} .$$

Then $t_P(M) \in \mathrm{Gen}\,(P_R)$ and $\mathrm{Hom}_R\,(P, M) \cong \mathrm{Hom}_R\,(P, t_P(M))$ canonically.

2.2. PROPOSITION. *Let ${}_AP_R$ be any bimodule. Then*:

a) $\mathrm{Im}\,(T) \subseteq C(P_R) \subseteq \mathrm{Gen}\,(P_R)$.

b) $\mathrm{Im}\,(H) \subseteq L(K_A) \subseteq \mathfrak{D}(K_A)$.

c) *For every* $M \in$ Mod-R, $H(M) \cong H(t_P(M))$ *canonically.*

PROOF. *a*) Let $L \in \text{Mod-}A$. There is an exact sequence of the form

$$A^{(X)} \to A^{(Y)} \to L \to 0 .$$

Tensoring by ${}_AP$ we get the exact sequence

$$P_R^{(X)} \to P_R^{(Y)} \to T(L) \to 0 .$$

Hence $T(L) \in C(P_R)$.

b) Let $M \in \text{Mod-}R$. There exists an exact sequence

$$0 \to M \to Q_R^X \to Q_R^Y .$$

Applying H we get the exact sequence

$$0 \to H(M) \to K_A^X \to K_A^Y$$

so that $H(M) \in L(K_A)$.

c) is obvious.

2.3. Let ${}_AP_R$ be any bimodule. Recall that for every $M \in \text{Mod-}R$ there exists a natural morphism in Mod-R

$$\varrho_M\colon \operatorname{Hom}_R(P_R, M) \underset{A}{\otimes} P \to M$$

given by $\varrho_M(f \otimes p) = f(p)$ $(f \in \operatorname{Hom}_R(P_R, M),\ p \in P)$.

It is also well known that, for every $L \in \text{Mod-}A$ there is a natural morphism in Mod-A

$$\sigma_L\colon L \to \operatorname{Hom}_R(P_R, L \underset{A}{\otimes} P)$$

given by

$$\sigma_L(l)\colon p \mapsto l \otimes p \qquad (l \in L\,,\ p \in P)\,.$$

The following remarks are useful:

a) *For every* $M \in \operatorname{Gen}(P_R)$, ϱ_M *is surjective.*

b) *For every* $L \in \mathfrak{D}(K_A)$, σ_L *is injective.*

Statement *a*) is obvious. Let us prove *b*). Let $L \in \mathfrak{D}(K_A)$. Then there exists an inclusion $L \hookrightarrow \mathrm{Hom}_R(P_R, Q_R)^X$. Thus for every $l \in L$, $l \neq 0$, there exists a $\xi \in \mathrm{Hom}_A(L, \mathrm{Hom}_R(P_R, Q_R))$ such that $\xi(l) \neq 0$. Hence there is a $p \in P$ such that $\xi(l)(p) \neq 0$. Let $\bar{\xi}\colon L \otimes_A P \to Q_R$ be the morphism defined by setting $\bar{\xi}(x \otimes y) = \xi(x)(y)$, $x \in L$, $y \in P$, Then $\bar{\xi}(l \otimes p) = \xi(l)(p) \neq 0$ and thus $l \otimes p \neq 0$ so that $l \notin \mathrm{Ker}(\sigma_L)$. Hence $\mathrm{Ker}(\sigma_L) = 0$.

2.4. Proposition. *Let ${}_AP_R$ be a bimodule which induces an equivalence between a subcategory $\mathfrak{D}_A$ of* Mod-A *and a subcategory $\mathcal{G}_R$ of* Mod-R. *Then, if $A \in \mathfrak{D}_A$, for every $M \in \mathcal{G}_R$ and for every $L \in \mathfrak{D}_A$ the morphisms ϱ_M and σ_L are isomophisms.*

Proof. Let $M \in \mathcal{G}_R$. Then, by Proposition 2.2, $M \in \mathrm{Gen}(P_R)$ and hence, by 2.3, ϱ_M is surjective. Set $N_A = \mathrm{Hom}_R(P_R, M)$.

By assumption there is an isomorphism

$$\varrho\colon TH(M) = T(N) \to M .$$

Let $\theta\colon N_A \to \mathrm{Hom}_R(P_R, M_R)$ be the morphism corresponding to ϱ because of the adjointness of T and H. Then for every $f \in N = \mathrm{Hom}_R(P_R, M_R)$ and for every $p \in P$:

$$\theta(f)(p) = \varrho(f \otimes p) . \tag{1}$$

Let $h \in \mathrm{Hom}_R(M_R, M_R)$ such that $\theta = \mathrm{Hom}_R(P_R, h)$.

Then, for every $f \in N$,

$$\theta(f) = h \circ f$$

and hence, for every $p \in P$,

$$\theta(f)(p) = h(f(p)) = (h \circ \varrho_M)(f \otimes p) . \tag{2}$$

From (1) and (2) we get

$$\varrho(f \otimes p) = (h \circ \varrho_M)(f \otimes p)$$

for every $f \in N$, $p \in P$.

Thus $\varrho = h \circ \varrho_M$. Then ϱ_M is injective as ϱ is injective.

Let now $L \in \mathfrak{D}_A$. Then, by Proposition 2.2, $L \in \mathfrak{D}(K_A)$ and hence by 2.3, σ_L is injective. Let $\xi \in \mathrm{Hom}_R\big(P, L \underset{A}{\otimes} P\big) = \mathrm{Hom}_R\,(T(A), T(L))$. Then there is an $f \in \mathrm{Hom}_A(A, L)$ such that $\xi = T(f)$. Let $x = f(1)$. Then, for every $p \in P$ we have

$$(\sigma_L(x))(p) = f(1) \otimes p = (T(f))(1 \otimes p) = \xi(p)\,.$$

Thus $\sigma_L(x) = \xi$ and σ_L is surjective.

2.5. Definition. A module ${}_AP \in$ Mod-A is called *weak generator* if, for every $L \in$ Mod-A,

$$L \neq 0 \Rightarrow L \underset{A}{\otimes} P \neq 0\,.$$

2.6. Lemma. *Let $M \in \mathrm{Gen}\,(P_R)$, $h\colon P_R^{(X)} \to M$ be an epimorphism. Let $h = (h_x)_{x \in X}$ with $h_x \in \mathrm{Hom}_R(P_R, M)$. Then the right A-submodule $\sum_{x \in X} h_x A$ of $\mathrm{Hom}_R(P_R, M)$ belongs to $\mathfrak{D}(K_A)$. Moreover if*

$$\sum_{x \in X} h_x A = \mathrm{Hom}_R(P_R, M)$$

then for every $f\colon P_R \to M$ there exists a $g \in \mathrm{Hom}_R(P_R, P_R^{(X)})$ such that $f = h \circ g$.

Proof. The first assertion follows by Proposition 2.2. Assume now that $\sum_{x \in X} h_x A = \mathrm{Hom}_R(P_R, M)$ and let $f\colon P_R \to M$ be an R-morphism. Then $f = \sum_{x \in X} h_x a_x$ where $a_x \in A$ and almost all a_x's vanish. Let $g\colon P_R \to P_R^{(X)}$ be the diagonal morphism of the a_x's, $x \in X$. Then for every $p \in P$ we have $(h \circ g)(p) = \sum h_x a_x(p) = f(p)$. Thus $f = h \circ g$.

2.7. Definitions. Let $P_R \in$ Mod-R, $A = \mathrm{End}\,(P_R)$. P_R is called *quasi-projective* if, for every diagram

$$\begin{array}{ccccc} & & P_R & & \\ & & \big\downarrow{\scriptstyle f} & & \\ P_R & \xrightarrow[h]{} & M & \longrightarrow & 0 \end{array}$$

with exact row, there exists an $a \in A$ such that $f = h \circ a$.

P_R is called *Σ-quasi-projective* if $P_R^{(X)}$ is quasi-projective for every set $X \neq \emptyset$. Clearly P_R is Σ-quasi-projective if and only if P_R is a projective object of $\mathrm{Gen}(P_R)$.

2.8. LEMMA. *Let* $P_R \in \text{Mod-}R$, $A = \mathrm{End}(P_R)$, $M \in \mathrm{Gen}(P_R)$, $h = (h_x)_{x\in X}$ *an epimorphism of* $P_R^{(X)}$ *onto* M *and assume that* ϱ_M *is injective. Then*:

a) *The morphism* $\varphi\colon \left(\sum_{x\in X} h_x A\right) \underset{A}{\otimes} P \to \mathrm{Hom}_R(P_R, M) \underset{A}{\otimes} P$ *given by the inclusion* $\sum_{x\in X} h_x A \hookrightarrow \mathrm{Hom}_R(P_R, M)$ *is surjective.*

b) *If* ${}_AP$ *is a weak generator, then* $\sum_{x\in X} h_x A = \mathrm{Hom}_R(P_R, M)$.

PROOF. See [*A*], Lemma 1 and Proposition 5.

2.9. LEMMA. *Let* $P_R \in \text{Mod-}R$, $A = \mathrm{End}(P_R)$. *If the functors* $T = - \underset{A}{\otimes} P$ *and* $H = \mathrm{Hom}(P_R, -)$ *subordinate an equivalence between* $\mathfrak{D}(K_A)$ *and* $\mathrm{Gen}(P_R)$ *and if* $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$, *then*:

a) *For every* $M \in \mathrm{Gen}(P_R)$ *and for every epimorphism*

$$h\colon P_R^{(X)} \to M\,, \qquad h = (h_x)_{x\in X}\,,$$

we have

$$\sum_{x\in X} h_x A = \mathrm{Hom}_R(P_R, M)\,.$$

b) P_R *is Σ-quasi-projective.*

PROOF. Since (T, H) is an equivalence between $\mathfrak{D}(K_A)$ and $\mathrm{Gen}(P_R)$, by Proposition 2.4 for every $M \in \mathrm{Gen}(P_R)$ and for every $L \in \mathfrak{D}(K_A)$, ϱ_M and σ_L are isomorphisms. Moreover, to prove *a*), it is enough to prove that the morphism

$$\varphi\colon \left(\sum_{x\in X} h_x A\right) \underset{A}{\otimes} P \to \mathrm{Hom}_R(P_R, M) \underset{A}{\otimes} P$$

defined in Lemma 2.8 is an isomorphism. Now, as ϱ_M is injective, by Lemma 2.8, φ is surjective.

Assume that φ is not injective. Set $L = \sum_{x \in X} h_x A$ and consider the inclusion $i\colon L \to \mathrm{Hom}_R(P_R, M)$. Applying $- \otimes_A P$ we get the exact sequence in Mod-R

$$0 \to Y \to T(L) \xrightarrow{T(i)} TH(M)\,, \tag{1}$$

where $T(i) = \varphi$ and $Y \neq 0$. Since $\mathrm{Gen}\,(P_R) = \overline{\mathrm{Gen}}\,(P_R)$, $Y \in \mathrm{Gen}\,(P_R)$. Applying H to (1) and setting $\bar{M} = H(M)$ we have the exact sequence

$$0 \to H(Y) \to HT(L) \xrightarrow{HT(i)} HT(\bar{M})\,. \tag{2}$$

On the other hand we have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\ i\ } & \bar{M} \\ {\scriptstyle \sigma_L}\downarrow & & \downarrow{\scriptstyle \sigma_{\bar{M}}} \\ HT(L) & \xrightarrow[HT(i)]{} & HT(\bar{M}) \end{array}$$

Since $L \in \mathfrak{D}(K_A)$, σ_L and $\sigma_{\bar{M}}$ are both isomorphisms. It follows that $HT(i)$ is injective so that, from (2), we get $H(Y) = 0$. Thus $Y = 0$ as $Y \in \mathrm{Gen}\,(P_R)$. Contradiction.

b) By *a*) and Lemma 2.6, it follows that for every diagram with exact row

$$\begin{array}{ccccc} & & P_R & & \\ & & \downarrow{\scriptstyle g} & & \\ P_R^{(H)} & \xrightarrow[h]{} & M & \longrightarrow & 0 \end{array}$$

there is a $g\colon P_R \to P_R^{(X)}$ such that $f = h \circ g$.

This means that P_R is Σ-quasi-injective.

3. The main result.

3.1. Representation Theorem. *Let A, R be two rings, $\mathfrak{D}_A$, $\mathfrak{G}_R$ full subcategories of* Mod-A *and* Mod-R *respectively, Q_R a fixed, but arbitrary, injective cogenerator of* Mod-R.

Assume that

a) $A_A \in \mathcal{D}_A$ *and* $\mathcal{D}_A$ *is closed under taking submodules.*

b) $\mathcal{G}_R$ *is closed under taking direct sums and epimorphic images*

c) *A category equivalence* $F\colon \mathcal{D}_A \to \mathcal{G}_R$, $G\colon \mathcal{G}_R \to \mathcal{D}_A$ *is given with* F, G *additive functors.*

Then there exists a bimodule ${}_AP_R$, *unique up to isomorphisms, with the following properties:*

1) $P_R \in \mathcal{G}_R$, $A \cong \operatorname{End}(P_R)$ *canonically.*

2) $\mathcal{D}_A = \mathcal{D}(K_A)$, *where* $K_A = \operatorname{Hom}_R(P_R, Q_R)$ *and* $\mathcal{G}_R = \operatorname{Gen}(P_R)$

3) *The functors* F *and* G *are naturally equivalent to the functors* $T = - \otimes_A P$ *and* $H = \operatorname{Hom}_R(P_R, -)$ *respectively.*

4) *For every* $L \in \mathcal{D}(K_A)$ *and for every* $M \in \operatorname{Gen}(P_R)$ *the canonical morphisms* σ_L *and* ϱ_M *are isomorphisms*

$$\varrho_M\colon \operatorname{Hom}_R(P_R, M) \otimes_A P \to M \qquad (\varrho_M(f \otimes p) = f(p)),$$

$$\sigma_L\colon L \to \operatorname{Hom}_R\big(P_R, L \otimes_A P\big) \qquad \sigma(l)(p \mapsto l \otimes p).$$

PROOF. Set $P_R = F(A)$. Then $A \cong \operatorname{End}(P_R)$ canonically and we have the bimodule ${}_AP_R$. For every $M \in \operatorname{Gen}(P_R)$ consider the canonical isomorphisms

$$G(M) \cong \operatorname{Hom}_A(A, G(M)) \cong \operatorname{Hom}_R\big(F(A), F(G(M))\big) \cong \operatorname{Hom}_R(P_R, M).$$

Thus, looking at the closure properties of $\mathcal{G}_R$, we deduce that

i) G *is naturally equivalent to the functor* $H = \operatorname{Hom}_R(P_R, -)$ *and* $\operatorname{Gen}(P_R) \subseteq \mathcal{G}_R$.

Consider now the functor $T\colon \mathcal{D}_A \to \mathcal{G}_R$ given by $T(L) = L \otimes_A P$ for every $L \in \mathcal{D}_A$. T is well defined by i) and by Proposition 2.2. By well known facts, the functors T and H are adjoints. Since (F, H) is an equivalence, F and H are adjoints. Therefore F and T are equivalent. Thus

ii) F *is naturally equivalent to the functor* $T = - \otimes_A P$.

Moreover, by i) and by Proposition 2.2 we get

iii) $\mathcal{G}_R = \mathrm{Gen}(P_R)$.

Set $K_A = \mathrm{Hom}_R(P_R, Q_R)$. Then:

iv) $\mathcal{D}_A = \mathcal{D}(K_A)$.

The proof is due to E. Gregorio. First of all let us prove that $\mathcal{D}_A$ is closed under taking direct products. Let $(L_\lambda)_{\lambda\in\Lambda}$ be a family of modules in $\mathcal{D}_A$. For every $\lambda \in \Lambda$ we have $L_\lambda = H(M_\lambda)$ where $M_\lambda \in \mathrm{Gen}(P_R)$. Now in Mod-$R$ we have the following natural isomorphisms:

$$\mathrm{Hom}_R\Big(P_R, t_R\Big(\prod_{\lambda\in\Lambda} M_\lambda\Big)\Big) \cong \mathrm{Hom}_R\Big(P_R, \prod_{\lambda\in\Lambda} M_\lambda\Big) \cong \cong \prod_{\lambda\in\Lambda} \mathrm{Hom}_R(P_R, M_\lambda) \cong \prod_{\lambda\in\Lambda} L_\lambda .$$

Since $t_R\Big(\prod_{\lambda\in\Lambda} M_\lambda\Big) \in \mathrm{Gen}(P_R) = \mathcal{G}_R$ and by *a*), it follows $\prod_{\lambda\in\Lambda} L_\lambda \in \mathcal{D}_A$. For similar reasons we have $K_A \in \mathcal{D}_A$. Indeed:

$$K_A = \mathrm{Hom}_R(P_R, Q_R) \cong \mathrm{Hom}_R\big(P_R, t_P(Q_R)\big) \in \mathcal{D}_A .$$

Therefore $\mathcal{D}(K_A) \subseteq \mathcal{D}_A$ by the closure properties of $\mathcal{D}_A$. On the other hand, by *a*) and by Proposition 2.2, $\mathcal{D}_A \subseteq \mathcal{D}(K_A)$. Statement 4) follows from Proposition 2.4 in view of *a*) and 3). Finally, since $A \cong \mathrm{End}(P_R)$ we have $A \cong H(P_R)$ so that $P_R \cong T(A)$ canonically. Thus P_R is unique up to isomorphisms.

From Theorem 3.1 we get the following important

3.2. PROPOSITION. *Suppose that the assumptions a), b), c) of Theorem* 3.1 *hold. Then the following conditions are equivalent*:

(*a*) $\mathcal{G}_R$ *is closed under taking submodules.*

(*b*) $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

(*c*) $\mathcal{D}(K_A) = \text{Mod-}A$.

(*d*) ${}_AP$ *is a weak-generator.*

(*e*) *For every* $M \in \mathrm{Gen}(P_R)$ *and for every epimorphism*

$$h\colon P_R^{(X)} \to M , \qquad \mathrm{Hom}_R(P_R, M) = \sum_{x\in X} h_x A .$$

(*f*) P_R *is* Σ*-quasi-projective.*

PROOF. $(a) \Leftrightarrow (b)$ is obvious in view of Theorem 3.1.

$(b) \Rightarrow (f)$ by Theorem 3.1 and Lemma 2.9.

$(f) \Rightarrow (b)$ Let $M \in \mathrm{Gen}(P_R)$, U a submodule of M and consider the exact sequence

$$0 \to U \to M \to N \to 0 .$$

As P_R is Σ-quasi-projective, it is a projective object of $\mathrm{Gen}(P_R)$ so that we get the exact sequence

$$0 \to H(U) \to H(M) \to H(N) \to 0 .$$

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & TH(U) & \longrightarrow & TH(M) & \longrightarrow & TH(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varrho_U} & & \downarrow{\scriptstyle \varrho_M} & & \downarrow{\scriptstyle \varrho_N} & & \\ 0 & \longrightarrow & U & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

ϱ_M and ϱ_N are isomorphisms so that ϱ_U is surjective. Since $\mathrm{Im}(T) \subseteq \subseteq \mathrm{Gen}(P_R)$ we get $U \in \mathrm{Gen}(P_R)$. Therefore $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

$(c) \Rightarrow (d)$ is clear in view of Theorem 3.1.

$(d) \Rightarrow (e)$ let $M \in \mathrm{Gen}(P_R)$. By Theorem 3.1 ϱ_M is an isomorphism. Thus (e) follows from Lemma 2.8.

$(e) \Rightarrow (f)$ by Lemma 2.6.

$(f) \Rightarrow (c)$ Let $L \in$ Mod-A. We have an exact sequence

$$A^{(X)} \to A^{(Y)} \to L \to 0 . \tag{1}$$

Tensoring (1) by ${}_AP$ we get the exact sequence

$$A^{(X)} \otimes_A P \to A^{(Y)} \otimes_A P \to L \otimes_A P \to 0 . \tag{2}$$

As P_R is Σ-quasi-projective applying $\mathrm{Hom}_R(P_R, -)$ to (2) we get the exact sequence

$$HT(A^{(X)}) \to HT(A^{(Y)}) \to HT(L) \to 0 .$$

Consider now the commutative diagram

$$\begin{array}{ccccccc} A^{(X)} & \longrightarrow & A^{(Y)} & \longrightarrow & L & \longrightarrow & 0 \\ \sigma_{A^{(X)}}\downarrow & & \sigma_{A^{(Y)}} & & \sigma_L & & \\ HT(A^{(X)}) & \longrightarrow & HT(A^{(Y)}) & \longrightarrow & HT(L) & \longrightarrow & 0 \end{array}$$

Since $\sigma_{A^{(X)}}$ and $\sigma_{A^{(Y)}}$ are both isomorphisms, σ_L is an isomorphism too. Since $\operatorname{Im}(H) \subseteq \mathfrak{D}(K_A)$, we have $L \cong H\big(L \underset{A}{\otimes} P\big) \in \mathfrak{D}(K_A)$. Thus $\mathfrak{D}(K_A) = \text{Mod-}A$.

3.3. Remark. The proposition above gives a non trivial generalization of Fuller's Theorem on equivalence (cf. [F], Theorem 1.1).

3.4. Proposition. *Let* ${}_AP_R$ *be a bimodule which induces an equivalence between* $\mathfrak{D}(K_A)$ *and* $\operatorname{Gen}(P_R)$ *and let* $(M_\lambda)_{\lambda\in\Lambda}$ *be a family of modules in* $\operatorname{Gen}(P_R)$. *Then*

1) $\operatorname{Hom}_R\big(P_R, \bigoplus_\lambda M_\lambda\big) \cong \bigoplus_\lambda \operatorname{Hom}_R(P_R, M_\lambda)$ *canonically.*

In particular

2) $A \cong \operatorname{End}(P_R)$.

3) *For every set* $X \neq \emptyset$, $\operatorname{Hom}_R(P_R, P_R^{(X)}) \cong A^{(X)}$.

Proof. 1) There exist the canonical isomorphisms:

$$\bigoplus_\lambda \operatorname{Hom}_R(P_R, M_\lambda) = \bigoplus_\lambda H(M_\lambda) \cong HT\big(\bigoplus_\lambda H(M_\lambda)\big) \cong$$
$$\cong H\big(\bigoplus_\lambda TH(M_\lambda)\big) \cong H\big(\bigoplus_\lambda M_\lambda\big) = \operatorname{Hom}_R\big(P_R, \bigoplus M_\lambda\big).$$

2) and 3) are now obvious.

3.5. Theorem 1.3 suggests the following natural question:

(*) *For a given ring* R *determine all modules* $P_R \in \text{Mod-}R$ *such that, setting* $A = \operatorname{End}(P_R)$, *the bimodule* ${}_AP_R$ *induces an equivalence between* $\mathfrak{D}(K_A)$ *and* $\operatorname{Gen}(P_R)$.

Suppose that ${}_AP_R$ is such a bimodule. Then the functors $T = - \underset{A}{\otimes} P$

and $H = \mathrm{Hom}_R(P_R, -)$ subordinate an equivalence between $\mathrm{Im}(H)$ and $\mathrm{Im}(T)$ and moreover, in view of Proposition 2.2, the subcategories $\mathrm{Im}(H) \subseteq \text{Mod-}A$ and $\mathrm{Im}(T) \subseteq \text{Mod-}R$ are the *largest possible.* To answer question (*) without further assumptions seems to be quite difficult.

Let ${}_AP_R$ be a bimodule, Q_R an injective cogenerator of Mod-R, $K_A = \mathrm{Hom}_R(P_R, Q_R)$ and the functors T, H have the usual meaning, M. Sato ([S], Theorem 1.3) has shown that the bimodule ${}_AP_R$ induces an equivalence between $\mathrm{Im}(H)$ and $\mathrm{Im}(T)$ if and only if $\mathrm{Im}(H) = L(K_A)$, $\mathrm{Im}(T) = C(P_R)$ and moreover ${}_AP_R$ induces an equivalence between $L(K_A)$ and $C(P_R)$ (see also the proof of Theorem 1.3 of [S]).

In this situation it could happen that $C(P_R) = \mathrm{Gen}(P_R)$ while $L(K_A) \neq \mathfrak{D}(K_A)$ as the following example shows.

3.6. AN EXAMPLE. Let p be a prime number, $\mathbb{Z}(p^\infty)$ the Prüfer group relative to p, J_p the ring of p-adic integers and consider the bimodule ${}_{J_p}\mathbb{Z}(p^\infty)_{J_p}$. Note that $\mathrm{End}(\mathbb{Z}(p^\infty)_{J_p}) = J_p$.

In this case $C(\mathbb{Z}(p^\infty)_{J_p}) = \mathrm{Gen}(\mathbb{Z}(p^\infty)) =$ the category of all divisible p-primary abelian groups.

On the other hand, since $\mathbb{Z}(p^\infty)$ is an injective cogenerator of Mod-J_p, we have

$$K_{J_p} = \mathrm{Hom}_{J_p}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) = J_P .$$

$\mathfrak{D}(J_p)$ is the category of all reduced torsion-free J-modules, while $L(J_p)$ is the category of all cotorsion and torsion-free J_p-modules, which are exactly all the direct summands of direct products of copies of J_p.

By well known results of Harrison [H], the functors $T = - \otimes_{J_p} \mathbb{Z}(p^\infty)$ and $H = \mathrm{Hom}_{J_p}(\mathbb{Z}(p^\infty), -)$ subordinate a category equivalence between

$$\mathrm{Im}(H) = L(J_p) \quad \text{and} \quad \mathrm{Im}(T) = \mathrm{Gen}(\mathbb{Z}(p^\infty)_{J_p}) = C(\mathbb{Z}(p^\infty)_{J_p}) .$$

Thus, in this case, $L(J_p) \subsetneq \mathfrak{D}(J_p)$.

There exists a condition in order that a bimodule ${}_AP_R$ induces an equivalence between $\mathfrak{D}(K_A)$ and $\mathrm{Gen}(P_R)$ which involves the whole categories Mod-A and Mod-R.

3.7. PROPOSITION. *Let* ${}_AP_R$ *be a bimodule with* $A = \mathrm{End}\,(P_R)$. *Then the following conditions are equivalent*:

(*a*) ${}_AP_R$ *induces an equivalence between* $\mathfrak{D}(K_A)$ *and* $\mathrm{Gen}\,(P_R)$.

(*b*) *For every* $L \in \mathrm{Mod}\text{-}A$ *the canonical morphism* σ_L *is surjective and for every* $M \in \mathrm{Mod}\text{-}R$ *the canonical morphism* ϱ_M *is injective.*

PROOF. $(a) \Rightarrow (b)$ Let $M \in \mathrm{Mod}\text{-}R$ and let $i\colon t_P(M) \to M$ be the canonical inclusion. Then $H(i)\colon H(t_P(M)) \to H(M)$ is an isomorphism and hence $TH(i)\colon TH(t_P(M)) \to TH(M)$ is an isomorphism too.

Consider the commutative diagram:

$$\begin{array}{ccc} TH(t_P(M)) & \xrightarrow[\sim]{TH(i)} & TH(M) \\ {\scriptstyle \varrho_{t_P(M)}}\downarrow & & \downarrow{\scriptstyle \varrho_M} \\ t_P(M) & \xrightarrow{i} & M \end{array}$$

As $t_P(M) \in \mathrm{Gen}\,(P_R)$, by Theorem 3.1, $\varrho_{t_P(M)}$ is an isomorphism. Thus ϱ_M is injective.

Let now $L \in \mathrm{Mod}\text{-}A$ and consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(\sigma_L) \longrightarrow L \xrightarrow{\pi} L/\mathrm{Ker}\,(\sigma_L) \longrightarrow 0$$

As $\mathrm{Ker}\,(\sigma_L) = \{x \in L\colon x \otimes p = 0 \text{ for every } p \in P\}$ the map

$$T(\pi)\colon T(L) \to T(L/\mathrm{Ker}\,(\sigma_L))$$

is an isomorphism hence

$$HT(\pi)\colon HT(L) \to HT(L/\mathrm{Ker}\,(\sigma_L))$$

is an isomorphism too.

Now $L/\mathrm{Ker}\,(\sigma_L)$ embedds into $\mathrm{Hom}_R\big(P, L \underset{A}{\otimes} P\big)$ and hence belongs to $\mathfrak{D}(K_A)$. Thus $\sigma_{L/\mathrm{Ker}(\sigma_L)}$ is an isomorphism so that from the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\pi} & L/\mathrm{Ker}\,(\sigma_L) \\ {\scriptstyle \sigma_L}\downarrow & & \downarrow{\scriptstyle \sigma_{L/\mathrm{Ker}(\sigma_L)}} \\ HT(L) & \xrightarrow[\sim]{HT(\pi)} & HT(L/\mathrm{Ker}\,(\sigma_P)) \end{array}$$

we get that σ_L is surjective.

$(b) \Rightarrow (a)$ follows in view of a) and b) of 2.3.

4. *w*-tilting modules.

4.1. In this section we will prove that under the assumptions *a*)' *b*), *c*) of Theorem 3.1 it holds, in general, that $\mathfrak{D}(K_A) \neq \text{Mod-}A$.

4.2. Let R be a ring. Generalizing the concept of tilting module in the sense of Happel and Ringel [HR$_2$] we say that a right R-module P_R is a *w-tilting module* if the following conditions hold:

1) P_R is finitely presented.

2) P_R has projective dimension $\leqslant 1$.

3) $\text{Ext}^1_R(P, P) = 0$.

4) There exists an exact sequence in Mod-R of the form

$$0 \to R \to P' \to P'' \to 0 \tag{1}$$

where P' and P'' are direct sums of direct summands of P_R.

Note that when R is a finite dimensional algebra over a field $\mathcal{K}$ any tilting module in the sense of Happel and Ringel is a w-tilting module.

The following theorem is modelled on Brenner-Butler Theorem on tilting modules (see [HR$_2$]). As in their setting all modules are finitely generated, we shall give the proof for our more general case.

4.3. THEOREM. *Let P_R be a w-tilting module, $A = \text{End}(P_R)$ and let $\mathfrak{D}_A = \{L \in \text{Mod-}A\colon \text{Tor}^R_1(L_A, {}_AP) = 0\}$. Then*

a) $\text{Gen}(P_R) = \{M \in \text{Mod-}R\colon \text{Ext}^1_R(P, M) = 0\}$.

b) *$A \in \mathfrak{D}_A$ and $\mathfrak{D}_A$ is closed under submodules.*

c) *For every $M \in \text{Gen}(P_R)$, $\text{Hom}_R(P_R, M) \in \mathfrak{D}_A$ and the functors $H\colon \text{Gen}(P_R) \to \mathfrak{D}_A$, $T\colon \mathfrak{D}_A \to \text{Gen}(P_R)$ given by $H(M) = \text{Hom}_R(P_R, M)$ and $T(L) = L \otimes_A P$, for every $M \in \text{Gen}(P_R)$ and $L \in \mathfrak{D}_A$, are an equivalence between $\text{Gen}(P_R)$ and $\mathfrak{D}_A$. Therefore if Q_R is an arbitrary cogenerator of* Mod-R, *then, by setting $K_A = \text{Hom}_R(P_R, Q_R)$, we have $\mathfrak{D}_A = \mathfrak{D}(K_A)$.*

PROOF. First of all we show that for every set X, $\mathrm{Ext}^1_R(P, P^{(X)}) = 0$. As P_R is finitely presented we have an exact sequence in Mod-R of the form

$$0 \to F_R \to R^n_R \to P_R \to 0 \tag{2}$$

where $n \in N$ and F_R is a finitely generated right R-module.

Applying $\mathrm{Hom}_R(-, P)$ we get the exact sequence

$$0 \to \mathrm{Hom}_R(P, P) \to \mathrm{Hom}_R(R^n, P) \to \mathrm{Hom}_R(F, P) \to 0 = \mathrm{Ext}^1_R(P, P)\,,$$

hence every morphism $F_R \to P_R$ can be extended to a morphism $R^n \to P$. Consider now a morphism $f\colon F \to P^{(X)}$. As F is finitely generated, f is a diagonal morphism of a finite family of morphisms from F into P and hence f extends to a morphism from R^n into $P^{(X)}_R$. Thus the sequence

$$0 \to \mathrm{Hom}_R(P, P^{(X)}) \to \mathrm{Hom}_R(R^n, P^{(X)}) \to \mathrm{Hom}_R(F, P^{(X)}) \to 0$$

is exact. Thus, as $\mathrm{Ext}^1_R(R^n, P^{(X)}) = 0$ we get $\mathrm{Ext}^1_R(P, P^{(X)}) = 0$. Now let $M \in \mathrm{Gen}(P_R)$. Then there exists an exact sequence

$$0 \to M' \to P^{(X)}_R \to M \to 0\,.$$

Applying $\mathrm{Ext}^1_R(P, -)$ we get the exact sequence

$$0 = \mathrm{Ext}^1_R(P, P^{(X)}) \to \mathrm{Ext}^1_R(P, M) \to \mathrm{Ext}^2_R(P, M')\,.$$

As $\mathrm{Ext}^2_R(P_R, M') = 0$ we get $\mathrm{Ext}^1_R(P, M) = 0$.

Conversely assume that $\mathrm{Ext}^1_R(P, M) = 0$ and consider the exact sequence

$$0 \to t_P(M) \to M \to M/t_P(M) \to 0\,.$$

Applying $\mathrm{Hom}_R(P, -)$ we get the exact sequence

$$0 \to \mathrm{Hom}_R(P, t_P(M)) \xrightarrow{\alpha} \mathrm{Hom}_R(P, M) \to$$
$$\to \mathrm{Hom}_R(P, M/t_P(M)) \to \mathrm{Ext}^1_R(P, t_P(M)) \to \mathrm{Ext}^1_R(P, M) \to$$
$$\to \mathrm{Ext}^1_R(P, M/t_P(M)) \to \mathrm{Ext}^2_R(P, t_P(M))\,.$$

As $t_P(M) \in \mathrm{Gen}(P_R)$, we have $\mathrm{Ext}^1_R(P, t_P(M)) = \mathrm{Ext}^2_R(P, t_P(M)) = 0$.

On the other hand, α, by the definition of $t_P(M)$, is surjective so that $\mathrm{Hom}_R(P, M/t_P(M)) = 0$.

Applying now the functor $\mathrm{Hom}_R(-, M/t_P(M))$ to the exact sequence (1) we get the exact sequence

$$0 \to \mathrm{Hom}_R(P'', M/t_P(M)) \to \mathrm{Hom}_R(P', M/t_P(M)) \to$$
$$\to \mathrm{Hom}_R(R, M/t_P(M)) \to \mathrm{Ext}^1_R(P'', M/t_P(M)) .$$

As $\mathrm{Hom}_R(P, M/t_P(M)) = 0 = \mathrm{Ext}^1_R(P, M/t_P(M))$ and as P' and P'' are direct sums of direct summands of P we get:

$$\mathrm{Hom}_R(P', M/t_P(M)) = 0 = \mathrm{Ext}^1_R(P'', M/t_P(M))$$

so that $\mathrm{Hom}_R(R, M/t_P(M)) = 0$ and hence $M = t_P(M) \in \mathrm{Gen}(P_R)$. Thus *a*) is proved.

Let now $M \in \mathrm{Gen}(P_R)$, $\Lambda = \mathrm{Hom}_R(P, M)$ and $f\colon P^{(\Lambda)} \to M$ the codiagonal map of the morphisms λ's.

Then $\mathrm{Im}(f) = t_P(M) = M$ as $M \in \mathrm{Gen}(P_R)$.

Consider now the exact sequence

$$(3) \qquad 0 \to M' \to P^{(\Lambda)} \xrightarrow{f} M \to 0 .$$

By applying $\mathrm{Hom}_R(P_R, -)$ to it, we get the exact sequence

$$(4) \qquad 0 \to \mathrm{Hom}_R(P_R, M') \to \mathrm{Hom}_R(P_R, P^{(\Lambda)}) \xrightarrow{f_*} \mathrm{Hom}_R(P_R, M) \to$$
$$\to \mathrm{Ext}^1_R(P, M') \to \mathrm{Ext}^1_R(P, P^{(\Lambda)}) = 0$$

where $f_* = \mathrm{Hom}_R(P, f)$. As f is the codiagonal map of the morphisms λ's, $\lambda \in \Lambda$, f_* is surjective and hence $\mathrm{Ext}^1_R(P, M') = 0$ so that $M' \in \mathrm{Gen}(P_R)$. Thus (3) is a sequence in $\mathrm{Gen}(P_R)$. Applying now the functor $- \otimes_A P$ to the exact sequence (4) we get:

$$\begin{array}{ccccccccc}
0 \to \mathrm{Tor}_1^A(H(M), P) & \to & H(M') \otimes_A P & \to & H(P^{(\Lambda)}) \otimes_A P & \to & H(M) \otimes_A P & \to & 0 \\
 & & \downarrow{\scriptstyle \varrho_{M'}} & & \downarrow{\scriptstyle \varrho_{P^{(\Lambda)}}} & & \downarrow{\scriptstyle \varrho_M} & & \\
0 & \to & M' & \to & P^{(\Lambda)} & \to & M & \to & 0
\end{array}$$

where $\varrho_{M'}$, ϱ_M are surjective and $\varrho_{P^{(A)}}$ is an isomorphism. Thus ϱ_M is an isomorphism for every $M \in \text{Gen}(P_R)$. Note that as $M' \in \text{Gen}(P_R)\varrho_{M'}$ is an isomorphism too and hence $\text{Tor}_1^A(H(M), P) = 0$ so that $H(M) \in \mathfrak{D}_A$. On the other hand recall that $N \otimes_A P \in \text{Gen}(P_R)$ for every $N \in \text{Mod-}A$.

Applying now the functor $\text{Hom}_R(-, P)$ to the exact sequence (1) we get

$$0 \to \text{Hom}_R(P'', P) \to \text{Hom}_R(P', P) \to \text{Hom}_R(R, P) \to \text{Ext}_R^1(P'', P) = 0 .$$

Thus ${}_AP \cong \text{Hom}_R(R, P)$ is a left A-module of projective dimension $\leqslant 1$ as $\text{Hom}_R(P', P)$ and $\text{Hom}_R(P'', P)$ are direct summands of free modules. In particular $\text{Tor}_2^A(L_A, {}_AP) = 0$ for every $L_A \in \text{Mod-}A$. Let now $L \in \mathfrak{D}_A$ and consider the injection

$$0 \to L' \to L$$

in Mod-A. By applying $\text{Tor}_1^A(-, P)$ we get the exact sequence

$$0 = \text{Tor}_2^A(L/L', P) \to \text{Tor}_1^A(L', P) \to \text{Tor}_1^A(L, P) = 0 .$$

Thus $L' \in \mathfrak{D}_A$ and therefore $\mathfrak{D}_A$ is closed under submodules. Let now $L \in \mathfrak{D}_A$ and consider the exact sequence

$$0 \to L' \to A^{(X)} \to L \to 0 . \tag{5}$$

Clearly $A \in \mathfrak{D}_A$ so that (5) is an exact sequence in $\mathfrak{D}_A$.

Applying T we get the exact sequence

$$0 = \text{Tor}_1^A(P, L) \to T(L') \to T(A^{(X)}) \to T(L) \to 0 .$$

Applying H to this sequence we get the exact sequence

$$0 \to HT(L') \to HT(A^{(X)}) \to HT(L) \to \text{Ext}_R^1(P, T(L')) = 0$$

as $T(L') \in \text{Gen}(P_R)$.

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L' & \longrightarrow & A^{(X)} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_{L'}} & & \downarrow{\scriptstyle \sigma_{A^{(X)}}} & & \downarrow{\scriptstyle \sigma_L} & & \\
0 & \longrightarrow & HT(L') & \longrightarrow & HT(A^{(X)}) & \longrightarrow & HT(L) & \longrightarrow & 0
\end{array}$$

Since $\sigma_{A(x)}$ is an isomorphism, σ_L is surjective. Thus, as $L' \in \mathfrak{D}_A$, $\sigma_{L'}$ is surjective too. Therefore σ_L is an isomorphism and hence *b*), and *c*) are proved.

The last assertion follows from Theorem 3.1.

4.4. PROPOSITION. *Let R be a ring, P_R a w-tilting module,* $A = \mathrm{End}\,(P_R)$, *Q_R an arbitrary cogenerator of* Mod-R, $K_A = \mathrm{Hom}_R\,(P_R, Q_R)$. *Then the following statements are equivalent*:

(*a*) $\mathrm{Gen}\,(P_R) = \overline{\mathrm{Gen}}\,(P_R)$.

(*b*) $\mathrm{Gen}\,(P_R) = \text{Mod-}R$.

(*c*) *P_R is Σ-quasi-projective.*

(*d*) *P_R is projective.*

(*e*) $\mathfrak{D}(K_A) = \text{Mod-}A$.

PROOF. (*a*) $\Leftrightarrow$ (*c*) $\Leftrightarrow$ (*e*). Follow by Theorem 4.3 and Proposition 3.2.

(*a*) $\Leftrightarrow$ (*b*). Follows in view of 4) of the definition of w-tilting module.

(*c*) $\Rightarrow$ (*d*) As P_R is Σ-quasi-projective it is a projective object of $\mathrm{Gen}\,(P_R)$. Since (*c*) $\Rightarrow$ (*a*) $\Rightarrow$ (*b*), $\mathrm{Gen}\,(P_R) = \text{Mod-}R$ so that P_R is projective.

(*d*) $\Rightarrow$ (*c*) Is trivial.

4.5. CONCLUSION. Let P_R be a w-tilting non projective module (for an example see [HR₁], pp. 126-127), $A = \mathrm{End}\,(P_R)$, Q_R an arbitrary cogenerator of Mod-R, $K_A = \mathrm{Hom}_R\,(P_R, Q_R)$. Then by Theorem 4.3 P_R gives rise to an equivalence between $\mathrm{Gen}\,(P_R)$ and $\mathfrak{D}_A$ which fulfils assumptions *a*), *b*), *c*) of Theorem 3.1 but, in view of Proposition 4.4, $\mathfrak{D}(K_A) = \mathfrak{D}_A \neq \text{Mod-}A$.

5. Quasi-progenerators.

5.1. In this section, under the assumptions of Theorem 3.1, we determine a number of sufficient conditions in order that $\mathfrak{D}(K_A) = \text{Mod-}A$.

5.2. Proposition. *Let* $P_R \in$ Mod-R, $A = \mathrm{End}\,(P_R)$. *The following conditions are equivalent*:

(*a*) *For every* $n \in \mathbb{N}$, P_R *generates all submodules of* P_R^n.

(*b*) $\mathrm{Gen}\,(P_R) = \overline{\mathrm{Gen}}\,(P_R)$.

(*c*) *For every* $M \in \overline{\mathrm{Gen}}\,(P_R)$, $M \simeq \mathrm{Hom}_R\,(P_R, M) \otimes_A P$ *canonically.*

(*d*) ${}_AP$ *is flat and for every* $M \in \mathrm{Gen}\,(P_R)$, $M \simeq \mathrm{Hom}_R\,(P_R, M) \otimes_A P$ *canonically.*

If these conditions are fulfilled, then the canonical image of R *in* $\mathrm{End}\,(P_R)$ *is dense in* $\mathrm{End}\,({}_AP)$ *endowed with its finite topology.*

Proof. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (d)$ are due to Zimmerman-Huisgen (cf. [ZH], Lemma 1.4).

The equivalence between (*b*) and (*c*) has been noted by Sato (cf. [S], Lemma 2.2). The last statement is due to Fuller (cf. [F], Lemma 1.3).

For another proof of this proposition see [MO], Proposition 5.5 •

5.3. Remark. It could happen that for every $M \in \mathrm{Gen}\,(P_R)$ $M \simeq \mathrm{Hom}_R\,(P_R, M) \otimes_A P$ canonically, but this is not true for every $M \in \overline{\mathrm{Gen}}\,(P_R)$.

In fact looking at Example 3.6 we have, for every $M \in \mathrm{Gen}\,(\mathbb{Z}(p^\infty)_\mathbb{Z})$ the required canonical isomorphism. On the other hand the cyclic group $\mathbb{Z}(p)$ of order p belongs to $\overline{\mathrm{Gen}}\,(\mathbb{Z}(p^\infty)_\mathbb{Z})$, but $\mathrm{Hom}_\mathbb{Z}\,(\mathbb{Z}(p^\infty), \mathbb{Z}(p)) = 0$.

5.4. Theorem. *Let* ${}_AP_R$ *be a bimodule and assume that* ${}_AP_R$ *induces an equivalence between* $\mathfrak{D}(K_A)$ *and* $\mathrm{Gen}\,(P_R)$. *Then* $A = \mathrm{End}\,(P_R)$ *and the following conditions are equivalent*:

(*a*) $\mathfrak{D}(K_A) = $ Mod-A.

(*b*) $\mathrm{Gen}\,(P_R) = \overline{\mathrm{Gen}}\,(P_R)$.

(*c*) P_R *is* Σ*-quasi-projective.*

(*d*) P_R *is quasi-projective and finitely generated.*

(*e*) P_R *is quasi-projective.*

(*f*) ${}_AP$ *is flat* ($\Leftrightarrow {}_AK$ *is injective*).

(*g*) ${}_AP$ *is a weak generator.*

PROOF. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (g)$ follow from Proposition 3.2.

$(b) \Leftrightarrow (f)$ Follows from Proposition 5.2 in view of Proposition 2.4.

$(c) \Leftrightarrow (d)$ Follows by Proposition 3.4 from Proposition 8 in [A].

$(d) \Rightarrow (e)$ Is trivial.

$(e) \Rightarrow (f)$ Follows from Proposition 4 in [A] in view of Proposition 3.7.

5.5. DEFINITION (Fuller [F]). A module $P_R \in \text{Mod-}R$ is called a *quasi-progenerator* if P_R is quasi-projective, finitely generated and generates all its submodules.

We are now ready to prove the concluding theorem of this section.

5.6. THEOREM. *Let ${}_AP_R$ be a bimodule. The following conditions are equivalent*:

(*a*) ${}_AP_R$ *induces an equivalence between* $\mathcal{D}(K_A)$ *and* $\overline{\text{Gen}}\,(P_R)$.

(*b*) ${}_AP_R$ *induces an equivalence between* Mod-A *and* Gen (P_R).

(*c*) P_R *is a quasi-progenerator and* $A = \text{End}\,(P_R)$.

(*d*) $A = \text{End}\,(P_R)$, P_R *is quasi-projective and generates all its submodules*, ${}_AP$ *is faithfully flat.*

If these conditions hold then:

1) $\text{Gen}\,(P_R) = \overline{\text{Gen}}\,(P_R)$ *and* K_A *is an injective cogenerator of* Mod-A.

2) *The canonical image of* R *in* End $({}_AP)$ *is dense whenever* End $({}_AP)$ *is endowed with its finite topology.*

PROOF. $(a) \Leftrightarrow (b)$ By Theorem 5.4 and Proposition 2.2.

$(b) \Rightarrow (c)$ By Theorem 5.4 P_R is quasi-projective and finitely generated and $\text{Gen}\,(P_R) = \overline{\text{Gen}}\,(P_R)$. Then P_R generates all its submodules.

$(b) \Rightarrow (d)$ Since $A \in \text{Im}\,(H)$, $A = \text{End}\,(P_R)$. By Theorem 5.4 ${}_AP$ is flat and a weak generator. Thus ${}_AP$ is faithfully flat.

$(d) \Rightarrow (a)$ By Lemma 2.2 of [F] $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}}(P_R)$. Hence by Proposition 5.2, for every $M \in \operatorname{Gen}(P_R)$, $M \simeq TH(M)$ canonically. Let $L \in \text{Mod-}A$. Since $T(L) \in \operatorname{Gen}(P_R)$ we have $TH(T(L)) \simeq T(L)$. It follows $T(HT(L)) \simeq T(L)$ hence $HT(L) \simeq L$ as ${}_AP$ is faithfully flat.

$(c) \Rightarrow (d)$ By Lemma 2.2 of [F] $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}}(P_R)$ so that, by Proposition 5.2, ${}_AP$ is flat. It is well known that ${}_AP$ is faithfully flat if and only if ${}_AP$ is flat and moreover for every right ideal I of A such that $IP = P$ we have $I = A$. Thus assume $IP = P$. As P_R is finitely generated there exist finitely many endomorphisms $a_1, \ldots, a_n \in I$ such that $\sum_{i=1}^{n} a_i P = P_R$. This yields an epimorphism $P_R^n \xrightarrow{\Sigma a_i} P_R \to 0$. Since P_R is quasi-projective this epimorphism splits so that there exists a morphism $\beta\colon P_R \to P_R^n$ such that $\beta \circ \sum_{i=1}^{n} a_i = 1$. We have $\beta = (b_1, \ldots, b_n)$ where $b_i \in A$. Thus $1 = \sum_{i=1}^{n} b_i a_i \in I$ so that $I = A$.

Suppose now that the conditions above hold.

1) We know that $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}}(P_R)$.

Moreover K_A is injective as ${}_AP$ is flat (recall that $K_A = \operatorname{Hom}_R(P_R, Q_R)$ where Q_R is an injective cogenerator of Mod-R) and K_A is a cogenerator as $\text{Mod-}A = \operatorname{Im}(H) \subseteq \mathfrak{D}(K_A)$.

2) Follows from Proposition 5.2.

5.7. Remark. The equivalences $(b) \Leftrightarrow (c) \Leftrightarrow (d)$ and statements $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}}(P_R)$ in 1) and 2) of 5.6 are due to Fuller (see [F] Theorem 2.6).

The equivalence $(a) \Leftrightarrow (b)$ is due to E. Gregorio ([G], Theorem 1.11).

5.8. Remark. Let P_R be a quasi-progenerator and $A = \operatorname{End}(P_R)$. Then ${}_AP_R$ induces an equivalence between $\mathfrak{D}(K_A)$ and $\operatorname{Gen}(P_R)$. While $\mathfrak{D}(K_A) = \text{Mod-}A$ in general $\operatorname{Gen}(P_R) \neq \text{Mod-}R$ as the following example shows.

Let R be a right primitive ring, P_R a faithful simple right R-module, $A = \operatorname{End}(P_R)$. Since P_R is a quasi-progenerator, ${}_AP_R$ induces an equivalence between Mod-A and $\operatorname{Gen}(P_R)$.

Mod-A is the category of right vector spaces over the division ring A and $\operatorname{Gen}(P_R)$ is the category of all semisimple modules of the form $P_R^{(X)}$. Thus, if R is not right artinian, $\operatorname{Gen}(P_R) \neq \text{Mod-}R$.

6. Equivalences and dualities.

6.1. Denote by R-CM the category of all compact Hausdorff left modules over the discrete ring R and let $\mathbb{T}$ be the topological quotient of the real field $\mathbb{R}$, endowed with the usual topology, modulo the group $\mathbb{Z}$ of rational integers. For every $N \in R$-CM, let $\Gamma_1(N) = \mathrm{Chom}_{\mathbb{Z}}(N, \mathbb{T})$ be the set of all continuous morphisms of abelian groups from N into $\mathbb{T}$.

$\Gamma_1(N)$ is an abelian group which has a natural structure of right R-module defined by setting

$$(\xi r)(x) = \xi(rx) \qquad (\xi \in \Gamma_1(N)\,,\ r \in R\,,\ x \in N)\,.$$

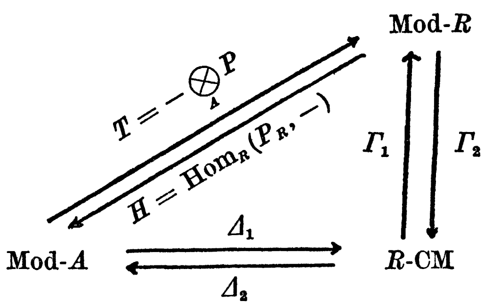
For every $M \in \text{Mod-}R$ let $\Gamma_2(M)$ be the left R-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{T})$ endowed with the topology of pointwise convergence.

It is well known that $\Gamma_2(M)$ is a compact group so that $\Gamma_2(M) \in R$-CM. In this way we obtain the contravariant functors

$$\Gamma_1\colon R\text{-CM} \to \text{Mod-}R \quad \text{and} \quad \Gamma_2\colon \text{Mod-}R \to R\text{-CM}$$

which coincide with the usual Pontryagin Duality functors when $R = \mathbb{Z}$. Clearly, from Pontryagin's classical results, we have that $\Gamma_2 \circ \Gamma_1 \approx 1_{R\text{-CM}}$ and $\Gamma_1 \circ \Gamma_2 \approx 1_{\text{Mod-}R}$: (Γ_1, Γ_2) is called the Pontryagin Duality over R. For more details about this duality see [MO].

6.2. Let ${}_AP_R$ be a bimodule with $A = \mathrm{End}(P_R)$. Consider the following commutative diagram of categories and functors:



where $\Delta_1 = \Gamma_2 \circ T$ and $\Delta_2 = H \circ \Gamma_1$. Set ${}_RK_A = \Gamma_2({}_AP_R)$ and for every

$N, N' \in R$-CM let $\mathrm{Chom}_R(N, N')$ be the group of all continuous R-morphisms from N into N'.

Clearly $A \cong \mathrm{Chom}_R({}_RK, {}_RK)$ canonically.

Denote by $\mathcal{C}({}_RK)$ the category of all topological modules M over the discrete ring R which are topologically isomorphic to closed submodules of topological powers of ${}_RK$.

As ${}_RK = \Gamma_2(P_R) \in R$-CM we have that $\mathcal{C}({}_RK) \subseteq R$-CM.

Let $L \in$ Mod-A. $N \in R$-CM. We have the canonical isomorphisms in R-CM and Mod-A respectively,

$$\Delta_1(L) = \Gamma_2\big(L \underset{A}{\otimes} P\big) = \mathrm{Hom}_{\mathbf{Z}}\big(L \otimes P, \mathrm{T}\big) \cong$$

$$\cong \mathrm{Hom}_A(L, \mathrm{Hom}_{\mathbf{Z}}(P, T)) \cong \mathrm{Hom}_A(L, K_A)$$

where $\mathrm{Hom}_A(L, K_A)$ is regarded as a topological submodule of the topological product ${}_RK^L$. Note that, since ${}_RK$ is compact and $\mathrm{Hom}_A(L, K_A)$ is closed in ${}_RK^L$, it follows that $\mathrm{Hom}_A(L, K_A)$ is compact. Clearly the topologically isomorphic modules $\Gamma_2(L \otimes P)$ and $\mathrm{Hom}_A(L, K_A)$ have the same characters.

$$\Delta_2(N) = H(\Gamma_1(N)) = \mathrm{Hom}_R(P_R, \Gamma_1(N)) \cong$$

$$\cong \mathrm{Chom}_R(\Gamma_2\Gamma_1(N), \Gamma_2(P_R)) \cong \mathrm{Chom}_R(N, {}_RK)\,.$$

Thus $\Delta_1(L) \in \mathcal{C}({}_RK)$ while $\Delta_2(N) \in \mathfrak{D}(K_A)$.

Let

$$\omega_L\colon L \to \mathrm{Chom}_R\big(\mathrm{Hom}_A(L, K_A), {}_RK\big)$$

and

$$\omega_N\colon N \to \mathrm{Hom}_A\big(\mathrm{Chom}_R(N, {}_RK), K_A\big)$$

be the canonical morphisms:

$$(\xi)\,\omega_L(x) = \xi(x)\,, \qquad \xi \in \mathrm{Hom}_A(L, K_A)\,, \qquad x \in L$$

and

$$\big((y)\,\omega_N\big)(\eta) = (y)\,\eta\,, \qquad \eta \in \mathrm{Chom}_R(N, {}_RK)\,, \qquad y \in N\,.$$

Then, by means of the Pontryagin duality over R, ω_L and ω_N correspond respectively to the canonical morphisms σ_L and $\varrho_{\Gamma_1(N)}$.

Indeed let $L \in$ Mod-A and consider the diagram:

$$L \xrightarrow{\sigma_L} \mathrm{Hom}_R\big(P, L \underset{A}{\otimes} P\big) \xrightarrow[\cong]{\varphi_L} \mathrm{Chom}_R\big(\Gamma_2(L \otimes P), \Gamma_2(P)\big) \xrightarrow[\cong]{\psi_L} \Delta_2\,\Delta_1(L)$$

where φ_L associates to every $\xi \in \mathrm{Hom}_R\left(P, L \underset{A}{\otimes} P\right)$ its transposed by Γ_2 and $\psi_L = \mathrm{Chom}_R(\xi_L^{-1}, 1_{\Gamma_2(P)})$ where

$$\xi_L\colon \Gamma_2\left(L \underset{A}{\otimes} P\right) =$$

$$= \mathrm{Hom}\left(L \underset{A}{\otimes} P, \mathbf{T}\right) \xrightarrow{\sim} \mathrm{Hom}_A(L, \mathrm{Hom}_{\mathbf{Z}}({}_AP, \mathbf{T})) = \mathrm{Hom}_A(L, K_A)$$

is the natural isomorphism. Set $\gamma_L = \psi_L \circ \varphi_L \circ \sigma_L$. We want to show that for every $\chi\colon L \to K_A$ we have

$$(1) \qquad (\chi)(\gamma_L(l)) = \chi(l) \qquad \text{for every } l \in L\,.$$

Let $\bar{\chi} = (\chi)\xi_L^{-1} \in \Gamma_2\left(L \underset{A}{\otimes} P\right)$. We have

$$\bar{\chi}(l \otimes p) = \chi(l)(p) \qquad \text{for every } l \in L\,,\ p \in P\,.$$

Therefore

$$(\bar{\chi} \circ \sigma_L(l))(p) = \bar{\chi}(l \otimes p) = \chi(l)(p) \qquad l \in L\,,\ p \in P$$

and we have $\bar{\chi} \circ \sigma_L(l) = \chi(l)$ for every $l \in L$ so that

$$(\chi)(\gamma_L(l)) = (\chi)[(\psi_L \circ \varphi_L \circ \sigma_L)(l)] = (\bar{\chi})[(\varphi_L \circ \sigma_L)(l)] = \bar{\chi} \circ \sigma_L(l) = \chi(l)\,.$$

Thus (1) is proved.

For $N \in R$-CM the correspondence between $\varrho_{\Gamma_1(N)}$ and ω_N is proved by an adjointness argument.

6.3. Corollary. *Let* $L \in \mathrm{Mod}\text{-}A$ $(N \in R_L\mathrm{CM})$. *Then* ω: (ω_N) *is an isomorphism (a topological isomorphism) if and only if* σ_L $(\varrho_{\Gamma_1(N)})$ *is an isomorphism.*

6.4. From Theorem 3.1 and from 6.2 we easily obtain a theorem of representations for dualities.

6.5. Theorem. *Let* R, A *be two rings,* $\mathfrak{D}_A$ *a subcategory of* Mod-A, ${}_R\mathcal{C}$ *a subcategory of* R-CM. *Assume that*

a) $A_A \in \mathfrak{D}_A$ *and* $\mathfrak{D}_A$ *is closed under taking submodules.*

b) $_R\mathfrak{C}$ *is closed under taking closed submodules and topological products.*

c) A duality $H_1\colon \mathfrak{D}_A \to {}_R\mathfrak{C}$, $H_2\colon {}_R\mathfrak{C} \to \mathfrak{D}_A$ *is given with* H_1, H_2 *additive functors.*

Then there exists a bimodule $_RK_A$ *with the following properties:*

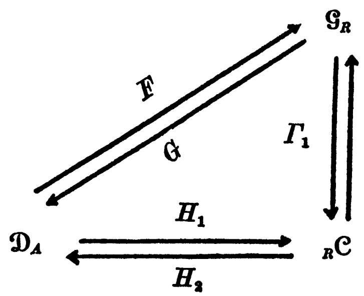
1) $_RK \in {}_R\mathfrak{C}$ *and* $A \simeq \mathrm{Chom}_R(K, K)$ *canonically.*

2) $\mathfrak{D}_A = \mathfrak{D}(K_A)$, $_R\mathfrak{C} = \mathfrak{C}(_LK)$ *where* $\mathfrak{C}(_RK)$ *consists of all compact modules which are closed submodules of topological powers of* $_RK$.

3) $H_1 \approx \Delta_1$, $H_2 \approx \Delta_2$.

4) *For every* $L \in \mathfrak{D}_A$, ω_L *is an isomorphism and for every* $N \in {}_R\mathfrak{C}$. ω_N *is a topological isomorphism.*

PROOF. Consider the commutative diagram



where $F = \Gamma_2 \circ H_1$, $G = H_2 \circ \Gamma_1$ and $\mathfrak{G}_R = \Gamma_2(_R\mathfrak{C})$.

Then $\mathfrak{G}_R$ is closed under taking homomorphic images and arbitrary direct sums. Set $_AP_R = \Gamma_1(_RK_A)$. Clearly $A \simeq \mathrm{End}\,(P_R)$ canonically. Set $Q_R = \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{T})$. Then Q_R is an injective cogenerator of Mod-R. We have the canonical isomorphisms:

$$\mathrm{Hom}_R(P_R, Q_R) = \mathrm{Hom}_R(P_R, \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{T})) \simeq \mathrm{Hom}_{\mathbf{Z}}\big(P_R \otimes_R R, \mathbf{T}\big) =$$

$$= \mathrm{Hom}_{\mathbf{Z}}(P_R, \mathbf{T}) = \Gamma_2(_AP_R) = {}_RK_A\,.$$

At this point we can apply Theorem 3.1 getting:

$$\mathfrak{D}_A = \mathfrak{D}(K_A)\,, \qquad \mathfrak{G}_R = \mathrm{Gen}\,(P_R)\,,$$

hence

$$_R\mathcal{C} = \mathcal{C}(_RK)\,, \qquad F \approx \Big(- \underset{A}{\otimes} P\Big)|\mathcal{D}_A\,, \qquad G \approx \mathrm{Hom}_R\,(P_R, -)|\mathcal{G}_L$$

and for every $L \in \mathcal{D}_A$, $M \in \mathcal{G}_R$ σ_L and ϱ_M are isomorphisms. Then $H_1 \approx \Delta_1$, $H_2 \approx \Delta_2$ and statement 4) follows by Corollary 6.3.

6.6. Let $_RK \in R$-CM, $A = \mathrm{Chom}_R\,(K, K)$. Assume that the couple of functors (Δ_1, Δ_2) induces a duality between $\mathcal{D}(K_A)$ and $\mathcal{C}(_RK)$. We say that this duality is *good* if $\mathcal{C}(_RK)$ has the extension property of K-characters. This means that for every $N \in \mathcal{C}(_RK)$ and for every (closed) submodule $N' \subseteq N$ every continuous morphism of N' in $_RK$ extends to a continuous morphism of N in $_RK$.

In [MO] it was proved that the considered duality is a good duality if and only if $\mathcal{D}(K_A) = \text{Mod-}A$.

The results of Section 4 solve an old problem of us: namely there exists dualities between $\mathcal{D}(K_A)$ and $\mathcal{C}(_RK)$ which are not good dualities.

REFERENCES

[A] G. Azumaya, *Some aspects of Fuller's theorem*, in: *Module Theory* (C. Faith and S. Wiegend, eds.), Lecture Notes in Math., Vol. **700**, pp. 34-45, New York-Heidelberg-Berlin, 1979.

[F] K. R. Fuller, *Density and equivalence*, J. Alg., **29** (1974), pp. 528-550.

[G] E. Gregorio, *A theorem of equivalences between categories of modules with some applications*, Rend. Cir. Mat. Palermo, Sez. II, **33** (1984), pp. 305-318.

[H] D. K. Harrison, *Infinite abelian groups and homological methods*, Ann. Math., **69** (1959), pp. 366-391.

[HR$_1$] D. Happel - C. M. Ringel, *Construction of tilted algebras*, in: *Representations of Algebras*, (M. Auslander and E. Lluis, eds.), Lecture Notes in Math., Vol. **903**, pp. 125-144, New-York-Heidelberg-Berlin, 1981.

[HR$_2$] D. Happel - C. M. Ringel, *Tilted algebras*, T.A.M.S., **274** (1982), pp. 399-443.

[M] K. Morita, *Localization in categories of modules - I*, Math. Z., **114** (1982), pp. 121-144.

[MO] C. Menini - A. Orsatti, *Good dualities and strongly quasi-injective modules*, Ann. Mat. Pura Appl., **127** (1981), pp. 182-230.

[S] M. SATO, *On equivalences between module subcategories*, J. Alg., **59** (1979), pp. 412-420.

[ZH] B. ZIMMERMANN-HUISGEN, *Endomorphism rings of self-generators*, Pac. J. Math., **61** (1975), pp. 587-602.

Manoscritto pervenuto in redazione il 12 ottobre 1988.