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The Lazarsfeld-Rao Problem for Buchsbaum Curves.

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0. Introduction.

In recent years there has been a great deal of work on the subject of Buchsbaum curves (sometimes called arithmetically Buchsbaum curves). These are the most natural curves, from a cohomological point of view, after the arithmetically Cohen-Macaulay ones, and they are a natural generalization of those: see, for instance, [SV], [BM1], [BM2], [EF], [A1], [A2], [GM1], [GM2]. In particular, in the last four papers quoted above the homogeneous ideal of a Buchsbaum curve is deeply investigated.

In this paper we concentrate on some geometrical properties of these curves and on their even liaison classes, and in particular on that we have called the LR-property (see below). This property is crucial for achieving a geometrical description of the structure of an even liaison class, but to date only certain very special even liaison classes are known to possess the property. On the other hand, it has been conjectured that *every* even liaison class possesses it. The central result of this paper is that every even Buchsbaum liaison class possesses the LR-property.

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To accomplish this we have to study two different topics and the combine the results: 1) how to give a structure to an even liaison class of curves in $\mathbb{P}^3$; and 2) Buchsbaum curves in $\mathbb{P}^3$ and their ideals.

The former is the first part of a research program to study the structure of even liaison classes in general: the case of curves in $\mathbb{P}^3$ is treated here in full detail. An extension of these techniques to codimension-two subschemes of $\mathbb{P}^n$ is outlined in [BM3] and applications of this approach (such as specialization to stick-figures; see e.g. [HH], [Se], [Ci]) are contained in [BM4].

After this paper was finished, the authors received a paper by M. C. Chang ([CH]), where a totally different approach to the study of Buchsbaum subschemes in codimension two is given. It seems reasonable to expect that the part of the present paper related to minimal Buchsbaum curves can also be obtained from this point of view.

Fix an even liaison class $\mathfrak{L}$. Roughly speaking, we say that the LR-property holds if it is possible to deform every curve in $\mathfrak{L}$ to a fixed « minimal » curve in $\mathfrak{L}$ with an arithmetically Cohen-Macaulay tail suitably attached by means of standard procedures (« basic double links »). Hence this property allows one to describe the geometrical structure of the whole class in terms of the minimal curve.

If in $\mathfrak{L}$ there exists a curve Y satisfying $e(Y) \leqslant \alpha(Y) - 4$, then $\mathfrak{L}$ has the LR-property with Y playing the role of the « minimal » curve. (This is the theorem by Lazarsfeld and Rao [LR] motivating our definition). Our paper is a first attempt to show that this property holds for other liaison classes.

In § 1 we outline a general approach to the LR-problem, and give a number of computational tools which are valid in every liaison class and which we use subsequently in the Buchsbaum case.

In § 2 we turn to Buchsbaum curves. Using facts from several recent papers, we combine the approaches used in [BSV] and [GM2]. Our main goal is to prove a key technical result (Corollary 2.17) about a special type of Buchsbaum curve, but as a consequence we give a careful description of them. In particular, we also give a great deal of information about the minimal curves: for instance, combining Theorem 2.9 with Corollary 2.18 one can produce the graded Betti numbers of a minimal free resolution for such a curve.

In § 3 we prove that every Buchsbaum liaison class has the LR-property, using the approach of § 1. We show that « almost » every

Buchsbaum curve can be easily handled with this approach. The difficult special case is disposed of using the technical results of § 2 and the computational tools of § 1.

Throughout this paper k is an algebraically closed field of characteristic zero, and curves are assumed to be locally Cohen-Macaulay and equidimensional. We refer to [R1] and [LR] for standard facts about liaison of curves in $\mathbf{P}^3$.

The idea of studying the LR and related problems was suggested to the second author by J. Harris, and in some sense the inductive approach used here has its origin in those conversations. The results of § 2, central to this paper, could not possibly have been obtained had it not been for many conversation with A. V. Geramita. We are extremely grateful to both for their mathematical influence and for their constant encouragement and enthusiasm.

We also thank the Department of Mathematics of Trento University for its hospitality during the preparation of this paper.

1. The Lazarsfeld-Rao problem.

In this section we discuss the general problem for liaison classes of curves in $\mathbf{P}^3$ which is suggested by the paper [LR]. We outline an approach which we shall use in this paper to solve the problem for arithmetically Bichsbaum liaison classes, and which we hope will be useful to prove the result in general.

Recall that if Z is a subscheme of $\mathbf{P}^n$ corresponding to a homogeneous (saturated) ideal I of $k[X_0, \dots, X_n]$ then the *Hilbert function* of Z is $H(Z, t) = \dim (k[X_0, \dots, X_n]/I)_t$. Also, the *first difference* (also called *Castelnuovo's function* in [D]) is $\Delta H(Z, t) = H(Z, t) - H(Z, t-1)$. In the case where $Z \subset \mathbf{P}^2$ of codimension 2, and especially where Z is the hyperplane section of a curve C in $\mathbf{P}^3$, ΔH has of course been studied extensively and used with great success to give results about Z and C. Notice that knowing $H(Z, t)$ is equivalent to knowing the *numerical character* ([GP]) of Z, and that from ΔH we can reconstruct $H(Z, t)$.

Recall also that for a curve C in $\mathbf{P}^3$, the *Hartshorne-Rao module* $M(C)$ is defined by $M(C) = \bigoplus_{n \in \mathbf{Z}} H^1(\mathbf{P}^3, \mathfrak{J}_C(n))$. This is of fundamental importance in the theory of liaison of curves in $\mathbf{P}^3$, and it has been generalized to other dimensions (cf. [R1], [R2], [Sc], [M4]).

From now on we assume that curves are in $\mathbb{P}^3$. We begin with a key observation. Recall

THEOREM 1.1 ([B], Corollary 2.3 and the subsequent remark). *Let X and Y be curves with $h^0(\mathbb{P}^3, \mathfrak{J}_X(n)) = h^0(\mathbb{P}^3, \mathfrak{J}_Y(n))$ for all n, and assume that $M(X) \cong M(Y)$ (as graded modules, with the same grading). The X and Y have the same degree d and arithmetic genus g, they are in the same irreducible component of* $\mathrm{Hilb}_{d,g}\,\mathbb{P}^3_k$, *and there is a deformation from one to the other through curves all having the same Hartshorne-Rao module.* $\square$

DEFINITION 1.2. Let X and Y be curves. We say that X and Y *have the same cohomology* if they satisfy the hypotheses of Theorem 1.1. Notice that this implies that they are in the same even liaison class.

COROLLARY 1.3. *Let X and Y be curves and let H be a hyperplane such that* $\dim(X \cap H) = \dim(Y \cap H) = 0$. *If $X \cap H$ and $Y \cap H$ have the same Hilbert function and $M(X) \cong M(Y)$ as above, then X and Y have the same cohomology (and hence the conclusion of Theorem* 1.1 *holds).*

PROOF. Let L be a linear form whose vanishing gives H. Let $\{\varphi_n(L)\colon M(X)_n \to M(X)_{n+1}\}$ be the collection of homomorphisms induced by L on $M(X)$. For any n we have the exact sequence

$$0 \to H^0(\mathfrak{J}_X(n)) \xrightarrow{\times L} H_0(\mathfrak{J}_X(n+1)) \to$$
$$\to H^0(\mathfrak{J}_{X\cap H}(n+1)) \to M(X)_n \xrightarrow{\varphi_n(L)} M(X)_{n+1}\,.$$

Let K_n be the kernel of $\varphi_n(L)$. Since $M(X) \cong M(Y)$, we have a similar exact sequence for Y, with the corresponding kernel of the same dimension as that for K_n, for each n. Abusively, we may thus write

$$0 \to H^0(\mathfrak{J}_X(n)) \xrightarrow{\times L} H^0(\mathfrak{J}_X(n+1)) \to H^0(\mathfrak{J}_{X\cap H}(n+1)) \to K_n \to 0,$$
$$0 \to H^0(\mathfrak{J}_Y(n)) \xrightarrow{\times L} H^0(\mathfrak{J}_Y(n+1)) \to H^0(\mathfrak{J}_{Y\cap H}(n+1)) \to K_n \to 0.$$

For every n we have $h^0(\mathfrak{J}_{X\cap H}(n+1) = h^0(\mathfrak{J}_{Y\cap H}(n+1))$, by hypothesis, and for small n the first two terms are 0. Therefore $h^0(\mathfrak{J}_X(n)) = h^0(\mathfrak{J}_Y(n))$ for all n, so we may apply Theorem 1.1. $\square$

Before giving the key definition of this paper we need to recall a particular way for constructing curves in a liaison class. Take a curve X in $\mathbf{P}^3$, a surface F containing X and a surface S meeting F properly. Choose a sufficiently general surface (of large degree) G passing through X and link X to $X^\#$ by using F and G, and then $X^\#$ to Y by using F and $S\cdot G$. Y does not depend on G, and set-theoretically we have $Y = X \cup (F \cap S)$. As homogeneous ideals, we have $I_Y = S\cdot I_X + (F)$. This procedure is called *basic double linkage* (cf. [LR]). We shall call Y a *basic double link* of X.

Recall also that given any graded module M of finite length, any sufficiently large leftward shift of M cannot be the Hartshorne-Rao module of any curve (cf. [Sw] or [M2]). Hence it makes sense to talk about the leftmost shift of M that is the Hartshorne-Rao module of some curve. The set of curves in the associated even liaison class $\mathfrak{L}$ which corresponds to this leftmost shift is denoted by $\mathfrak{L}^0$, and those corresponding to subsequent shifts by $\mathfrak{L}^1$, $\mathfrak{L}^2$, etc.

DEFINITION 1.4. Let $\mathfrak{L}$ be an even liaison class. We say that $\mathfrak{L}$ has the *LR-property* in the following conditions hold:

i) If X and Y are curves in $\mathfrak{L}^0$ then X and Y have the same cohomology (in the sense of Definition 1.2).

ii) Given a curve X in $\mathfrak{L}^0$ and a curve Y in $\mathfrak{L}^h$, $h \geqslant 1$, then there exists a sequence of curves $X = X_1, X_2, \ldots, X_m$ such that Y is a deformation of X_m through curves with fixed Hartshorne-Rao module, and every X_i is obtained from X_{i-1} by a basic double linkage. □

Notice that the hypothesis in i) implies that every two curves in $\mathfrak{L}^0$ have the same degree and genus, and that they lie in the same irreducible component of the Hilbert scheme.

The historical motivation for this definition is the paper by Lazarsfeld and Rao [LR], where they prove the following theorem:

THEOREM 1.5 ([LR], 1.4). *Let $\mathfrak{L}$ be an even liaison class where there exists a curve X satisfying $e(X) \leqslant s(X) - 4$. Then $\mathfrak{L}$ has the LR-property.*

It is clear that when a liaison class has the LR-property, then a quite complete description of the structure of the liaison class can be given in terms of the curves lying in the minimal shift. A natural question then is the following: which even liaison classes have the *LR*-property?

CONJECTURE ([M1]). Every even liaison class has the LR-property.

As we have seen, the *LR* problem consists in showing that, at least cohomologically, the term «minimal» is uniquely defined for any even liaison class $\mathcal{L}$, and that for every non-minimal curve $C \in \mathcal{L}$ there is a curve Y obtained from a minimal curve by a sequence of basic double links, such that Y has the same cohomology as C. Our next step is to show that in fact all we need consider is basic double links using planes (i.e. linear forms).

LEMMA 1.6. *Let X be a curve, $F \in I_X$ and $G_1, G_2 \in k[X_0, ..., X_3]_d$. Suppose C_i is a curve obtained from X by a basic double link using F and G_i. Then*

a) C_1 and C_2 have the same cohomology.

b) Given any sequence of curves $X = X_0, X_1, ..., X_d$ such that X_j is obtained from X_{j-1} by a basic double link using the same F and a linear form L_j, it follows that X_d and C_i, $i = 1, 2$, have the same cohomology.

PROOF.

a) It follows from [BM4], Lemma 3.8.

b) Let C be either C_1 or C_2 and G be either G_1 or G_2. Note that $I_C = G \cdot I_X + (F)$. Now, $I_{X_1} = L_1 \cdot I_X + (F)$. Hence

$$I_{X_2} = L_2 \cdot [L_1 \cdot I_X + (F)] + (F) = L_2 L_1 \cdot I_X + (F) .$$

Continuing in this way, $I_{X_d} = (L_d \dots L_1) I_X + (F)$. That is, X_d can be obtained from X by a single basic double link using F and the surface $(L_d \dots L_1)$, which also has degree d. Then by *a*) we get that X_d and C have the same cohomolohy. □

LEMMA 1.7. *Let X_1, X_2 be curves with the same cohomology and let $F_1 \in I_{X_1}$, $F_2 \in I_{X_2}$ be forms of the same degree. Perform basic double links on each of X_1, X_2 using F_1, F_2 respectively and a linear form L (as above), producing curves Y_1, Y_2. Then Y_1 and Y_2 have the same cohomology.*

PROOF. Choose $G_1 \in I_{X_1}$, $G_2 \in I_{X_2}$ general forms of the same degree $\gg 0$. Perform the two links on each (as in the definition of basic double linkage, with L playing the role of S) which produce the curves Y_1, Y_2. Of course $M(Y_1) \cong M(Y_2)$. Note furthermore that our condition that two curves «have the same cohomology» also forces the equality $h^0(\omega_{X_1}(n)) = h^0(\omega_{X_2}(n))$ for all n. Therefore, since the corresponding complete intersections use surfaces of the same degrees, we get after two links for each that $h^0(\mathfrak{J}_{Y_1}(n)) = h^0(\mathfrak{J}_{Y_2}(n))$ for all n. □

REMARK 1.8. (*a*) The property that two curves X_1, X_2 have the same cohomology does not guarantee that any link possible for X_1 can also be done for X_2 with surfaces of the same degree. (See Remark 1.10 (*b*).) However, it does guarantee that $h^0(\mathfrak{J}_{X_1}(n)) = h^0(\mathfrak{J}_{X_2}(n))$ for all n, by definition, so we do not lose any generality in our hypothesis that $\deg F_1 = \deg F_2$. And since the G_i play no real role in the resulting curves Y_i, it is enough that we choose $\deg G_i \gg 0$.

(*b*) Of course, Lemma 1.7 is true if we replace L by a form of arbitrary degree, or if we use different forms (but of the same degree) for X_1 and X_2. □

We now come to our general plan of attack for the LR problem, which we shall use in the special case of arithmetically Buchsbaum curves in the next two sections.

THEOREM 1.9. *Let $\mathfrak{L}$ be an even liaison class with the following two properties:*

(1) *Any two curves $X, X' \in \mathfrak{L}^0$ have the same cohomology.*

(2) *For every curve $X \in \mathfrak{L}^h$ $(h > 0)$ there exists a curve $Y \in \mathfrak{L}^h$ with the same cohomology as that of X (possibly X itself), satisfying the following condition: There exist surfaces of some degrees a and b linking Y to a curve Y_1, and surfaces of degrees a and $b-1$ linking Y_1 to a curve $Y_2 \in \mathfrak{L}^{h-1}$.*

Then $\mathfrak{L}$ has the LR *property.*

PROOF. We only need to show that given any curve $X \in \mathfrak{L}^h$ $(h > 0)$ and any minimal curve X_0, there exists a sequence of curves

$X_0, X_1, \ldots, X_h$ such that for each i $X_i \in \mathfrak{L}^i$ is a basic double link of $X_{i-1} \in \mathfrak{L}^{i-1}$, and X_h has the same cohomology as that of X. The proof is by induction on h.

Let $X_0 \in \mathfrak{L}^0$, $X \in \mathfrak{L}^1$. Let Y be as in the statement of (2) and perform the indicated pair of links to a curve $Y_2 \in \mathfrak{L}^0$. Suppose the pairs of surfaces used in (2) are $F_1, G_1 \in I_Y$ and $F_2, G_2 \in I_{Y_2}$, where $\deg F_1 = \deg F_2 = a$ and $\deg G_1 = \deg G_2 + 1 = b$.

Now, let L be a general linear form and perform a pair of links starting with Y_2 using $F_2, G_2 \in I_{Y_2}$ and then $F_2, LG_2 \in I_{Y_1}$ to a curve $Z \in \mathfrak{L}^1$. Note that Z is a basic double link of Y_2. But by assumption (1) and Lemma 1.7, Z has the same cohomology as a similar basic double link of X_0. On the other hand, by considering the degrees of the surfaces used in the links, we see that Z has the same cohomology as that of Y, i.e. that of X.

The proof of the inductive step is identical to that for $h = 1$, simply with Y_2 being not a minimal curve but rather a curve having the same cohomology as a «basic double link curve». □

REMARK 1.10. (*a*) The underlying idea, that we can attack the LR problem by linking down to smaller curves as in Theorem 1.9 (2), was discussed in the last chapter of [M1] (although the results there are not nearly as strong as Theorem 1.9). This discussion, in turn, was motivated by work of Gaeta ([G1]).

Notice that to achieve the hypotheses of Theorem 1.9 it is enough to show the following:

(1) For any two curves $X, X' \in \mathfrak{L}^0$ there exists a sequence of pairs of links, beginning with X and ending with X', such that for each pair the degrees of the surfaces in the first link are the same as those in the second.

(2) For any $X \in \mathfrak{L}^h$ $(h > 0)$ there exists a similar sequence of pairs of links starting with X and ending with a curve Y as in Theorem 1.9.

We believe that these conditions are true in any even liaison class, and indeed that there exist minimal generators at each step which do the job. We do not know that this is true even in the case of arithmetically Buchsbaum curves, but examples where at least (1) holds are contained in [M2] and [M3].

(*b*) In Theorem 1.9 (2), it is necessary to bring in the curve Y. That is, the property that there exist surfaces of degrees a and b

which give a link is not preserved under deformation, even through curves with the same shift of the same Hartshorne-Rao module. For example, let Y be the union of a plane cubic and a line, meeting at one point. Let Y' be the complete intersection of two quadrics. Y and Y' are arithmetically Cohen-Macaulay, so the notion of « shift of the Hartshorne-Rao module » makes no sense. However, let Z be the disjoint union of two lines. Let C (resp. C') be the curves obtained by performing the liaison addition of Z and Y (resp. Z and Y'), using two (not necessarily irreducible) quadrics $F_1 \in I_Z$, $F_2 \in I_Y$ (resp. $G_1 \in I_Z$, $G_2 \in I_{Y'}$).

For a general hyperplane H, we can sketch $C \cap H$ and $C' \cap H$:

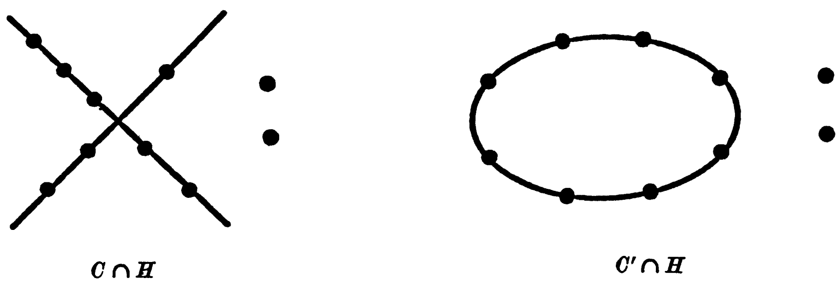


Figure 1

One can check that $C \cap H$ and $C' \cap H$ have the same Hilbert function. Also, $M(C) \cong M(C')$ by Liaison Addition. Hence by Corollary 1.3, C and C' have the same cohomology.

But $I_C = F_2 \cdot I_Z + F_1 \cdot I_Y$ and $I_{C'} = G_2 \cdot I_Z + G_1 \cdot I_{Y'}$. Now, $I_{C'}$ contains a link (i.e. a regular sequence) in degree 4 while I_C does not (since everything of degree 4 contains the plane of the cubic as a component). In fact, even the number ν of minimal generators is not preserved: C has 6 while C' has 5. □

In § 3 it will be important to understand how the function ΔH behaves when we perform a basic double link. In fact the behavior is quite simple, and we now derive the basic facts.

Let C be a curve contained in a surface Σ_b of degree b. Perform a basic double link by using Σ_b and a plane Π. Let Z be $\Sigma_b \cap \Pi$, and Y the curve whose ideal is defined by

$$I_Y = \Pi \cdot I_C + (\Sigma_b) .$$

Recall that

$$\Delta H(C \cap H, t) = t + 1 - h^0(H, \mathfrak{J}_{C \cap H}(t)) + h^0(H, \mathfrak{J}_{C \cap H}(t-1))$$

where H is a general plane.

By construction we have an exact sequence

$$(\#) \qquad 0 \to \mathcal{O}_C(-1) \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0 .$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{\mathbf{P}^3}(-1) & \to & \mathcal{O}_{\mathbf{P}^3} & \to & \mathcal{O}_\Pi & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_C(-1) & \to & \mathcal{O}_Y & \to & \mathcal{O}_Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

from which we get the exact sequence of the kernels:

$$0 \to \mathfrak{J}_C(-1) \to \mathfrak{J}_Y \to \mathfrak{J}_{Z,\Pi} \to 0 .$$

Now let H be a general plane, and tensor this sequence with $\mathcal{O}_H$, thus getting

$$(\#\#) \qquad 0 \to \mathfrak{J}_{C \cap H,H}(-1) \to \mathfrak{J}_{Y \cap H,H} \to \mathfrak{J}_{Z \cap H, \Pi \cap H} \to 0 ,$$

Notice that $L = H \cap \Pi$ is a line, and that $Z \cap H$ is a set of points on L; it is the set of points on L cut out by the curve $\Sigma_b \cap H \subset H$. Then from the exact sequence

$$0 \to \mathfrak{J}_{Z \cap H,H} \to \mathcal{O}_L \to \mathcal{O}_{Z \cap H} \to 0$$

we get

$$\mathfrak{J}_{Z \cap H, \Pi \cap H} = \mathcal{O}_L(-b) .$$

Hence (##) becomes

$$(\#\#') \qquad 0 \to \mathfrak{J}_{C \cap H,H}(-1) \to \mathfrak{J}_{Y \cap H,H} \to \mathcal{O}_L(-b) \to 0 .$$

Passing to the long exact cohomology sequence we have

$$0 \to H^0(H, \mathfrak{J}_{C\cap H,H}(t-1)) \to H^0(H, \mathfrak{J}_{Y\cap H,H}(t)) \to H^0(L, \mathcal{O}_L(t-b)) \to$$
$$\to H^1(H, \mathfrak{J}_{C\cap H,H}(t-1)) \to H^1(H, \mathfrak{J}_{Y\cap H,H}(t)) \to 0\,.$$

If $t < b$, then it follows that

$$h^0(H, \mathfrak{J}_{Y\cap H,H}(t)) = h_0(H, \mathfrak{J}_{C\cap H,H}(t-1))\,.$$

If $t \geqslant b$, the sequence gives

$$\begin{aligned} h^0(H, \mathfrak{J}_{Y\cap H,H}(t)) &= h_0(H, \mathfrak{J}_{C\cap H,H}(t-1)) + h_0(L, \mathcal{O}_L(t-b)) - \\ &\quad - h^1(H, \mathfrak{J}_{C\cap H,H}(t-1)) + h^1(H, \mathfrak{J}_{Y\cap H,H}(t)) = \\ &= t-b+1+h^0(H, \mathfrak{J}_{C\cap H,H}(t-1)) - \\ &\quad - h^1(H, \mathfrak{J}_{C\cap H,H}(t-1)) + h^1(H, \mathfrak{J}_{Y\cap H,H}(t))\,. \end{aligned}$$

Hence we must calculate the h^1's. From the exact sequences

$$0 \to \mathfrak{J}_Y(t-1) \to \mathfrak{J}_Y(t) \to \mathfrak{J}_{Y\cap H}(t) \to 0\,,$$
$$0 \to \mathfrak{J}_C(t-2) \to \mathfrak{J}_C(t-1) \to \mathfrak{J}_{C\cap H}(t-1) \to 0\,,$$

we have

$$0 \to K \to H^1(\mathbf{P}^3, \mathfrak{J}_Y(t)) \to H^1(H, \mathfrak{J}_{Y\cap H,H}(t)) \to$$
$$\to H^2(\mathbf{P}^3, \mathfrak{J}_Y(t-1)) \to H^2(\mathbf{P}^3, \mathfrak{J}_Y(t)) \to 0$$

and

$$0 \to M \to H^1(\mathbf{P}^3, \mathfrak{J}_C(t-1)) \to H^1(H, \mathfrak{J}_{C\cap H,H}(t-1)) \to$$
$$\to H^2(\mathbf{P}^3, \mathfrak{J}_C(t-2)) \to H^2(\mathbf{P}^3, \mathfrak{J}_C(t-1)) \to 0\,,$$

where

$$K = \operatorname{Im}\left(H^1(\mathbf{P}^3, \mathfrak{J}_Y(t-1)) \xrightarrow{\times H} H^1(\mathbf{P}^3, \mathfrak{J}_Y(t))\right)$$

and

$$M = \operatorname{Im}\left(H^1(\mathbf{P}^3, \mathfrak{J}_C(t-2)) \xrightarrow{\times H} H^1(\mathbf{P}^3, \mathfrak{J}_C(t-1))\right)\,.$$

Since C and Y are evently linked, then $\dim K = \dim M$. Hence we have

$$h^1(H, \mathfrak{J}_{Y\cap H,H}(t)) = h^2(\mathbf{P}^3, \mathfrak{J}_Y(t-1)) - h^2(\mathbf{P}^3, \mathfrak{J}_Y(t)) + h^1(\mathfrak{J}_Y(t)) - \dim K ,$$

$$h^1(H, \mathfrak{J}_{C\cap H,H}(t-1)) =$$
$$= h^2(\mathbf{P}^3, \mathfrak{J}_C(t-2)) - h^2(\mathbf{P}^3, \mathfrak{J}_C(t-1)) + h^1(\mathbf{P}^3\mathfrak{J}_C(t-1)) - \dim M .$$

On the other hand, by (#) we find (Z is a complete intersection)

$$0 \to H^1(C, \mathcal{O}_C(t-1)) \to H^1(Y, \mathcal{O}_Y(t)) \to H^1(Z, \mathcal{O}_Z(t)) \to 0$$

and hence, $\forall t$,

$$h^2(\mathbf{P}^3, \mathfrak{J}_Y(t)) = h^2(\mathbf{P}^3, \mathfrak{J}_C(t-1)) + h^2(\mathbf{P}^3, \mathfrak{J}_Z(t)) .$$

But it follows from a standard computation that

$$h^2(\mathbf{P}^3, \mathfrak{J}_Z(t)) = 0 \qquad \text{if } t \geqslant b-2 .$$

Hence, for $t \geqslant b-2$ we get

$$h^2(\mathbf{P}^3, \mathfrak{J}_Y(t)) = h^2(\mathbf{P}^3, \mathfrak{J}_C(t-1)) .$$

Moreover, $\forall t$, we have

$$h^1(\mathbf{P}^3, \mathfrak{J}_Y(t)) = h^1(\mathbf{P}^3, \mathfrak{J}_C(t-1)) ,$$

and thus, for $t \geqslant b-1$, we have

$$h^1(H, \mathfrak{J}_{Y\cap H,H}(t)) = h^1(H, \mathfrak{J}_{C\cap H,H}(t-1)) .$$

Hence, for $t \geqslant b-1$, we get

$$h_0(H, \mathfrak{J}_{Y\cap H,H}(t)) = h_0(H, \mathfrak{J}_{C\cap H,H}(t-1)) + (t-b+1) .$$

In this way we have proved the following

PROPOSITION 1.11. *Let C, b and Y as above. Then*

$$\Delta H(Y\cap H, t)=\begin{cases}\Delta H(C\cap H, t-1)+1 & \text{if } t<b\,,\\ \Delta H(C\cap H, t-1) & \text{if } t\geqslant b\,.\end{cases}\qquad\square$$

This proposition has an immediate corollary:

COROLLARY 1.12. *Let C be a curve, and perform a basic double link using a surface Σ_b of degree b containing X and a surface Σ_r of degree r meeting Σ_b properly. If Y_r is the curve thus obtained, then*

$$\Delta H(Y_r\cap H, t)=$$

$$=\begin{cases}\Delta H(C\cap H, t-r)+r & \text{if } t\leqslant b-1\,,\\ \Delta H(C\cap H, t-r)+b+r-1-t & \text{if } b\leqslant t\leqslant b+r-2\,,\\ \Delta H(C\cap H, t-r) & \text{if } t\geqslant b+r-1\,.\end{cases}$$

PROOF. Thanks to Lemma 1.6 (*b*), it is enough to consider a basic double link performed by using Σ_b and the union of r planes $\bigcup\limits^{r}\Pi_i=\Sigma_r$. Let us call $Y_0=C$ and $Y_i=Y_{i-1}\cup(\Sigma_b\cap\Pi_i)$ (this means that Y_i is obtained from Y_{i-1} by a basic double link using Σ_b and Π_i). But now it is enough to iterate r times the procedure of Proposition 1.11. $\square$

Later on it will be useful to handle the differences between two consecutive values, so we state it as a corollary:

COROLLARY 1.13. *Let C, Σ_b, Π and Y be as in Proposition* 1.11, *and let H be a general plane. Then* ($t\geqslant 2$):

$$\Delta H(Y\cap H, t-1)-\Delta H(Y\cap H, t)=$$

$$=\begin{cases}\Delta H(C\cap H, t-2)-\Delta H(C\cap H, t-1) & \text{if } t\neq b\,,\\ \Delta H(C\cap H, b-2)-\Delta H(C\cap H, b-1)+1 & \text{if } t=b\,.\end{cases}\qquad\square$$

Finally, we derive another useful computational tool. Recall that for a curve C, $e(C)=\max\{n|\ h^0(\omega_C(-n))\neq 0\}$.

LEMMA 1.14. *Let C be a curve, $F \in (I_C)_b$. Let Y be the result of a basic double link performed on C by F and a general plane*

(*a*) *If* $b \leqslant e(C) + 3$, *then* $e(Y) = e(C) + 1$.

(*b*) *If* $b \geqslant e(C) + 4$, *then* $e(Y) = b - 3$.

PROOF. Assume that C is linked to Y_0 by F and a form of degree a, and that Y_0 is linked to Y by F and a form of degree $a + 1$ (namely the original form of degree a times a linear form). Let X be the complete intersection linking C to Y_0 and X' that linking Y_0 to Y. Note that $h^0(\mathfrak{I}_X(t)) = h^0(\mathfrak{I}_{X'}(t))$ for $t \leqslant a - 1$. We have

$$0 \to \mathfrak{I}_X(t) \to \mathfrak{I}_{Y_0}(t) \to \omega_C(4 - a - b + t) \to 0\,,$$

$$0 \to \mathfrak{I}_{X'}(t) \to \mathfrak{I}_{Y_0}(t) \to \omega_Y(3 - a - b + t) \to 0\,.$$

Hence for $t \leqslant a - 1$, we have

$$h^0(\omega_C(4 - a - b + t)) = h^0(\omega_Y(3 - a - b + t))\,.$$

In particular,

$$h^0(\omega_C(3 - b)) = h^0(\omega_Y(2 - b)) \qquad \text{and} \qquad h^0(\omega_C(-k)) = h^0(\omega_Y(-k - 1))$$

for $k \geqslant b - 3$. This proves (*a*). For (*b*), set $t = a$:

$$h^0(\omega_C(3 - b)) = h^0(\mathfrak{I}_{Y_0}(a)) - h^0(\mathfrak{I}_{X'}(a)) > h^0(\mathfrak{I}_{Y_0}(a)) - h^0(\mathfrak{I}_X(a)) \geqslant 0\,. \qquad \square$$

2. Basic results on Buchsbaum curves.

In this section we first review a number of facts about Buchsbaum curves. Our main goal is a technical result (Proposition 2.16, Corollary 2.17), which will allow us in § 3 to handle the only difficult special case in the proof.

Recall the following standard definitions for a subscheme Z of $\mathbb{P}^n$ corresponding to a homogeneous (saturated) ideal I in $k[X_0, \ldots, X_n]$:

1) $\alpha(Z) = \min\{d\colon \exists F \in I,\ F \neq 0,\ \deg F = d\}$;

2) $\nu(Z) =$ number of minimal generators of I.

Now, a curve C in $\mathbb{P}^3$ is called *arithmetically Buchsbaum* (or simply *Buchsbaum*) if its Hartshorne-Rao module is annihilated by the maximal ideal $\mathfrak{m}$ of $k[X_0, \dots, X_3]$. While there has been a surge of activity on the subject of Buchsbaum curves recently, an important reference on these and on the broader subject of Buchsbaum rings is the beautiful book [SV]. Up to shift, the Hartshorne-Rao module of a Buchsbaum curve is determined by the dimensions of its graded components (as k-vector spaces). We will call the associated liaison class a *Buchsbaum liaison class.* This motivates the following definition:

DEFINITION 2.1 ([BM2]). Let $(n_1, \dots, n_r)$ be a sequence of non-negative integers, where $n_1 \neq 0$ and $n_r \neq 0$. Then $\boldsymbol{L}_{n_1 \dots n_r}$ is the even Buchsbaum liaison class associated to a graded module whose components have dimension $n_1, \dots, n_r$ respectively, and which is annihilated by $\mathfrak{m}$. If M is such a module then $\operatorname{diam} M = r$. The *Buchsbaum type* of $\boldsymbol{L}_{n_1 \dots n_r}$ is the integer $N = n_1 + \dots + n_r$. □

From the point of view of liaison, an important first step is to know exactly which shifts of the Hartshorne-Rao module can actually occur for curves in the even liaison class. In the case of Buchsbaum curves we have the following:

THEOREM 2.2. *Let* $C \in \boldsymbol{L}_{n_1 \dots n_r}$, $\alpha = \alpha(C)$.

- (a) *The Hilbert function* $H(C \cap H, t)$ *is independent of* H, *as long as* $\dim(C \cap H) = 0$.
- (b) $\alpha \geqslant 2N$.
- (c) $M(C)_t = 0$ *for* $t \leqslant \alpha - 3$.
- (d) $M(C)_{\alpha-2} \neq 0$ *if and only if* $\alpha(C \cap H) = \alpha - 1$ $\big($*and then* $h^0(\mathfrak{J}_{C \cap H}(\alpha - 1)) = n_1\big)$.
- (e) *Curves* C *for which* $\dim M(C)_{2N-2} = n_1$ *exist and can be constructed directly.*

PROOF. (a), (c), and (d) are from [GM1], (b) is from [A1] (and a new proof appears in [GM2]), while (e) is from [BM2] and uses Liaison Addition ([Sw]). □

As a result of Theorem 2.2, the leftmost possible non-zero component for the Hartshorne-Rao module of a Buchsbaum curve in

$\boldsymbol{L}_{n_1 \dots n_r}$ is in degree $2N-2$. Given $C \in \boldsymbol{L}_{n_1 \dots n_r}$, then, it is natural to measure the shift of $M(C)$ in terms of this extremal value, as discussed in § 1:

DEFINITION 2.3 ([BM2]). $\boldsymbol{L}^h_{n_1 \dots n_r}$ $(h \geqslant 0)$ is the subset of $\boldsymbol{L}_{n_1 \dots n_r}$ consisting of those curves C whose first non-zero component (which has dimension n_1) occurs in degree $2N-2+h$. □

REMARK 2.4. For $C \in \boldsymbol{L}^h_{n_1 \dots n_r}$ we have $\alpha(C) \leqslant 2N+h$ by Theorem 2.2 (*c*). Furthermore, $\alpha(C) = 2N+h$ if and only if C satisfies the condition of Theorem 2.2 (*d*). These curves play a special role in the theory of Buchsbaum curves, as we will see shortly (and as was noted in [GM1] and [BM2]). □

In [LR], an important role was played by the integer $e(C)$ (see § 1). In the case of Buchsbaum curves, we have certain limitations on e:

THEOREM 2.5. *Let* $C \in \boldsymbol{L}^h_{n_1 \dots n_r}$ $(h \geqslant 0)$.

(*a*) *If C lies on a surface of degree $2N$ then* $e(C) = 2N + h + r - 5$ *and C can be directly linked to a curve in* $\boldsymbol{L}^0_{n_1 \dots n_r}$.

(*b*) *In any case,* $e(C) \geqslant 2N + h + r - 5$.

(*c*) $\boldsymbol{L}_{n_1 \dots n_r}$ *contains a curve satisfying the hypothesis of Theorem* 1.5 *if and only if* $r = 1$.

PROOF. (*a*) and (*c*) are from [BM2] while (*b*) is from [GM2]. As we shall see, there are many possibilities for extremal curves for (*b*) in addition to those described in (*a*). The fact that $\boldsymbol{L}_n$ has the LR property was used heavily in [BM1]. □

An important question in general, and for us in particular, is to describe the degrees of the minimal generators of I_C, and the number $\nu(C)$. For special (and important!) curves we shall say more shortly, but for now we recall some results.

THEOREM 2.6. *Let* $C \in \boldsymbol{L}^h_{n_1 \dots n_r}$.

(*a*) $3N + 1 \leqslant \nu(C) \leqslant \alpha(C) + N + 1 \leqslant 3N + 1 + h$.

(*b*) *Let* $t = \max\{d \colon \Delta H(C \cap H) \neq 0\}$. *Then I_C is generated in degree* $\leqslant t + 1$.

PROOF. For (a), the first inequality is from [BSV] (and we shall examine their approach shortly) the second is obtained immediately from [A1] (and reproved in [GM2] by a different method), and the third is from [GM2] (and follows immediately from Remark 2.4). Finally, (b) is from [GM1]. □

COROLLARY 2.7 ([GM2]). *If $C \in \boldsymbol{L}_{n_1 \dots n_r}$ lies on a surface of degree $2N$ then $\nu(C) = 3N + 1$.* □

EXAMPLE 2.8. The converse to Corollary 2.7 is false. In fact, we now show how to produce a curve in $\boldsymbol{L}^h_{n_1 \dots n_1}$ (with h rather large) with $\alpha(C) = 2N + h$ (cf. Remark 2.4) and having exactly $3N + 1$ minimal generators *all in degree* $2N + h$.

Begin with $C \in \boldsymbol{L}^0_{n_1 \dots n_1}$, which has exactly $3N + 1$ minimal generators. We shall see (Corollary 2.18) that I_C has $3n_i$ minimal generators in degree $2N + i - 1$ (except for $i = 1$, where there are $3n_1 + 1$). Say that $I_C = (F_1, \dots, F_{3N+1})$. Perform a basic double link using a minimal generator F_k of degree $> 2N$, and a general plane H. The new curve C_1 has

$$I_{C_1} = H \cdot (F_1, \dots, F_{3N+1}) + (F_k) =$$
$$= (HF_1, \dots, HF_{k-1}, F_k, HF_{k+1}, \dots, HF_{3N+1}) .$$

Now $\alpha(C_1) = 2N + 1$, $\nu(C_1) = 3N + 1$, and the degree of each generator except F_k has increased by one. Repeating this often enough « collects » all the generators in the minimal degree. □

Our idea in this section is to use the above groundwork to compare two different approaches to the ideal of a Buchsbaum curve: that suggested by [BSV] and that of [GM2]. Accordingly, we begin with the former.

THEOREM 2.9 ([R1]). *Let C be a curve in $\mathbf{P}^3$. Assume that $M(C)$ has a minimal free resolution*

$$0 \to L_4 \xrightarrow{\sigma} L_3 \to L_2 \to L_1 \to L_0 \to M(C) \to 0 .$$

Then I_C has a minimal resolution of the form

$$0 \to L_4 \xrightarrow{(\sigma, 0)} L_3 \oplus \bigoplus_{i=1}^{t} S(-l_i) \to \bigoplus_{i=1}^{m} S(-e_i) \to I_C \to 0 . \qquad \square$$

In our case, following [BSV], we have the Koszul resolution of k, suitably tensored and shifted. This gives L_i $(0 \leqslant i \leqslant 4)$ as above, where L_i are free S-modules of rank $H \cdot \binom{4}{i}$: But now, let K be the cokernel of σ:

$$\begin{array}{ccccccccccccc} 0 & \to & L_4 & \xrightarrow{\sigma} & L_3 & \to & L_2 & \to & L_1 & \to & L_0 & \to M(C) \to 0 \\ & & & \searrow & & \nearrow & & & & & & & \\ & & & & K & & & & & & & & \\ & & & \nearrow & & \searrow & & & & & & & \\ & & 0 & & & & 0 & & & & & & \end{array}$$

Once we fix the shift h we know σ exactly, so we can compute all the cohomology of K (sheafified). Now, by Theorem 2.9 we have

$$(*) \qquad 0 \to K \oplus \bigoplus_{i=1}^{t} S(-l_i) \to \bigoplus_{i=1}^{3N+1+t} S(-e_i) \to I_C \to 0 .$$

As a special case, note that if $\nu(C) = 3N + 1$ (for example as in Corollary 2.7) then $t = 0$ and the kernel can be easily computed. The case $t > 0$ is somewhat harder since we must consider the l_i as well.

Ler $C \in L^h_{n_1 \dots n_r}$ and let us examine the corresponding K.

$$0 \to \begin{array}{c} n_1 S(-2N-2-h) \\ \oplus \\ n_2 S(-2N-3-h) \\ \oplus \\ \vdots \\ \oplus \\ n_r S(-2N-r-1-h) \end{array} \to \begin{array}{c} 4n_1 S(-2N-1-h) \\ \oplus \\ 4n_2 S(-2N-2-h) \\ \oplus \\ \vdots \\ \oplus \\ 4n_r S(-2N-r-h) \end{array} \to K \to 0 .$$

We conclude

LEMMA 2.10.

(*a*) $h^0(K(2N+h)) = 0$,

$h^0(K(2N+h+1)) = 4n_1$,

...

$h^0(K(2N+h+d)) =$

$$= 4n_1\binom{d+2}{3} + \dots + 4n_d - \left[n_1\binom{d+1}{3} + \dots + n_{d-1}\right].$$

(*b*) $h^1(K(t)) = 0$ *for all* t. □

LEMMA 2.11. *Let* $C \in L^h_{n_r \dots n_1}$.

(*a*) *If* $\alpha(C) = 2N + h$ *then* $l_i \geqslant 2N + h + 1$ *for all* i. (*See the sequence* (*).)

(*b*) *For any degree* d, *the size of the image in* $(I_C)_{d+1}$ *of* $(I_C)_d$ *is*

$\dim (I_C)_{d+1} -$ (number of minimal generators in degree $d + 1$) $=$

$$= \dim \left[\bigoplus S(-e_i) \colon e_i \leqslant d\right]_{d+1} - \dim \left[K \bigoplus_1^t S(-l_i)\right]_{d+1}. \qquad \square$$

Inasmuch as this section will be primarily concerned with curves $C \in L^h_{n_1 \dots n_r}$ for which $M(C)_{\alpha-2} \neq 0$ (i.e. $\alpha(C) = 2N + h$), we adopt a special notation for these:

NOTATION 2.12. Let $C \in L^h_{n_1 \dots n_r}$ satisfying $\alpha(C) = 2N + h$. Then

$\nu_d =$ number of minimal generators in degree $2N + h + d$ $(d \geqslant 0)$,

$s_d =$ number of components of $\bigoplus S(-l_i)$ such that $l_i = 2N + h + d$ $(d \geqslant 1)$ (see the sequence (*)).

Note also that $n_d = \dim M(C)_{2N+h+d-3}$ $(1 \leqslant d \leqslant r)$. $\square$

Now we turn to the approach in [GM1] and [GM2]. From the exact sequence

$$0 \to H^0(\mathfrak{I}_C(n)) \xrightarrow{\times L} H^0(\mathfrak{I}_C(n+1)) \to H^0(\mathfrak{I}_{C \cap H}(n+1)) \to M(C)_n \to 0$$

(where L is a linear form not vanishing on any component of C, H is the corresponding hyperplane, and C as usual is Buchsbaum), we obtain

$$\frac{I_{C \cap H}}{[I_C / L \cdot I_C]} \cong M(C)(-1).$$

Let $C \in L^h_{n_1 \dots n_r}$ be a Buchsbaum curve for which $\alpha(C) = 2N + h$. As in [GM2], we can schematically represent $I_{C \cap H}$ (thanks to the above isomorphism and Theorem 2.2 (*d*)) as follows:

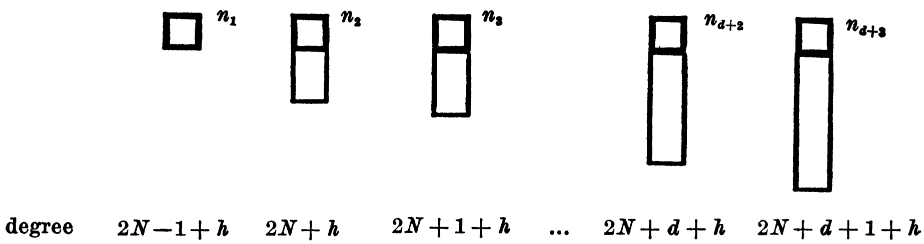


Figure 2

That is, in each degree we choose a (vector space) basis for $I_C/L\cdot I_C$ and extend it to a basis for $I_{C\cap H}$ by adding n_i vectors corresponding to $M(C)(-1)$. (Note n_i may be 0 for $i\neq 1, r$.) However, we modify the notation of [GM2] somewhat, as follows. First, note:

LEMMA 2.13 ([GM2]).

(*a*) *The minimal generators of* $I_C/L\cdot I_C$ *are in one-to-one, degree preserving correspondence with those of* I_C.

(*b*) *Let* $J_d = (I_C/L\cdot I_C)_{2N+d+h}$ $(d\geqslant 0)$, *so*

$$\dim J_d = h^0(\mathfrak{J}_{C\cap H}(2N+d+h)) - n_{d+2}\,.$$

Let $\mathfrak{m}$ *be the maximal ideal in the coordinate ring of the plane* H. *Then*

$$\dim \mathfrak{m}_1\cdot J_d \geqslant 2[\dim J_d] + 1 - h^0(\mathfrak{J}_{C\cap H}(2N+d-1+h))\,.$$

(*c*) *If* $\nu(C) = \alpha(C) + N + 1$ *then the bound of* (*b*) *is sharp for all* d. □

Now,

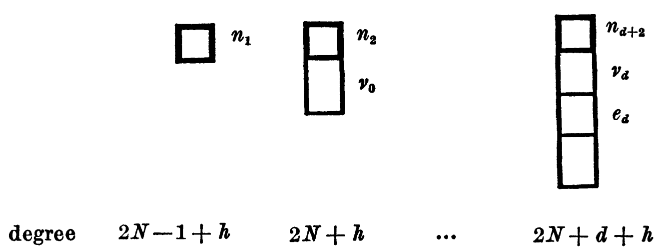


Figure 3

in fig. 3 the bottom section represents the smallest possible dimension of the image $\mathfrak{m}_1 \cdot J_{d-1}$ (Lemma 2.13 (*b*)) and e_d represents the amount that the dimension od $\mathfrak{m}_1 \cdot J_{d-1}$ actually exceeds this lower bound ($e_d = 0$ if $\nu(C) = \alpha(C) + N + 1$ by Lemma 2.13 (*c*).) Then the remaining part of the basis for $(I_C/L\cdot I_C)_{2N+d+h}$ consists of minimal generators, the number of which by Lemma 2.13 (*a*) is what we have called ν_d. Formally, we now add

NOTATION 2.14.

$$\begin{aligned} e_d = h^0(\mathfrak{J}_{C\cap H}(2N+d+h)) - 2h^0(\mathfrak{J}_{C\cap H}(2N+d-1+h)) + \\ + h^0(\mathfrak{J}_{C\cap H}(2N+d-2+h)) - 1 - \nu_d - n_{d+2} + 2n_{d+1} = \\ = \Delta H(C\cap H, 2N+d-1+h) - \Delta H(C\cap H, 2N+d+h) - \\ - \nu_d - n_{d+2} + 2n_{d+1}\,. \qquad \square \end{aligned}$$

The symbols e_d, s_d, ν_d and n_d (cf. Notation 2.12) will be used frequently in what follows.

We are now ready to combine these two approaches. What we shall do is to compute the dimension of the *image* $\mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_d]$ in $(I_C/L\cdot I_C)_{d+1}$ in two different ways and set the two answers equal to each other. We begin with the lowest degree, $2N+h$.

LEMMA 2.15. *If* $C \in L^h_{n_1\ldots n_r}$ *with* $\alpha(C) = 2N + h$ *then*

$$h^0(\mathfrak{J}_C(2N+h)) = 3n_1 + 1 + e_1 + s_1\,.$$

PROOF. By the second approach we have

$$\dim \mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_{2N+h}] = 2\nu_0 + 1 - n_1 + e_1\,.$$

By the first approach we have that the dimension of the image in $(I_C)_{2N+h+1}$ of $(I_C)_{2N+h}$ is $4\nu_0 - 4n_1 - s_1$. Hence

$$\dim \mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_{2N+h}] = (4\nu_0 - 4n_1 - s_1) - \nu_0 = 3\nu_0 - 4n_1 - s_1\,.$$

Therefore $\nu_0 = 2n_1 + 1 + e_1 + s_1$ as desired. This answers affirmatively a question of [GM1] (Remark 4.4), as has also been noted by Amasaki ([A3]). $\square$

Our next result is the key technical point of the paper.

Proposition 2.16. *If $C \in L^h_{n_1 \dots n_r}$ with $\alpha(C) = 2N + h$ then for any $d \geq 1$ we have*

$$(a) \quad 3n_{d+1} + e_{d+1} + s_{d+1} = e_d + \nu_d ,$$

$$(b) \quad \dim \mathfrak{m}_1 \cdot [(I_C/L \cdot I_C)_{2N+d+h}] =$$

$$= e_{d+1} + (2d+5)n_1 + (2d+3)n_2 + \dots + 5n_{d+1} + \binom{d+3}{2} +$$

$$+ (d+2)(e_1+s_1) + (d+1)(e_2+s_2) + \dots + 3(e_d+s_d) + 2(e_{d+1}+s_{d+1}) .$$

Proof. Note again that the vector space in (*b*) is a subspace of $(I_{C \cap H})_{2N+h+d+1}$. The proof is by induction on d. For $d = 1$ we have (using Lemma 2.15)

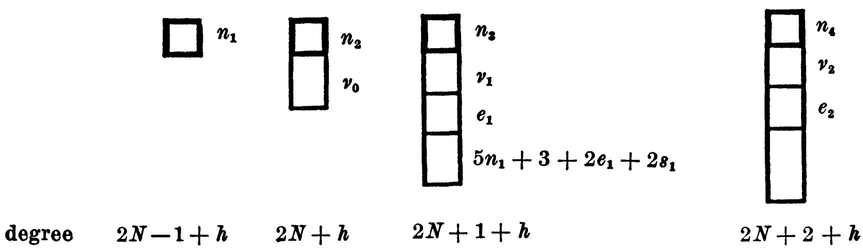


Figure 4

Now, a quick calculation gives

$$\dim \mathfrak{m}_1 \cdot [I_C/L \cdot I_C)_{2N+1+h}] = e_2 + 7n_1 - n_2 + 6 + 5e_1 + 3s_1 + 2\nu_1 .$$

And by the first approach,

$$\dim \mathfrak{m}_1 \cdot [(I_C/L \cdot I_C)_{2N+1+h}] =$$

$$= [10\nu_0 + 4\nu_1 - (15n_1 + 4n_2 + 4s_1 + s_2)] - [4\nu_0 + \nu_1 - 4n_1 - s_1] =$$

$$= 7n_1 + 6 + 6e_1 + 3s_1 + 3\nu_1 - 4n_2 - s_2 .$$

Hence $3n_2 + e_2 + s_2 = e_1 + \nu_1$. Thus is (*a*), and for (*b*) we substitute $e_1 + \nu_1 = 3n_2 + e_2 + s_2$ into the previous computation. (Note that there is an extra e_{d+1}, in this case e_2.)

Now the inductive step. Assume the statement is true for all $t < d$. Then in particular we have in degree $2N + h + d$

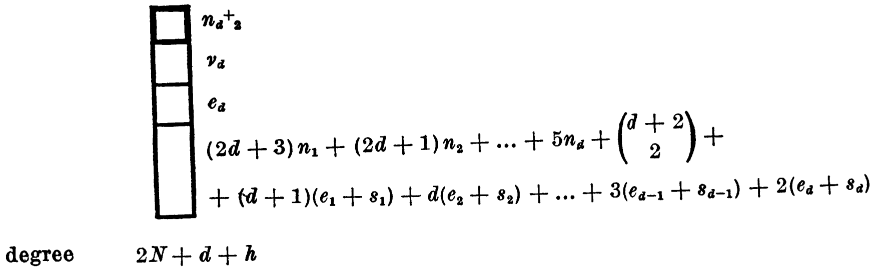


Figure 5

$$\begin{aligned}\dim \mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_{2N+d+h}] &= \\ &= e_{d+1} + 2\Big[\nu_d + e_d + (2d+3)n_1 + \dots + 5n_d + \\ &+ \binom{d+2}{2} + (d+1)(e_1+s_1) + \dots + 2(e_d+s_d)\Big] + \\ &+ 1 - \Big[n_{d-1} + \nu_{d-1} + e_{d-1} + (2d+1)n_1 + \dots + 5n_{d-1} + \\ &\qquad + \binom{d+1}{2} + d(e_1+s_1) + \dots + 2(e_{d-1}+s_{d-1})\Big].\end{aligned}$$

Substituting $\nu_{d+1} + e_{d+1} = 3n_d + e_d + s_d$, a substitution gives

$$\begin{aligned}(1)\qquad \dim \mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_{2N+d+h}] &= 2(\nu_d + e_d) + (2d+5)n_1 + \dots + 7n_d + \\ &+ \binom{d+3}{2} + (d+2)(e_1+s_1) + \dots + 3(e_d+s_d) - n_{d+1} + e_{d+1}.\end{aligned}$$

On the other hand, using the first approach we get

$$\begin{aligned}\dim \mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_{2N+d+h}] &= \\ &= \binom{d+4}{3}\nu_0 + \dots + 4\nu_d - \Big[\binom{d+3}{3}s_1 + \dots + 4s_d + s_{d+1}\Big] -\end{aligned}$$

$$-h^0(K(2N+h+d+1))-\left\{\binom{d+3}{3}\nu^0+\ldots+4\nu_{d+1}+\nu_d-\right.$$

$$\left.-\left[\binom{d+2}{3}s_1+\ldots+4s_{d-1}+s_d\right]-h_0(K(2N+h+d))\right\}=$$

$$=\binom{d+3}{2}\nu_0+\ldots+3\nu_d-\left[\binom{d+2}{2}s_1+\ldots+6s_{d-1}+3s_d+s_{d+1}\right]-$$

$$-\left[4n_1\binom{d+2}{2}+4n_2\binom{d+1}{2}+\ldots+4n_{d-1}(6)+4n_d(3)+4n_{d+1}\right]+$$

$$+\left[n_1\binom{d+1}{2}+n_2\binom{d}{2}+\ldots+3n_{d-1}+n_d\right].$$

By induction we have

$$\begin{aligned}\nu_0 &= 3n_1+1+e_1+s_1\,,\\ \nu_1 &= 3n_2+e_2+s_2-e_1\,.\\ &\ldots\\ \nu_{d-1} &= 3n_d+e_d+s_d-e_{d-1}\,.\end{aligned}$$

Making these substitutions and simplifying, we get

$$\text{(2)}\qquad \dim \mathfrak{m}_1\cdot[(I_C/L\cdot I_C)_{2N+d+h}]=(2d+5)n_1+(2d+3)n_2+\ldots+$$
$$+7n_d-4n_{d+1}+(d+2)(e_1+s_1)+\ldots+4(e_{d-1}+s_{d-1})+$$
$$+3(e_d+s_d)+3e_d-s_{d+1}+\binom{d+3}{2}+3\nu_d\,.$$

Combining (1) with (2) gives

$$\nu_d+e_d=3n_{d+1}+e_{d+1}+s_{d+1}$$

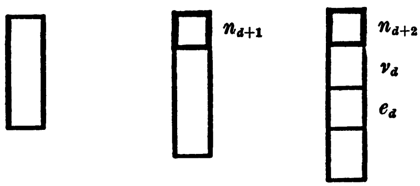
as desired. This proves (*a*), and (*b*) follows by substitution. □

COROLLARY 2.17. *If $C\in \boldsymbol{L}^h_{n_1\ldots n_r}$ with $\alpha(C)=2N+h$ then for any $d\geqslant 0$,*

$$\Delta H(C\cap H, 2N+h+d-1)-\Delta H(C\cap H, 2N+h+d)\geqslant n_{d+1}+n_{d+2}\,.$$

Proof. A quick calculation gives that the left-hand side of the inequality is

$$h^0(\mathfrak{J}_{C\cap H}(2N+h+d)) - 2h^0(\mathfrak{J}_{C\cap H}(2N+h+d-1)) + \\ + h_0(\mathfrak{J}(2N+h+d-2)) - 1 .$$



degree $2N+h+d-2$ $2N+h+d-1$ $2N+d+h$

Figure 6

By Lemma 2.13 (*b*) and definition,

$$h^0(\mathfrak{J}_{C\cap H}(2N+h+d)) - n_{d+2} - v_d - e_d = \\ = 2[h^0(\mathfrak{J}_{C\cap H}(2N+h+d-1)) - n_{d+1}] + 1 - h^0(\mathfrak{J}_{C\cap H}(2N+h+d-2)).$$

That is,

$$h^0(\mathfrak{J}_{C\cap H}(2N+h+d-1)) - \\ - 2h^0(\mathfrak{J}_{C\cap H}(2N+h+d-1)) + h^0(\mathfrak{J}_{C\subset H}(2N+h+d-2)) - 1 = \\ = v_d + e_d + n_{d+2} - 2n_{d+1} \geqslant n_{d+1} + n_{d+2}$$

(the latter by Proposition 2.16). □

Corollary 2.18. *Let* $C \in L^0_{n_1 \dots n_r}$. *Then*

(*a*) $v_0 = 3n_1 + 1$ *and* $v_d = 3n_{d+1}$ *for* $d \geqslant 1$,

(*b*) $\Delta H(C \cap H, 2N+k) =$

$$= \begin{cases} n_{k+2} + 2n_{k+3} + \dots + 2n_r , & -1 \leqslant k \leqslant r-3 , \\ n_r , & k = r-2 \, ' \\ 0 , & k \geqslant r+1 . \end{cases}$$

PROOF. C clearly satisfies the hypothesis of Lemma 2.15, Proposition 2.16. Now, since $\alpha(C) = 2N$ we know that $\nu(C) = 3N + 1$ (Corollary 2.7). Then $s_i = 0$ for all i since $t = 0$ in Theorem 2.9. On the other hand, $e_i = 0$ for all i by Lemma 2.13 (*c*). This proves (*a*).

For (*b*), note that

$$\Delta H(C \cap H, 2N-2) = 2N-1\,,$$
$$\Delta H(C \cap H, 2N-1) = 2N-n_1\,.$$

Also, $\Delta H(C \cap H, 2N + r - 1) = 0$ by Theorem 2.5 (*a*) and a simple calculation. Now add:

$$\begin{aligned} 2N - n_1 = {} & [\Delta H(C \cap H, 2N-1) - \Delta H(C \cap H, 2N)] + \\ & + [\Delta H(C \cap H, 2N) - \Delta H(C \cap H, 2N)] + \dots + \\ & + [\Delta H(C \cap H, 2N + r - 2) - \Delta H(C \cap H, 2N + r - 1)] \geqslant \\ & \geqslant (n_1 + n_2) + (n_3 + n_3) + \dots + (n_{r-1} + n_r) + (n_r + 0)\,. \end{aligned}$$

Therefore we have equality at each step in Corollary 2.17, and the result is a simple calculation. □

REMARK 2.19.

(1) This verifies the conjecture at the end of [GM1]. M. Amasaki has informed us ([A3]) that he can derive Corollary 2.18 (*b*) with his techniques.

(2) Of course, Corollary 2.18 shows that the Hilbert function of a curve C in $L^0_{n_1 \dots n_r}$ is uniquely determined, since $\alpha(C \cap H) = 2N - 1$.

(3) We believe that a result very much like Proposition 2.16 should be true in general. In any case, we have verified that any curve in $L_{n_1 \dots n_r}$ lying on a surface of degree $2N$ satisfies Corollary 2.18, suitably re-indexed. (Recall Corollary 2.7.) □

3. The LR-property for Buchsbaum curves.

We have now laid the groundwork, and are prepared to prove the main result of the paper.

THEOREM 3.1. *Every Buchsbaum laison class has the* LR-*property.*

PROOF. The approach is via Theorem 1.9, and we now outline it. Consider the even liaison class $\boldsymbol{L}_{n_1 \ldots n_r}$.

(1) If $X, X' \in \boldsymbol{L}^0_{n_1 \ldots n_r}$ then they have the same cohomology by Corollary 2.18 and Corollary 1.3.

(2) For $h > 0$, we have three cases for $X \in \boldsymbol{L}^h_{n_1 \ldots n_r}$ (recall Theorem 2.5 (*b*), Theorem 2.2 and Remark 2.4):

(*a*) If $e(X) > 2N + h + r - 5$ then X can be linked in two steps to $Y_2 \in \boldsymbol{L}^{h-1}_{n_1 \ldots n_r}$ as required.

(*b*) If $e(X) = 2N + h + r - 5$ and $\alpha(X) < 2N + h$ then X can be linked in two steps to $Y_2 \in \boldsymbol{L}^{h-1}_{n_1 \ldots n_r}$ as required.

(*c*) If $e(X) = 2N + h + r - 5$ and $\alpha(X) = 2N + h$ then we will identify all the possible cohomologies for X (in terms of ΔH) and show that for each there exists the desired curve Y with the same cohomology. In fact, Y itself is obtained from a minimal curve by a sequence of h basic double links using planes.

Parts (*a*) and (*b*) are contained in the proof of Proposition 2.7 of [BM2], but since the proofs are short we repeat them here for completeness.

If $e = e(X) > 2N + h + r - 5$, then it follows from Theorem 2.6 (or from Castelnuovo-Mumford) that we can find a link using surfaces of degree $\alpha = \alpha(X)$ and $e + 3$. Let C be the complete intersection and Y_1 the residual curve. We have

$$0 \to \mathfrak{J}_C(\alpha - 1) \to \mathfrak{J}_{Y_1}(\alpha - 1) \to \omega_X(-e) \to 0$$

so by definition of e we get that Y_1 admits a link using surfaces of degree $\alpha - 1$ and $e + 3$. This proves (*a*).

If $e = e(X) = 2N + h + r - 5$ and $\alpha = \alpha(X) < 2N + h$, then again by Theorem 2.6 we get a link using surfaces of degree α and $2N + h + r - 1$. Let C be the complete intersection and Y_1 the residual curve as before. From the exact sequence

$$0 \to \mathfrak{J}_C(\alpha - 1) \to \mathfrak{J}_X(\alpha - 1) \to \omega_{Y_1}(4 - 2N - h - r) \to 0$$

we get $e(Y_1) \leqslant 2N + h + r - 5$. On the other hand, a computation gives that the rightmost non-zero component of $M(Y_1)$ occurs in degree $\alpha + r - 3 < 2N + h + r - 3$. Hence again invoking Theorem 2.6 we get a link for Y_1 using a surface of degree α and one of degree $2N + h + r - 2$, proving (*b*).

We now turn to (*c*). Let $C \in \boldsymbol{L}^h_{n_1 \dots n_r}$, $e(C) = 2N + h + r - 5$, $\alpha(C) = 2N + h$.

Claim. The condition that $e(C) = 2N + h + r - 5$ implies

$$\Delta H(C \cap H, 2N + h + r - 2) = n_r \quad \text{and} \quad \Delta H(C \cap H, t) = 0$$

for $t > 2N + h + r - 2$.

From a standard exact sequence we get

$$h^1(\mathfrak{J}_{C \cap H}(2N + h + r - 3)) = n_r \quad \text{and} \quad h^1(\mathfrak{J}_{C \cap H}(2N + h + r - 3)) = 0 .$$

Hence

$$\Delta H(C \cap H, 2N + h + r - 2) = \\ = h^1(\mathfrak{J}_{C \cap H}(2N + h + r - 3)) - h^1(\mathfrak{J}_{C \cap H}(2N + h + r - 2)) = n_r ,$$

and $\Delta H(C \cap H, t) = 0$ for $t > 2N + h + r - 2$, proving the claim.

So we have:

(1) $\Delta H(C \cap H, t) = t + 1$ for $t \leqslant 2N + h - 2$;

(2) $\Delta H(C \cap H, 2N + h - 1) = 2N + h - n_1$;

(3) $\Delta H(C \cap H, 2N + h + r - 2) = n_r$;

(4) $\Delta H(C \cap H, t) = 0$ for $t \geqslant 2N + h + r - 1$;

(5) $\Delta H(C \cap H, 2N + h + d - 1) - \Delta H(C \cap H, 2N + h + d) \geqslant$
$\geqslant n_{d+1} + n_{d+2}$ for all $d \geqslant 0$ (Corollary 2.17).

Note that for $C \in \boldsymbol{L}^h_{n_1 \dots n_r}$, (1) and (2) together are equivalent to the condition that $\alpha(C) = 2N + h$. Our task will be complete if we can produce a « basic double link curve » $Y \in \boldsymbol{L}^h_{n_1 \dots n_r}$ for any Hilbert function allowed by (1)-(5) (by Corollary 1.3).

The first main observation to make is that (5) contains an implicit upper bound, since the total drop in ΔH from degree $2N + h - 1$ to $2N + h + r - 1$ is $2N + h$ (including a drop of n_1 in degree $2N + h - 1$), so there is a total surplus of exactly h in all the inequalities (5) combined, considering also the conditions (2) and (3).

For example, for $h = 1$ we have equality in (5) for every d but one, and that one can only differ by 1.

The second main observation to make is that for any h, if $Y_1 \in L^{h-1}_{n_1 \ldots n_r}$ satisfies $e(Y_1) = 2N + h + r - 6$ (the minimum possible) and $\alpha(Y_1) = 2N + h - 1$ (the maximum possible), then a basic double link $Y \in L^h_{n_1 \ldots n_r}$ of Y_1 is limited if it is to continue to have the extremal value for e and α. That is, for $e(Y) = 2N + h + r - 5$ we must use a surface of degree $\leqslant 2N + h + r - 2$ by Lemma 1.14, and a simple check gives that for $\alpha(Y) = 2N + h$ we need to use a surface of degree $\geqslant 2N + h$ for the basic double link.

We can proceed by induction. For $h = 0$ note that the minimal curve satisfies (1)-(5) (with equality in (5)) by previous calculations (cf. Theorem 2.5 (a), Corollary 2.18 (b)).

Let $C \in L^h_{n_1 \ldots n_r}$, $h > 0$, with $e(C) = 2N + h + r - 5$ and $\alpha(C) = 2N + h$, and assume that every Hilbert function allowed by (1)-(5) in $L^{h-1}_{n_1 \ldots n_r}$ is obtained by some basic double link curve Y_1 with

$$e(Y_1) = 2N + h + r - 6 , \qquad \alpha(Y_1) = 2N + h - 1 .$$

Since $h > 0$ we have

$$\Delta H(C \cap H, 2N + h + k - 1) - \Delta H(C \cap H, 2N + h + k) > n_{k+1} + n_{k+2}$$

for some $0 \leqslant k \leqslant r - 2$. But by Corollary 1.13 and the inductive hypothesis, there is a suitable basic double link curve $Y_1 \in L^{h-1}_{n_1 \ldots n_r}$ such that $e(Y_1) = 2N + h + r - 6$, $\alpha(Y_1) = 2N + h - 1$, and such that the curve Y obtained from Y_1 by a basic double link using a surface of degree $2N + h + k$ has the same Hilbert function as that of C. Note that the range $0 \leqslant k \leqslant r - 2$ is exactly what we required to preserve the extremal values of e and α. □

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