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A note on the generic solvability of the Navier-Stokes equations

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A Note on the Generic Solvability of the Navier-Stokes Equations.

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1. – Introduction.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain in $\mathbf{R}^3$ with boundary $\partial\Omega$ of class C^2. Consider the Navier-Stokes equations

$$(1.1)\qquad \begin{aligned} &u' + (u\cdot\nabla)u - \Delta u + \nabla\pi = f && \text{in } Q_T \equiv (0,T)\times\Omega\,,\\ &\operatorname{div} u = 0 && \text{in } Q_T\,,\\ &u = 0 && \text{on } \Sigma_T \equiv (0,T)\times\partial\Omega\,,\\ &u(0) = u_0 && \text{in } \Omega\,, \end{aligned}$$

with some $T > 0$. By a strong solution $(u, \nabla\pi)$ of (1.1) we mean a solution with

$$u \in W_p^{2,1}(Q_T) \equiv L^p(0,T;\, W_p^2(\Omega)^3) \cap W_p^1(0,T;\, L^p(\Omega)^3)\,,$$
$$\nabla\pi \in L^p(Q_T) \equiv L^p(0,T;\, L^p(\Omega)^3)$$

for some p with $2 \leqslant p < \infty$. Let $\overset{\circ}{J}{}_p^{2-2/p}(\Omega)$ denote the closure in the norm of $W_p^{2-2/p}(\Omega)^3$ of the set of smooth finite solenoidal vectors equal

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to zero on $\partial\Omega$. Consider $u_0 \in \mathring{J}_p^{2-2/p}(\Omega)$ and $f \in L^p(Q_T)$. Then it is known that these assumptions on the data assure the existence of a local in time unique strong solution of (1.1) (see [4]). The existence of strong solutions for arbitrary $T > 0$ is an important open problem. Therefore it is interesting to know properties of the set

$$R(u_0) =$$
$$= \{f \in L^p(Q_T)/(1.1) \text{ has a unique strong solution } (u, \nabla\pi) \text{ with data } u_0, f\}$$

for a fixed initial value $u_0 \in \mathring{J}_p^{2-2/p}(\Omega)$. It is not known whether or not $R(u_0) = L^p(Q_T)$; however it is interesting to prove some density properties of this set, since this gives information about how many f do exist such that (1.1) is strongly solvable. In this concern H. Sohr and W. von Wahl [3] have proved the following interesting result: *the set* $R(u_0) \subset L^p(Q_T)$ *is dense in the norm of* $L^s(0, T; L^q(\Omega)^3)$ *for all* $s, q \in (1, \infty)$ *with* $4 < 2/s + 3/q$ (see also [2] for a weaker previous result). Their result is proved by a regularization procedure for (1.1) using an approximation of Yosida type and an estimate of the nonlinear term $(u\cdot\nabla)u$ using the exponent $p = 5/4$ (see [3]). The aim of the present note is to prove the same result with a completely different method. We use an approximation method due to H. Beirão da Veiga [1] plus Sobolev imbedding and Hölder inequality. This approach is particularly simple and so we think it is of interest, even if the result is not new. Denote by $|\cdot|_p$ the norm in $L^p(\Omega)^3$ and by $\|\cdot\|_{s,q,T}$ the norm in $L^s(0, T; L^q(\Omega)^3)$. Our result reads as follows

THEOREM A. *Let* $2 \leq p < \infty$ *and* $u_0 \in \mathring{J}_p^{2-2/p}(\Omega)$. *Then the set* $R(u_0) \subset L^p(Q_T)$ *is dense in* $L^s(0, T; L^q(\Omega)^3)$ *for all* $s, q \in (1, \infty)$ *with* $4 < 2/s + 3/q$. *Therefore, for every* $f \in L^p(Q_T)$ *and every* $\varepsilon > 0$ *there exists* $g_\varepsilon \in L^p(Q_T)$ *with* $\|g_\varepsilon\|_{s,q,T} \to 0$ *as* $\varepsilon \to 0$ *and such that*

$$\begin{aligned}
&u' + (u\cdot\nabla)u - \Delta u + \nabla\pi = f + g_\varepsilon && \text{in } Q_T, \\
&\operatorname{div} u = 0 && \text{in } Q_T, \\
&u = 0 && \text{on } \Sigma_T, \\
&u(0) = u_0 && \text{in } \Omega,
\end{aligned}$$

has a unique strong solution $(u, \nabla\pi)$.

2. – Proof of Theorem A.

Following [1] we define the set of vectors

$$\Lambda \equiv \{v \in C^\infty(\overline{Q_T}) / v(t) \in C_0^\infty(\Omega)^3,\ \operatorname{div} v(t) = 0 \text{ in } \Omega \text{ for all } t \in [0, T]\},$$

where $T > 0$ arbitrary, and consider the linearized system

$$(2.1)\qquad \begin{aligned} &u' + (v\cdot\nabla)u - \Delta u + \nabla\pi = f && \text{in } Q_T\,,\\ &\operatorname{div} u = 0 && \text{in } Q_T\,,\\ &u = 0 && \text{on } \Sigma_T\,,\\ &u(0) = u_0 && \text{in } \Omega\,, \end{aligned}$$

for $v \in \Lambda$. For convenience define

$$(2.2)\qquad \begin{aligned} A(u_0, f) &= |u_0|_2 + \|f\|_{1,2,T}\,,\\ A_1^2(u_0, f) &\equiv |u_0|_2^2 + 2\|f\|_{1,2,T}^2\,. \end{aligned}$$

From [4] (Theorem 4.2, p. 487) we have the following preliminary result:

THEOREM 1. *Let* $v \in \Lambda$, $u_0 \in \mathring{J}_p^{2-2/p}(\Omega)$, $f \in L^p(Q_T)$. *Then there exists a unique solution* $(u, \nabla\pi)$ *of proeblem* (2.1) *such that*

$$(2.3)\qquad u \in W_p^{2,1}(Q_T)\,,\qquad \nabla\pi \in L^p(Q_T)\,.$$

We quote now the result which gives us the approximating solutions we shall use later. It is proved in [1] (see Theorem 1.6, p. 329) as a consequence of a very interesting general approximation theorem.

THEOREM 2 ([1]). *Let* $u_0 \in H_0^1(\Omega)$ *and* $f \in L^2(0, T; L^2(\Omega)^3)$ *be given and let* $1 < q \leqslant 5/4$. *Then, in correspondence to every* $\varepsilon > 0$, *there exist*

$$u_\varepsilon \in \Lambda\,,\qquad u_\varepsilon \in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; H_0^1(\Omega)) \cap W_q^{2,1}(Q_T)\,,$$

$$\pi_\varepsilon \in L^q(0, T; W_q^1(\Omega)^3)$$

verifying the system

$$(2.4)_\varepsilon \qquad \begin{aligned} &u'_\varepsilon + (v_\varepsilon\cdot\nabla)u_\varepsilon - \Delta u_\varepsilon + \nabla\pi_\varepsilon = f && \textit{in } Q_T\,, \\ &\operatorname{div} u_\varepsilon = 0 && \textit{in } Q_T\,, \\ &u_\varepsilon = 0 && \textit{on } \Sigma_T\,, \\ &u_\varepsilon(0) = u_0 && \textit{in } \Omega\,, \end{aligned}$$

and for which

$$(2.5) \qquad \|u_\varepsilon - v_\varepsilon\|_{2,2,T} < \varepsilon\,.$$

Moreover the following estimates hold

$$(2.6) \qquad \begin{aligned} &\|u_\varepsilon\|_{\infty,2,T} \leqslant A(u_0, f)\,, \\ &\|\nabla u_\varepsilon\|_{2,2,T} \leqslant A_1(u_0, f) \end{aligned}$$

Estimates $(2.6)_1$ *and* $(2.6)_2$ *hold also for* v_ε *and* ∇v_ε *respectively.*

Let now u_0, f and s, q as in Theorem A. A combination of Theorems 1 and 2 gives us that the approximating solution given by Theorem 2 satisfies also (2.3). We are now in position to prove our result. We write $(2.4)_\varepsilon$ in the form

$$(2.7) \qquad \begin{aligned} &u'_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon - \Delta u_\varepsilon + \nabla\pi_\varepsilon = f + (u_\varepsilon\cdot\nabla)u_\varepsilon - (v_\varepsilon\cdot\nabla)u_\varepsilon && \text{in } Q_T\,, \\ &\operatorname{div} u_\varepsilon = 0 && \text{in } Q_T\,, \\ &u_\varepsilon = 0 && \text{on } \Sigma_T\,, \\ &u_\varepsilon(0) = u_0 && \text{in } \Omega\,. \end{aligned}$$

Hence $(u_\varepsilon, \nabla\pi_\varepsilon)$ is a strong solution of (1.1) with external force $f + g_\varepsilon$, $g_\varepsilon \equiv (u_\varepsilon\cdot\nabla)u_\varepsilon - (v_\varepsilon\cdot\nabla)u_\varepsilon$. Observe that, because $4 < 2/s + 3/q$, we have $s < 2$, $q < 3/2$; it follows that $L^p(Q_T)$, $2 \leqslant p < \infty$, is densely contained in $L^s(0, T; L^q(\Omega)^3)$. Hence the theorem is proved if we show that $g_\varepsilon \in L^p(Q_T)$ and $\|g_\varepsilon\|_{s,q,T} \to 0$ as $\varepsilon \to 0$. Since $v_\varepsilon \in \Lambda$ and $(u_\varepsilon, \nabla\pi_\varepsilon)$ satisfies (2.3), using some Sobolev imbeddings it easily follows that $g_\varepsilon \in L^p(Q_T)$. On the other hand, using the Hölder inequality gives

$$(2.8) \qquad \|g_\varepsilon\|_{s,q,T} = \|((u_\varepsilon - v_\varepsilon)\cdot\nabla)\,u_\varepsilon\|_{s,q,T} \leqslant \|u_\varepsilon - v_\varepsilon\|_{s_1,q_1,T}\|\nabla u_\varepsilon\|_{2,2,T}$$

where $(1/s_1) + (1/2) = 1/s$, $(1/q_1) + (1/2) = 1/q$. Since $1 < s < 2$, $1 < q < 3/2$ with $2/s + 3/q > 4$ we obtain $2 < s_1$, $2 < q_1 < 6$ with $2/s_1 + 3/q_1 > 3/2$. Let s_2, q_2 be the solution of

$$2/s_2 + 3/q_2 = 3/2 \tag{2.9}$$

$$(1 - 2/q_1)1/s_2 - (1 - 2/s_1)1/q_2 = 1/s_1 - 1/q_1 . \tag{2.10}$$

The estimates on s_1, q_1 yield $2 < s_1 < s_2$, $2 < q_1 < q_2 < 6$. Using the Hölder inequality gives

$$\|u_\varepsilon - v_\varepsilon\|_{s_1,q_1,T} \leqslant \|u_\varepsilon - v_\varepsilon\|_{2,2,T}^a \|u_\varepsilon - v_\varepsilon\|_{s_2,q_2,T}^b \tag{2.11}$$

where a, b must satisfy

$$\begin{aligned} a + b &= 1 , \\ a/2 + b/q_2 &= 1/q_1 , \\ a/2 + b/s_2 &= 1/s_1 . \end{aligned} \tag{2.12}$$

System (2.12) has a solution if and only if the determinant of the corresponding complete matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1/2 & 1/q_2 & 1/q_1 \\ 1/2 & 1/s_2 & 1/s_1 \end{pmatrix}$$

is zero. This condition is satisfied since (2.10) holds. Hence we find a solution of (2.12)

$$a = (1/q_1 - 1/q_2)/(1/2 - 1/q_2) , \qquad b = (1/2 - 1/q_1)/(1/2 - 1/q_2)$$

with $a, b > 0$. Since we have

$$1/q_2 = (1 - 2/s_2)/2 + (2/s_2)/6$$

with $0 < (2/s_2) < 1$ it follows

$$|u_\varepsilon - v_\varepsilon|_{q_2} \leqslant |u_\varepsilon - v_\varepsilon|_2^{1-(2/s_2)} |u_\varepsilon - v_\varepsilon|_6^{2/s_2} \leqslant C_1 |u_\varepsilon - v_\varepsilon|_2^{1-(2/s_2)} |\nabla(u_\varepsilon - v_\varepsilon)|_2^{2/s_2} ,$$

where C_1 is a positive constant. Integrating in time at the s_2-th power and the Young's inequality give

$$\|u_\varepsilon - v_\varepsilon\|_{s_2,q_2,T} \leqslant C_2[\|u_\varepsilon - v_\varepsilon\|_{\infty,2,T} + \|\nabla(u_\varepsilon - v_\varepsilon)\|_{2,2,T}]\,, \tag{2.13}$$

where C_2 is a positive constant; the right-hand side of (2.13) is bounded because of (2.6). Hence from (2.5), (2.6) (also for v_ε), (2.8), (2.11), (2.13) we obtain

$$\|g_\varepsilon\|_{s,q,T} \leqslant C_3\,\varepsilon^a\,,$$

where C_3 is a positive constant independent of ε. The theorem is proved.

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