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# Some Questions on the Number of Generators of a Finite Group.

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## Introduction.

In [9] it is proved that if each Sylow subgroup of an arbitrary finite group can be generated by $\mu$ elements then the group itself can be generated by $\mu + 1$ elements. In other words if we define by $d(G)$ the minimal number of generators for a finite group $G$ and with $d_p(G)$ the minimal number of generators of a Sylow $p$-subgroup of $G$, we get the relation

$$(*) \qquad d(G) \leqslant \max_{p||G|} d_p(G) + 1 \, .$$

Our aim is to give some more precise informations about the minimal number of generators of a finite group; in particular we will try to answer the following questions:

1) For what classes of groups the bound given by $(*)$ can be improved?

2) It is possible to characterize the finite groups for which the relations $(*)$ holds as equality?

An important role in this problem is played by an invariant that is called presentation rank and that is usually indicated by pr $(G)$: the

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definition comes from the study of relation modules but it can also be defined as the non negative integer that one gets as difference between $d(G)$ and $d_{\mathbb{Z}G}(I_G)$, the minimal number of generators of $I_G$, the augmentation ideal of $G$, as $\mathbb{Z}G$-module.

The results about the minimal number of generators of a finite group $G$ are different according as pr $(G)$ is or is not equal to zero. We analyze the first possibility in section 1. In this case $d(G) = d_{\mathbb{Z}G}(I_G)$ and so it is possible to apply a formula proved by Cossey, Gruenberg and Kovács [2] that gives $d_{\mathbb{Z}G}(I_G)$ as a function of the internal structure of the group $G$. In particular from this result we will deduce the answer to question (2) in the case pr $(G) = 0$ proving:

THEOREM 1. *Let $G$ be a finite group with* pr $(G) = 0$; *if* $d(G) = \max_{p||G|} d_p(G) + 1$ *then $G$ contains a normal subgroup $N$ such that $G/N$ is solvable, $d(G/N) = d(G)$ and $G/N$ is the semidirect product $P\langle x\rangle$ of an elementary abelian $p$-group $P$ of rank $d(G) - 1$ with a cyclic group $\langle x\rangle$ such that $x$ acts on $P$ as a non trivial power.*

In section 2 we will study the groups with non-zero presentation rank. In this case it becomes very difficult to express $d(G)$ in terms of the internal structure of $G$. But an interesting improvement to the bound given by (*) holds for the class of the groups with non zero presentation rank. Precisely we will prove the following result:

THEOREM 2. *If $G$ is a finite group with* pr $(G) \neq 0$ *then* $d(G) \leqslant d_2(G) + 1$.

A consequence of that is:

COROLLARY 3. *If $G$ is a finite group then* $d(G) \leqslant \max\left(d_{\mathbb{Z}G}(I_G), d_2(G) + 1\right)$.

With the help of the results proved in section 1 and 2, we will be concerned in section 3 with the class of finite perfect groups. In this class the bound given by (*) can be improved: if we set $d = \max_{p||G|} d_p(G)$ we can show that if $G$ is a perfect group then $d(G) \leqslant d$, but in particular also the following is true:

THEOREM 4. *If $G$ is a perfect group then* $d(G) \leqslant \max\left((d+4)/2, d_2(G)\right)$.

## 1. – Groups with zero presentation rank.

The assumption pr $(G) = 0$ means that $d(G) = d_{\mathbf{Z}G}(I_G)$, so for the class of groups with zero presentation rank the informations about the minimal number of generators for a group $G$ can be deduced from the study of its augmentation ideal $I_G$. We recall some important results about $d_{\mathbf{Z}G}(I_G)$.

The first is that $I_G$ is a Swan module, that is the following is truc ([10] 5.3):

1.1 $d_{\mathbf{Z}G}(I_G) = \max\limits_{p||G|} d_{\mathbf{Z}G}(I_G/pI_G)$.

The second is a formula proved in [2] by Cossey, Gruenberg and Kovács that allows to express $d_{\mathbf{Z}G}(I_G/pI_G)$ as a function of some integers coming from the study of the structure of the $G$-modules: precisely, given an irreducible $GF(p)G$-module $M$, we define the integer numbers $q, r(M), s(M)$, by setting $q = |\mathrm{Hom}_G(M, M)|$, $q^{r(M)} = |\mathrm{Hom}_G(GF(p)G, M)| = |M|$, $q^{s(M)} = |H^1(G, M)|$. The formula is:

1.2. $d_{\mathbf{Z}G}(I_G/pI_G) = \max\{d(G/G'G^p), [\![s(M)/r(M) + 1]\!]\}$ *where $M$ varies over all non trivial irreducible $GF(p)G$ modules and, if $a$ is a rational number, with $[\![a]\!]$ it is denoted the smallest integer $\geqslant a$.*

In order to use (1.2) we recall some results that make it easier to calculate $|H^1(G, M)|$.

In [1] it is proved:

1.3. *Let $M$ be an irreducible $G$-module, $q = |\mathrm{Hom}_G(M, M)|$ and $\Delta(G, M)$ the set of the $G$-invariant subgroups $I$ of $C = C_G(M)$ such that $C/I$ is $G$-isomorphic to $M$ and $C/I$ has a complement in $G/I$. Then if $K = \bigcap\limits_{I\in\Delta} I$,*

*a) $K$ is a normal subgroup of $G$;*

*b) $C/K$ is complemented in $G/K$;*

*c) $C/K$ is $G$-isomorphic to $M^n$ for some natural number $n$;*

*d) $|H^1(G, M)| = q^n|H^1(G/C, M)|$.*

The number $n$ in ($c$) and ($d$) will be denoted in the following discussion by $n(M)$.

In [1] it is also proved the following powerful result:

1.4. *If $G$ is a finite group and $M$ is an irreducible faithful $G$-module, then* $|H^1(G, M)| < |M|$.

The proof of (1.4) uses the classification of finite simple groups and so all the results that we will prove applying (1.4) will depend on the classification.

We need the following lemma:

LEMMA 1.5. *Let $M$ be an irreducible non trivial $GF(p)$ $G$-module*:

i) *if* $n(M) = 0$ *then* $[\![s(M)/r(M) + 1]\!] \leq 2$.

ii) *if* $n(M) \neq 0$ *then* $[\![s(M)/r(M) + 1]\!] \leq n(M) + 1$;

iii) *if* $r(M) = 1$ *then $H$, a complement for $M^{n(M)}$ in $G/K$, is isomorphic to a subgroup of $GF(p^\alpha)^*$ where $p^\alpha$ is the order of $M$*;

iv) *if* $r(M) > 1$ *then* $[\![s(M)/r(M) + 1]\!] \leq n(M)/2 + 2$.

PROOF. By (1.3, *d*)

$$q^{s(M)} = |H^1(G, M)| = q^{n(M)}|H^1(G/C, M)| = q^{n(M)}q^{t(M)}$$

if we set $q^{t(M)} = |H^1(G/C, M)|$. Hence $s(M) = n(M) + t(M)$. Since $M$ is an irreducible faithful $G/C$-module by (1.4) $q^{t(M)} = |H^1(G/C, M)| < |M| = q^{r(M)}$ and this implies $t(M) < r(M)$.

We distinguish two possibilities:

$$n(N) = 0: \quad [\![s(M)/r(M) + 1]\!] = [\![t(M)/r(M) + 1]\!] \leq 2$$

since $t(M) < r(M)$. This proves (i).

$$n(M) \neq 0: \quad s(M) = n(M) + t(M) < n(M) + r(M)$$

implies $s(M) \leq n(M) - 1 + r(M)$ and so

$$s(M)/r(M) \leq \big(n(M) - 1\big)/r(M) + 1 \leq n(M)\,.$$

We conclude $[\![s(M)/r(M) + 1]\!] \leq n(M) + 1$; also (ii) is proved.

If $r(M) = 1$, $|\mathrm{Hom}_G(M, M)| = |M| = p^\alpha$. By Schur's lemma $\mathrm{Hom}_G(M, M) \cong GF(p^\alpha)$. Since $C_H(M) = 1$, $H \leqslant \mathrm{Aut}(M) = GL(\alpha, p)$. The action of $H$ on $M$ in induced by the structure of $M$ as $G$-module, so it must be $H \leqslant C_{GL(\alpha,p)}(\mathrm{Hom}_G(M, M) \setminus \{0\}) = C_{GL(\alpha,p)}(GF(p^\alpha)^*) = GF(p^\alpha)^*$.

Proof of (iv): if $n(M) = 0$ then by (i) $[\![ s(M)/r(M) + 1 ]\!] \leqslant 2 \leqslant n(M)/2 + 2$. If $n(M) \neq 0$,

$$\frac{s(M)}{r(M)} + 1 \leqslant \frac{n(M) - 1 + r(M)}{r(M)} + 1 = \frac{n(M) - 1}{r(M)} + 2 \leqslant \frac{n(M) - 1}{2} + 2$$

and so $[\![ s(M)/r(M) + 1 ]\!] \leqslant n(M)/2 + 2$. #

We can now prove the first result about the augmentation ideal.

Proposition 1.6.

1) *For each prime number dividing* $|G|$ $d_{ZG}(I_G/pI_G) \leqslant d_p(G) + 1$ *and if the equality* $d_{ZG}(I_G/pI_G) = d_p(G) + 1$ *occurs then either* $d_p(G) = 1$ *or* $G$ *contains a normal subgroup* $N$ *such that* $G/N = P\langle x \rangle$, *the semi-direct product of an elementary abelian p-group* $P$ *of rank* $d_p(G)$ *with a cyclic group* $\langle x \rangle$ *where* $x$ *acts on* $P$ *as a non trivial power automorphism.*

2) $d_{ZG}(I_G/2I_G) \leqslant d_2(G)$ *and the equality* $d_{ZG}(I_G/2I_G) = d_2(G)$ *occurs only if either* $d_2(G) \leqslant 2$ *or* $G$ *contains a normal subgroup* $N$ *such that* $G/N$ *is an elementary abelian 2-group of rank* $d_2(G)$.

Proof. The relation $d_{ZG}(I_G/pI_G) \leqslant d_p(G) + 1$ is known (see [11]). Suppose $d_{ZG}(I_G/pI_G) = d_p(G) + 1$. By 1.2

$$d_p(G) + 1 = \max \{ d(G/G'G^p), [\![ s(M)/r(M) + 1 ]\!] \}$$

where $M$ varies over all the non trivial irreducible $GF(p)G$-modules; since $d(G/G'G^p) \leqslant d_p(G)$, an irreducible non trivial $GF(p)G$-module, $M$ say, must exist such that $[\![ s(M)/r(M) + 1 ]\!] = d_p(G) + 1$. We are in the situation described in 1.3 and 1.5. There are two possibilities:

i) $n(M) = 0$; by lemma (1.5, i) $d_p(G) + 1 = [\![ s(M)/r(M) + 1 ]\!] \leqslant 2$ and we get $d_p(G) \leqslant 1$.

ii) $n(M) \neq 0$: by lemma (1.5, ii) $d_p(G) + 1 = [\![ s(M)/r(M) + 1 ]\!] \leqslant n(M) + 1$; $G/K \cong M^{n(M)} H$ where $M^{n(M)}$ is the direct product of

$n(M)$ $H$-invariant factors, each of them $G$-isomorphic to $M$, and $C_H(M) = 1$. A Sylow $p$-subgroup of $G/K$ is a semidirect product $M^{n(M)}P$ of $M^{n(M)}$ with a Sylow $p$-subgroup $P$ of $H$. In particular it must be $d(M^{n(M)}P) \leqslant d_p(G)$: since $n(M) \geqslant d_p(G)$ it follows $P = 1$, $d_p(G) = n(M)$ and $M \simeq C_p$. Since $C_H(M) = 1$, $H$ can be thought of as a subgroup of Aut $(M)$, hence it is cyclic. So we have shown that $G/K$ is the semidirect product of an elementary abelian $p$-group of rank $d_p(G)$ with a cyclic group that acts as a non trivial power, as we stated.

We have now to prove that $d_{\mathbf{Z}G}(I_G/2I_G) \leqslant d_2(G)$. By what we have just shown $d_{\mathbf{Z}G}(I_G/2I_G) \leqslant d_2(G) + 1$ and if, by contradiction, $d_{\mathbf{Z}G}(I_G/2I_G) = d_2(G) + 1$, one of the two following cases occurs:

i) $G$ contains a normal subgroup $N$ such that $G/N = P\langle x \rangle$ where $P$ is an elementary abelian 2-group on which $x$ acts as a non trivial power: a contradiction since every power automorphism of an elementary abelian 2-group is trivial.

ii) $d_2(G) = 1$: a Sylow 2-subgroup of $G$ is cyclic, so $G$ is 2-nilpotent. We again reach a contradiction since it can be proved:

1.7. *If $M$ is a normal subgroup of a finite group $G$ and $p$ is a prime number dividing $|G|$ but not $|M|$ then $d_{\mathbf{Z}G}(I_G/pI_G) \leqslant d(G/M)$.*

(So, in our case, taking as $M$ a normal 2-complement in $G$, $d_{\mathbf{Z}G}(I_G/2I_G) \leqslant d(G/M) = d_2(G)$.) To prove 1.7 observe that since $(|M|, p) = 1$ there exist two integer $r$ and $s$ such that $r|M| + sp = 1$ and define $e = r \sum_{\substack{x \in M \\ x \neq 1}} (1 - x) + pI_G$. It is easy to verify

$$(*) \qquad e^2 = e$$

and

$$(**) \qquad e(1 - y + pI_G) = 1 - y + pI_G$$

for every $y \in M$. Let $G/M = \langle g_1 M, \ldots, g_1 M \rangle$ and let $B$ the $G$-submodule of $I_G/pI_G$ generated by the elements $e + g_i - 1 + pI_G$, $1 \leqslant i \leqslant l$; $(e + g_i - 1 + pI_G)e \in B$ but $(e + g_i - 1)e = g_i e =$ (since $M$ is a normal subgroup of $G$) $eg_i$; this implies that also $e = (eg_i)g_i^{-1} \in B$. So $B$ contains the $G$-submodule generated by the elements $e$ and $(g_i - 1) + pI_G$, $1 \leqslant i \leqslant l$. By $(**)$ $B$ contains also all the elements $x - 1 + pI_G$ for $x \in M$. Since $G = \langle M, g_1, \ldots, g_l \rangle$ we conclude $B = I_G/pI_G$.

To complete the proof of proposition 1.6 it remains now to discuss when the equality $d_2(G) = d_{ZG}(I_G/2I_G)$ occurs. By (1.2) $d_2(G) = \max\{d(G/G'G^2), [\![s(M)/r(M)+1]\!]\}$ where $M$ runs over the set of the irreducible non trivial $GF(2)G$-modules.

If $d_2(G) = d(G/G'G^2)$ then $G$ has a quotient that is isomorphic to an elementary abelian 2-group of rank $d_2(G)$, as claimed.

Suppose $d_2(G) = [\![s(M)/r(M)+1]\!]$ for a given $GF(2)G$-module $M$. We are in the situation described in 1.3 and 1.5. We distinguish the two possibilities for $n(M)$:

if $n(M) = 0$ by (1.5, i) $d_2(G) = [\![s(M)/r(M)+1]\!] \leqslant 2$;

if $n(M) \neq 0$ by (1.5, ii) $d_2(G) = [\![s(M)/r(M)+1]\!] \leqslant n(M)+1$;

it must be also $n(M) \leqslant d_2(G) - 1$: in fact a Sylow 2-subgroup of $G/K$ cannot be generated by less than $n(M)$ elements, and if, by contradiction, $n(M) = d_2(G)$, then $|M| = 2$ and $M$ is trivial as $G$-module. So we conclude $n(M) = d_2(G) - 1$. If $r(M) = 1$ by (1.5, iii) $G/K \simeq M^{n(M)}H$ where $H \leqslant GF(2^\alpha)^*$ and $\alpha$ is the rank of $M$ as an elementary abelian 2-group. So $|H|$ is odd and a Sylow 2-subgroup of $G/K$ is an elementary abelian 2-group of rank $\alpha d_2(G)$. Since $\alpha(d_2(G)-1) = d(M^{n(M)}) \leqslant d_2(G)$ it follows that either $d_2(G) \leqslant 2$ or $\alpha = 1$, but the latter case must be excluded since it implies that $M$ is a trivial $G$-module.

If $r(M) \neq 1$ by (1.5, iv)

$$d_2(G) = [\![s(M)/r(M)+1]\!] \leqslant n(M)/2 + 2 \leqslant \frac{d_2(G)-1}{2} + 2$$

and so $d_2(G) \leqslant 3$.

To conclude it remains to discuss the case $d_2(G) = 3$: since $n(M) = d_2(G) - 1 = 2$ a Sylow 2-subgroup of $G/K$ is a semidirect product $(M_1 \times M_2)P$ where $M_1 \times M_2$ is $G$-isomorphic to $M^2$ and $P$ is a Sylow 2-subgroup of $H$, a complement of $M_1 \times M_2$ in $G/K$. This subgroup can be generated by 3 elements and this implies:

1) $P = \langle x \rangle$ is a cyclic group;

2) $M/[M, P] \simeq C_2$.

It is $s(M) = n(M) + t(M) = 2 + t(M)$ where $q^{t(M)} = |H^1(H, M)|$; we calculate $t(M)$:

$$H^1(H, M) \leqslant H^1(P, M) =$$
$$= \{m \in M : m(1 + x \ldots + x^{|P|-1}) = 0\}/[P, M] \leqslant M/[P, M] \simeq C_2;$$

so $t(M) \leqslant 1$ and if $t(M) = 1$ then also $q = 2$. From

$$3 = d_2(G) = [\![(s(M))/r(M) + 1]\!] = [\![(2 + t(M))/r(M) + 1]\!]$$

we conclude $t(M) = 1$ and $r(M) = 2$. We have therefore $M = C_2 \times C_2$ and $H \leqslant GL(2, 2)$. $P$ is non trivial, otherwise a Sylow 2-subgroup of $G/K$ would be isomorphic to $(C_2)^4$ and could not be generated with 3 elements, so only the two following cases are possible:

1) $H = C_2$;

2) $H = GL(2, 2)$.

In the first case $G/K$ is a 2-group that can be generated by 3 elements and so $G/G'G^2$ is an elementary abelian 2-group of rank 3.

The second case must be excluded since $H^1(GL(2, 2), M) = 1$ while we have proved before that $|H^1(H, M)| = q^{t(M)} = q$. #

Define $d = \max_{p|G} d_p(G)$. We can now characterize the finite groups $G$ with $\mathrm{pr}\,(G) = 0$ and such that $d(G) = d + 1$ by proving:

THEOREM 1.8. *Let $G$ be a finite group with* $\mathrm{pr}\,(G) = 0$. *If $d(G) = d + 1$ then $G$ contains a normal subgroup $N$ such that $G/N$ is the semidirect product $P\langle x\rangle$ of an elementary abelian $p$-group of rank $d$ with a cyclic group $\langle x\rangle$ where $x$ acts on $P$ as a non trivial power automorphism.*

PROOF. Since $\mathrm{pr}\,(G) = 0$ we have $d + 1 = d(G) = d_{\mathbf{Z}G}(I_G) = \max_{p|G} d_{\mathbf{Z}G}(I_G/pI_G)$. So there exists a prime $p$ such that $d_p(G) + 1 = d_{\mathbf{Z}G}(I_G/pI_G) = d + 1$: by (1.6, 1) either $G$ contains a normal subgroup $N$ such that $G/N$ has the described structure or $d_p(G) = d = 1$: in this latter case all the Sylow subgroup of $G$ are cyclic: this implies that $G$ is solvable and the conclusion follows by ([9] th. 2). #

## 2. – Groups with nonzero presentation rank.

For groups with nonzero presentation rank our invariant $d(G)$ is more difficult to study: in particular it is not known whether there is some general result espressing the minimal number of generators in terms of the internal structure of the group, as formulas 1.1 and 1.2 do for groups with zero presentation rank.

One reason is that the class of groups with non zero presentation rank is difficult to study or characterize: there are few informations available about it and, although it is known that there exist groups with arbitrarily large presentation rank (see [10] lemma 5.16), the examples of groups of this kind are not many. On the other hand in all known examples the minimal number of generators seems not too large and one expects that in the class of groups with non zero presentation rank the general bound « $d(G) \leqslant d + 1$ » can be improved. Actually the results that we will prove are in this direction.

We will apply the following results:

2.1. (see [4] p. 218). *If $N$ is a soluble normal subgroup of $G$ and* $\mathrm{pr}\,(G) > 0$ *then* $d(G) = d(G/N)$.

2.2 (see [7] p. 222). *Suppose $G$ contains a non-trivial normal perfect subgroup $P$ all of whose abelian chief factors are cyclic; then:*

1) *if $G/P$ is cyclic then* $\mathrm{pr}\,(G) = d(G) - 2$;

2) *if $G/P$ is not cyclic then* $\mathrm{pr}\,(G) = d(G) - d(G/P) + \mathrm{pr}\,(G/P)$.

The meaning of 2.2 is that extending above semisimple groups $\mathrm{pr}\,(G)$ and $d(G)$ increase in the same way. Therefore in the study of groups with non zero presentation rank it is useful to get informations about the contribution given to the growth of the number of generators by the perfect minimal normal subgroups. Our next results have this aim.

We will often use the following result, proved by Guralnick, about the generation of finite simple groups [5]:

2.3. *If $G$ is a non abelian finite simple group then there exists $x \in G$ and $P \in \mathrm{Syl}_2\, G$ such that $G = \langle P, P^x \rangle$.*

We need also the following lemma, supplementing the informations given by [9], th. 1.

Lemma 2.4. *If $G$ is a finite group and $g_1, \ldots, g_n$ are $n$ elements of such that $\langle g_1, \ldots, g_n \rangle = G$ and $n > d_2(G)$ then there exist $n$ elements $h_1, \ldots, h_n$ in $G$ such that $\langle h_1, \ldots, h_n \rangle = G$ and the subgroup $H = \langle h_1, \ldots, h_{d_2(G)} \rangle$ contains a Sylow 2-subgroup of $G$.*

PROOF. We proceed by induction on the order of $G$. Let $N$ be a minimal normal subgroup of $G$; by induction

$$G/N = \langle g_1 N, \ldots, g_n N\rangle = \langle h_1 N, \ldots, h_n N\rangle$$

for suitable elements $h_1, \ldots, h_n$ of $G$ such that $\langle h_1 N, \ldots, h_{d_2} N\rangle$ contains a Sylow 2-subgroup of $G/N$.

We distinguish three possibilities:

1) $N$ is a 2-group. Set $H = \langle h_1, \ldots, h_{d_2(G)}, N\rangle$; a Sylow 2-subgroup of $H$ is also a Sylow 2-subgroup of $G$ and so it can be generated by $d_2(G)$ elements: by [9] lemma 1.*b* there exist $y_1, \ldots, y_{d_2(G)}$ in $H$ such that $H = \langle y_1, \ldots, y_{d_2(G)}\rangle$. But then the elements $y_1, \ldots, y_{d_2(G)}$, $h_{d_2(G)+1}, \ldots, h_n$ are those we are looking for.

2) $N$ is a $p$-group with $p \neq 2$. By a theorem of Gaschütz ([10], prop. 5.18) there exist $u_1, \ldots, u_n$ in $N$ such that $G = \langle h_1 u_1, \ldots, h_n u_n\rangle$. Let $H = \langle h_1 u_1, \ldots, h_{d_2(G)} u_{d_2(G)}\rangle$. $HN$ contains a Sylow 2-subgroup of $G$ and $HN/N \cong H/H \cap N$: since $|N|$ is odd $H$ contains a Sylow 2-subgroup of $G$ and so the elements $h_1 u_1, \ldots, h_n u_n$ satisfy the conditions of the lemma.

3) $N$ is not soluble: $N$ is a direct product of isomorphic non abelian simple groups. By 2.3 there exist $x \in N$, $P \in \mathrm{Syl}_2 N$, such that $N = \langle P, P^x\rangle$. Since $G = N N_G(P)$ it is not restrictive to assume $h_i \in N_G(P)$, $1 \leqslant i \leqslant n$. Let $H = \langle h_1, \ldots, h_{d_2(G)}, P\rangle$; 2 does not divide $|NH:H|$ since $H \cap N \geqslant P$: so $H$ contains a Sylow 2-subgroup of $G$. But then each Sylow 2-subgroup of $H$ can be generated by $d_2(G)$ elements; since $P$ is normal in $H$, applying [9] lemma 1, (*b*), it follows that there are in $H$ $d_2(G)$ elements $y_1, \ldots, y_{d_2(G)}$ with $H = \langle y_1, \ldots, y_{d_2(G)}\rangle$. The subgroup $\langle y_1, \ldots, y_{d_2(G)}, h_{d_2(G)+1}, \ldots, h_n x\rangle$ contains $P$, $P^{h_n x} = P^x$ and so contains $N$ and, consequently, it is $G$: we conclude that $y_1, \ldots$ $\ldots, y_{d_2(G)}, h_{d_2(G)+1}, \ldots, h_n x$ are requested elements.

LEMMA 2.5. *If $P$ is a normal 2-subgroup of $G$ with $d(G/P) < d_2(G)$ then*:

i) $d(G) \leqslant d_2(G)$;

ii) *if $d(G) = d_2(G)$ then either $d_2(G) = 2$ or $G$ is 2-nilpotent.*

PROOF. We distinguish two cases:

1) $\mathrm{pr}\,(G) > 0$. By (2.1) $d(G) = d(G/P) < d_2(G)$.

2) pr $(G) = 0$. It is $d(G) = d_{ZG}(I_G) = \max_{p||G|} d_{ZG}(I_G/pI_G)$. By 1.7, if $p \neq 2$, $d_{ZG}(I_G/pI_G) \leqslant d(G/P)$ and by (1.6, 2) $d_{ZG}(I_G/2I_G) \leqslant d_2(G)$. In other words $d(G) \leqslant \max\,(d(G/P)), d(I_G/2I_G) \leqslant \max\,(d(G/P), d_2(G)) \leqslant d_2(G)$. Furthermore if $d(G) = d_2(G)$ then $d_{ZG}(I_G/2I_G) = d_2(G)$ and this, again by (1.6, 2), implies that either $d_2(G) \leqslant 2$ or there exists a normal subgroup $N$ of $G$ such that $G/N$ is an elementary abelian 2-group of rank $d_2(G)$. In the latter case, since for each Sylow 2-subgroup $Q$ of $G$, it is $Q \cap N \leqslant \mathrm{Frat}\,(Q)$, by a theorem of Tate (see [6] pag. 431) $M$ is a 2-nilpotent group; but $G/N$ is a 2-group: we conclude that $G$ itself is 2-nilpotent. #

In last lemma of this section we are again concerned with the growth of the minimal number of generators when we go from a proper factor group $G/M$ to $G$. We now deal with the case where $M$ is generated by two conjugate 2-subgroups; by 2.3 this covers non abelian simple minimal normal subgroups.

LEMMA 2.6. *Let $M$ be a normal subgroup of $G$ such that there exist $x \in G$ and $P \in \mathrm{Syl}_2 M$ with $M = \langle P, P^x \rangle$; then:*

i) *if $d(G/M) > d_2(G)$ then $d(G) = d(G/M)$;*

ii) *if $d(G/M) < d_2(G)$ then $d(G) \leqslant d_2(G) + 1$;*

iii) *if $d(G/M) < d_2(G)$ and $d_2(G) > 2$ then $d(G) = d_2(G) + 1$ only if there exists a subgroup $H$ of $G$ such that $H$ is 2-nilpotent, $P$ is a normal subgroup of $H$ and $G = HM$;*

iv) *if $d(G/M) = d_2(G)$ and $d_2(G/M) < d_2(G)$ then $d(G) \leqslant d_2(G) + 1$;*

v) *if $d(G/M) = d_2(G)$, $d_2(G/M) < d_2(G)$ and $d_2(G) > 2$ then $d(G) = d_2(G) + 1$ only if there exist an element $g$ of $G$ and a subgroup $H$ of $G$ such that $H$ is 2-nilpotent, $P$ is a normal subgroup of $H$ and $G = \langle H, g \rangle M$.*

PROOF. If

$$d(G/M) > d_2(G) \geqslant d_2(G/M)\,,$$

by 2.4, there exist $g_1, \dots, g_{d(G/M)}$ in $G/M$ such that $\langle g_1 M, \dots, g_{d(G/M)} M \rangle = G/M$ and the subgroup $\langle g_1 M, \dots, g_{d_2(G)} M \rangle$ contains a Sylow 2-subgroup of $G/M$. By Frattini's argument, it is not restrictive to assume $g_i \in N_G(P)$, $1 \leqslant i \leqslant d(G/M)$. Let $T = \langle g_1, \dots, g_{d_2(G)}, P \rangle$: $P$ is normal in $T$ and $T$ contains a Sylow 2-subgroup of $G$ so by [9] lemma 1.$b$, there exist in $G$ $d_2(G)$ elements $h_1, \ldots, h_{d_2(G)}$ such that $\langle h_1, \ldots, h_{d_2(G)}\rangle = T$. It results

$$\langle h_1, \ldots, h_{d_2(G)}, g_{d_2(G)+1}, \ldots, g_{d(G/M)}x\rangle =$$
$$= \langle g_1, \ldots, g_{d(G/M)-1}, g_{d(G/M)}x, P\rangle = \langle g_1, \ldots, g_{d(G/M)-1}, g_{d(G/M)}x, M\rangle = G\,.$$

This proves (i).

If $d(G/M) < d_2(G)$, again by Frattin's argument, we may suppose that there are in $N_G(P)$ $d(G/M)$ elements, $g_1, \ldots, g_{d(G/M)}$, such that $\langle g_1M, \ldots, g_{d(G/M)}M\rangle = G/M$. Let $H = \langle g_1, \ldots, g_{d(G/M)}, P\rangle$. Clearly $G = HM = \langle H, x\rangle$ and so $d(G) \leqslant d(H) + 1$. Since $d(H/P) < d_2(G) = d_2(H)$, by (2.5, i) $d(H) \leqslant d_2(H)$ and so $d(G) \leqslant d_2(G) + 1$. Furthermore if $d(G) = d_2(G) + 1$ then $d(H) = d_2(G) = d_2(H)$ and so, by (2.5, ii), if we suppose $d_2(G) = d_2(H) > 2$, we conclude that $H$ is 2-nilpotent. This proves ii) and iii).

Finally if $d(G/M) = d_2(G)$ and $d_2(G/M) < d_2(G)$, by 2.4, there exist $d(G/M)$ elements, $g_1, \ldots, g_{d(G/M)}$, such that $\langle g_1M, \ldots, g_{d(G/M)}M\rangle = G/M$ and the subgroup $\langle g_1M, \ldots, g_{d_2(G)-1}M\rangle$ contains a Sylow 2-subgroup of $G/M$. By Frattini's argument we may choose these elements to be in $N_G(P)$. If we now set $H = \langle g_1, \ldots, g_{d_2(G)-1}, P\rangle$ then

$$G = \langle H, g_{d(G/M)}\rangle M = \langle H, g_{d(G/M)}x\rangle\,.$$

It follows $d(G) \leqslant d(H) + 1$. Since $H$ contains a Sylow 2-subgroup of $G$ $d_2(G) = d_2(H)$ and, by (2.5, i) $d(H) \leqslant d_2(H) = d_2(G)$; hence $d(G) \leqslant d_2(G) + 1$. And again $d(G) = d_2(G) + 1$ implies $d(H) = d_2(H)$: if we suppose $d_2(H) > 2$, from (2.5, ii), it follows that $H$ is a 2-nilpotent group. #

REMARK 2.7. The hypothesis $d_2(G/M) < d_2(G)$ that appears in iv) and v) of the previous lemma is verified when $M$ is perfect group; to prove this it is enough to remark that, by Tate's theorem ([6] p. 431), if $P$ is a Sylow 2-subgroup of $G$ then $P \cap M \nleqslant \mathrm{Frat}\,(P)$, otherwise $M$ would be 2-nilpotent. Furthermore if in particular $G/M$ is perfect, and this is the case that we will consider in the next section, then, since a perfect group has even order and its Sylow 2-subgroups are not cyclic, $d_2(G) > d_2(G/M) \geqslant 2$: so in this case also the hypothesis $d_2(G) > 2$ that appears in iii) and v) is satisfied.

We can now prove the main result of this section:

THEOREM 2.8. *If $G$ is a finite group then*

$$d(G) \leqslant \max\left(d_{\mathbf{Z}G}(I_G),\, d_2(G)+1\right).$$

PROOF. – By induction on $|G|$. If $\operatorname{pr}(G)=0$ then $d(G)=d_{\mathbf{Z}G}(I_G)$. Suppose $\operatorname{pr}(G)\neq 0$ and let $N$ be a minimal normal subgroup of $G$. If $N$ is solvable, by 2.1, we obtain $d(G)=d(G/N)\leqslant$ (by induction)$\cdot$ $\cdot\max\left(d_{\mathbf{Z}G/N}(I_{G/N}),\, d_2(G/N)+1\right)\leqslant\max\left(d_{\mathbf{Z}G}(I_G),\, d_2(G)+1\right)$. If $N$ is not solvable then it is a direct product of non abelian simple groups and so, by 2.3, there exist $x\in N$ and $P\in\operatorname{Syl}_2(N)$ such that $N=\langle P, P^x\rangle$. We distinguish two possibilities:

i) $d(G/N)>d_2(G)$: by (2.6 i)

$$d(G)=d(G/N)\leqslant(\text{by induction})\max\left(d_{\mathbf{Z}G/N}(I_{G/N}),\, d_2(G/N)+1\right)\leqslant$$
$$\leqslant\max\left(d_{\mathbf{Z}G}(I_G),\, d_2(G)+1\right).$$

ii) $d(G/N)\leqslant d_2(G)$: as remarked in (2.7) we can apply Lemma 2.6 to conclude that $d(G)\leqslant d_2(G)+1$. $\#$

COROLLARY 2.9. *If $G$ is a finite group with* $\operatorname{pr}(G)\neq 0$ *then* $d(G)\leqslant$ $\leqslant d_2(G)+1$.

The previous results show that in the class of the groups with non zero presentation rank the general bound for the minimal number of generators can be improved but it remains an open question whether the relation « $d(G)\leqslant d_2(G)+1$ » is or not the best that is possible to prove for this class; one of the reasons for which it is not easy to solve this problem is that it is difficult to construct examples of groups with positive presentation rank.

In theorem 2.8 we have described the structure of a group $G$ with $\operatorname{pr}(G)=0$ and such that $d(G)=d+1$. Using the above results we can generalize that theorem proving:

THEOREM 2.10. *Let $G$ be a finite group: if $d(G)=d+1$ then either $d=d_2(G)$ or $G$ contains a normal subgroup $N$ such that $G/N$ is the semidirect product $P\langle x\rangle$ of an elementary abelian $p$-group $P$ of rank $d$ with a cyclic group $\langle x\rangle$ where $x$ acts on $P$ as a non trivial power automorphism.*

PROOF. If $\mathrm{pr}(G) = 0$ then the conclusion follows from th. 1.8. If $\mathrm{pr}(G) \neq 0$ we have $d + 1 = \max_{p||G|} d_p(G) + 1 = d(G) \leqslant$ (by corollary 2.9) $d_2(G) + 1$: hence $d(G) = d_2(G) + 1$ and $d = d_2(G)$. #

## 3. – The case of perfect groups.

In this section we apply the results proved above to the class of perfect groups: in this case we will find a bound for $d(G)$ in terms of $d_2(G)$ and the augmentation ideal $I_G$. This will then imply the bound $d(G) \leqslant d$. We are interested in this class since it represents the opposite of the class of the solvable groups from which the study of the minimal number of generators started and produced the most complete results. But we think that this can be useful also as an example of how one can work in some other class of finite groups.

We begin by studying the augmentation ideal of a perfect group.

LEMMA 3.1. *If $G$ is a finite perfect group and $p$ is a prime dividing $|G|$ then*

$$d_{\mathbf{Z}G}(I_G/pI_G) \leqslant \max\left(2, \frac{d_p(G)}{2} + 1\right).$$

PROOF. By (1.2) $d_{\mathbf{Z}G}(I_G/pI_G) = \max\{d(G/G'G^p), [\![s(M)/r(M) + 1]\!]\}$, where $M$ varies over all non-trivial irreducible $GF(p)$ modules. Since $G$ is perfect $G/G'G^p = 1$, hence there exists a non-trivial irreducible $GF(p)G$-module $M$ with $d_{\mathbf{Z}G}(I_G/pI_G) = [\![s(M)/r(M) + 1]\!]$. We are in the situation described in 1.3 and 1.5.

We distinguish two cases:

*a*) $n(M) = 0$: By (1.5, i) $d(G) \leqslant 2 \leqslant (d_p(G) + 3)/2$ since $d_p(G) \geqslant 1$.

*b*) $n(M) \neq 0$. A Sylow $p$-subgroup of $G/K$ is a semidirect product $M^{n(M)}P$ where $M^{n(M)}$ is a direct product of $G$-isomorphic factors and $P$ is Sylow $p$-subgroup of a complement, $H$ say, of $M^{n(M)}$ in $G/K$. Since $d(M^{n(M)}P) \leqslant d_p(G/K) \leqslant d_p(G)$ it must be $n(M) \leqslant d_p(G)$. On the other hand $n(M) = d_p(G)$ only if $P = 1$ and $M \simeq C_p$ but this would be imply $H$ cyclic in contradiction with $G$ perfect. Therefore $n(M) \leqslant d_p(G) - 1$. By (1.5, iii) $r(M) \neq 1$ otherwise $H$ would be cyclic, but then, from (1.5, iv) it follows

$$d_{\mathbf{Z}G}(I_G/pI_G) = [\![s(M)/r(M) + 1]\!] \leqslant \frac{n(M)}{2} + 2 \leqslant \frac{d_p(G) - 1}{2} + 2 \leqslant \frac{d_p(G) + 3}{2}.$$

To conclude we have to show that the equality

$$(*)\qquad d_{\mathbf{Z}G}(I_G/pI_G)=\frac{d_p(G)+3}{2}$$

can occur only when $d_{\mathbf{Z}G}(I_G/pI_G)\leqslant 2$. If $P=1$, since $M$ cannot be cyclic, $d_p(G)\geqslant d(M^{n(M)})\geqslant 2n(M)=2d_p(G)-2$ whence $d_p(G)\leqslant 2$: but then, by (1,6.1), $d_{\mathbf{Z}G}(I_G/pI_G)\leqslant d_p(G)\leqslant 2$.

If $P\neq 1$, from $d(M^{n(M)}P)\leqslant d_p(G)$ it follows:

*a*) $P$ is a cyclic $p$-group: $P=\langle x\rangle$;

*b*) $M/[M,P]\cong C_p$.

It is $s(M)=n(M)+t(M)$ being $q^{t(M)}=|H^1(H,M)|$. On the other hand

$$H^1(H,M)\leqslant H^1(P,M)\leqslant\{m\in M\colon m(1+x+\ldots+x^{|P|-1})=0\}/[M,P]\leqslant \leqslant M/[M,P]\cong C_p:$$

so either $t(M)=0$ or $t(M)=1$ and in the latter case it is also $p=q$; since

$$[\![s(M)/r(M)+1]\!]=[\![(t(M)+d_p(G)-1/r(M))+1]\!]=\frac{d_p(G)+3}{2}$$

we conclude that either $d_p(G)=2$, but then $d_{\mathbf{Z}G}(I_G/pI_G)\leqslant 2$, or $t(M)=1$ and $r(M)=2$. Therefore the situation that remains to discuss is the following: $M$ is a 2-dimensional vector space over $GF(p)$ and $H$ is an irreducible perfect subgroup of $GL(2,p)$ all of whose Sylow $p$-subgroups are cyclic and non-trivial. $H\leqslant SL(2,p)$ otherwise

$$H/H\cap SL(2)\cong HSL(2,p)/SL(2,p)\cong GL(2,p)/SL(2,p)\cong (GF(p))^*$$

and $H$ would have an abelian quotient in contradiction with the assumption $H$ perfect. Let $\pi$ be the projection of $SL(2,p)$ over $PSL(2,p)$: $H^\pi$ is a perfect subgroup of $PSL(2,p)$; but the perfect subgroups of $PSL(2,p)$ are classified in ([6] p. 213) and from this classification follows that there are only the following possibilities:

1) $H^\pi\cong PSL(2,p)$;

2) $H^\pi\cong A_5$ and $p^2\equiv 1 \bmod 5$.

If (2) is the case then $|H| = |A_5||\ker(\pi)|$ and $|\ker(\pi)| = (2, p-1)$, hence either $|H| = 60$ or $|H| = 120$: in both cases, since $p^2 \equiv 1 \bmod 5$, $p$ does not divide $|H|$ in contradiction with the hypothesis that $H$ has a non-trivial Sylow $p$-subgroup. So (1) holds and consequently $H \cong SL(2, p)$ with $p \neq 2$ becouse it is perfect. We can conclude applying

(**) If $V$ is a $G$-module and $N$ is normal subgroup of $G$ such that $C_V(N) = 0$ then $|H^1(G, V)| \leqslant |H^1(N, V)|$. (see [1] (2.7, 1)).

Applying this result to the case

$$G = SL(2, p), \quad V = M, \quad N = Z(SL(2, p)) \cong C_2,$$

since from $p \neq 2$ it follows $C_M(N) = 0$, we obtain $H^1(SL(2, p), M) \leqslant$ $\leqslant H^1(N, M) = 1$: a contradiction since we have proved before $|H^1(H, M)| = p$.

This conclude the proof. #

An immediate consequence is:

Proposition 3.2. *If $G$ is a finite perfect group then $d_{ZG}(I_G) \leqslant d/2 + 1$.*

The previous result can be stated also in the following way:

Corollary 3.3. *If $G$ is a finite perfect group with* $\operatorname{pr}(G) = 0$ *then $d(G) \leqslant d/2 + 1$.*

We have so proved that in the class of perfect group with zero presentation rank a stronger bound holds for the minimal numbers of generators. For this class it is also true:

Proposition 3.4. *If $G$ is a perfect group with* $\operatorname{pr}(G) = 0$ *then $d(G) \leqslant d$ and the equality $d(G) = d$ holds only if $d(G) = 2$.*

Proof. Since $G$ is perfect $d \geqslant 2$ (a group all of whose Sylow subgroups are cyclic is soluble) and so the conclusion follows from corollary 3.3. #

The previous results hold in the class of the perfect groups with zero presentation rank. Before studying the perfect groups with non zero presentation rank we need some lemma.

Lemma 3.5. *Let $G$ be a perfect group: if $d_{ZG}(I_G) = 2$ then $d(G) \leqslant d_2(G)$.*

PROOF. Let $G$ be a minimal counterexample. Since $G$ is perfect $d_2(G) \geqslant 2$. If $\operatorname{pr}(G) = 0$ then $d(G) = d_{\mathbf{Z}G}(I_G) = 2 \leqslant d_2(G)$. Suppose $\operatorname{pr}(G) \neq 0$: since a simple group has presentation rank equal to zero, $G$ contains a minimal normal subgroup $N$ with $G/N \neq 1$; $d_{\mathbf{Z}G/N}(I_{G/N}) \leqslant d_{\mathbf{Z}G}(I_G) = 2$ and $d_{\mathbf{Z}G/N}(I_{G/N}) \neq 1$ otherwise $G/N$ would be cyclic contrary to the fact that $G$ is perfect: therefore $d_{\mathbf{Z}G/N}(I_{G/N}) = 2$ and so, by induction, $d(G/N) \leqslant d_2(G/N)$. We distinguish two possibilities:

*a*) $N$ is solvable: by (2.1) $d(G) = d(G/N) \leqslant d_2(G/N) \leqslant d_2(G)$.

*b*) $N$ is a direct product of non abelian simple groups: by (2.3) $N = \langle P, P^x \rangle$ for $x \in N$ and $P \in \operatorname{Syl}_2(N)$; as remarked in 2.7 we can applay lemma 2.6 and conclude that either $d(G) \leqslant d_2(G)$ or there exists a 2-nilpotent subgroup $H$ of $G$ with $G = HN$ but in this latter case $G/N$ would be soluble in contradiction with the hypothesis that $G$ is perfect. $\#$

LEMMA 3.6. *If $G$ is a finite perfect group then* $\operatorname{pr}(G) \leqslant d_2(G) - 2$.

PROOF. Since $G$ is perfect $d_2(G) \geqslant 2$; hence if $\operatorname{pr}(G) = 0$ we can conclude $\operatorname{pr}(G) = 0 \leqslant d_2(G) - 2$. Suppose now $\operatorname{pr}(G) \neq 0$. Consider a minimal normal subgroup $N$ of $G$. The two following cases are possible:

*a*) $N$ is solvable: $\operatorname{pr}(G) + d_{\mathbf{Z}G}(I_G) = d(G) =$ (by 2.1) $d(G/N) = d_{\mathbf{Z}G/N}(I_{G/N}) + \operatorname{pr}(G/N)$: hence, from $d_{\mathbf{Z}G/N}(I_{G/N}) \leqslant d_{\mathbf{Z}G}(I_G)$ it follows $\operatorname{pr}(G) \leqslant \operatorname{pr}(G/N) \leqslant$ (by induction) $d_2(G/N) - 2 \leqslant d_2(G) - 2$.

*b*) $N \simeq S^n$ where $S$ in a non-abelian simple group: $N$ does not have abelian composition factors and $G/N$ is not cyclic, so, by (2.2), $d(G) - \operatorname{pr}(G) = d(G/N) - \operatorname{pr}(G/N)$. Since $N$ is the normal closure of $S$ in $G$ and, (by [1] th A) $d(S) = 2$, we obtain $d(G) \leqslant d(G/N) + 2$: it follows that it is also $\operatorname{pr}(G) \leqslant \operatorname{pr}(G/N) + 2$. Suppose, by contradiction, $\operatorname{pr}(G) \geqslant d_2(G) - 1$. Since $d_{\mathbf{Z}G}(I_G) = 1$ implies $G$ cyclic and if $d_{\mathbf{Z}G}(I_G) = 2$ then by 3.5 $d(G) \leqslant d_2(G)$ whence

$$\operatorname{pr}(G) = d(G) - d_{\mathbf{Z}G}(I_G) = d(G) - 2 \leqslant d_2(G) - 2 \, ,$$

we may assume $d_{\mathbf{Z}G}(I_G) \geqslant 3$. Therefore

$$d(G) = \operatorname{pr}(G) + d_{\mathbf{Z}G}(I_G) \geqslant d_2(G) - 1 + 3 = d_2(G) + 2$$

and so $d(G/N) \geqslant d(G) - 2 \geqslant d_2(G)$. If $d(G/N) > d_2(G)$, by (2.6, i) $d(G) =$

$= d(G/N)$: it follows $\mathrm{pr}(G) = \mathrm{pr}(G/N) \leqslant$ (by induction) $\leqslant d_2(G/N) - 2 \leqslant d_2(G) - 2$, a contradiction. If $d(G/N) = d_2(G)$, by (2.6, iv) $d(G) \leqslant d_2(G) + 1$ contrary to the relation $d(G) \geqslant d_2(G) + 2$ obtained before. $\#$

LEMMA 3.7. *If $G$ is a finite perfect group and 2 is an augmentation prime for $G$, that is $d_{\mathbf{Z}G}(I_G) = d_{\mathbf{Z}G}(I_G/2I_G)$, then $d(G) \leqslant d_2(G)$.*

PROOF. Let $G$ be a minimal counterexample. If $\mathrm{pr}(G) = 0$ then $d(G) = d_{\mathbf{Z}G}(I_G) = d_{\mathbf{Z}G}(I_G/2I_G) \leqslant d_2(G)$ by (1.6, 2). So we may assume $\mathrm{pr}(G) > 0$. Since $G$ is perfect by (1.2) there exists a nontrivial irreducible $GF(2)G$-module, $M$, such that

$$d_{\mathbf{Z}G}(I_G) = d_{\mathbf{Z}G}(I_G/2I_G) = [\![ s(M)/r(M) + 1 ]\!] .$$

Let $K$ be the normal subgroup of $G$ introduced in (1.3) and let $H$ be a complement for $M^{n(M)}$ in $G/K$. We distinguish two cases:

*a*) $K \neq 1$.

Let $N$ be a minimal normal subgroup of $G$ contained in $K$: we have

$$d_{\mathbf{Z}G/N}(I_{G/N}/2I_{G/N}) = d_{\mathbf{Z}G}(I_G/2I_G) = d_{\mathbf{Z}G}(I_G) \geqslant d_{\mathbf{Z}G/N}(I_{G/N})$$

hence 2 is an augmentation prime for $G/N$ and by induction $d(G/N) \leqslant d_2(G/N)$.

There are two possibilities:

i) $N$ is soluble: since $\mathrm{pr}(G) > 0$, by (2.1) $d(G) = d(G/N) \leqslant d_2(G/N) \leqslant d_2(G)$.

ii) $N$ is a direct product of non-abelian simple groups: so, as remarked in 2.7, we can apply (2.6, iii) to conclude that either $d(G) \leqslant d_2(G)$ or $G = HN$ with $H$ 2-nilpotent: but we exclude the latter case since it is in contradiction with the assumption that $G$ is perfect.

*b*) $K = 1$.

$G = M^{n(M)}H$ where $M^{n(M)}$ is a direct product of $H$-invariant factors and $C_H(M) = 1$. Since $\mathrm{pr}(G) \neq 0$,

$$d(G) = d(G/M^{n(M)}) = d(H) = d_{\mathbf{Z}H}(I_H) + \mathrm{pr}(H);$$

by 3.6 $\mathrm{pr}(H) \leqslant d_2(H) - 2$. Furthermore $d_2(H) + n(M) \leqslant d_2(G)$; hence $d(G) = d(H) \leqslant d_{ZH}(I_H) + d_2(H) - 2 \leqslant d_{ZG}(I_G) + d_2(G) - n(M) - 2$.

If $n(M) = 0$, by (1.5, i) $d_{ZG}(I_G) \leqslant 2$: it follows $d(G) \leqslant d_2(G)$.

If $n(M) \neq 0$, by (1.5, ii) $d_{ZG}(I_G) \leqslant n(M) + 1$ and so $d(G) \leqslant n(M) + 1 + d_2(G) - n(M) - 2 \leqslant d_2(G) - 1$. #

With the help of the previous lemmas we can now prove the following bound for the minimal number of generators of a perfect group.

THEOREM 3.8. *If $G$ is a finite perfect group then*

$$d(G) \leqslant \max\left(d_{ZG}(I_G) + 1, d_2(G)\right) .$$

PROOF. Let $G$ be a minimal counterexample. Since $d(G) = d_{ZG}(I_G) + \mathrm{pr}(G)$, if $\mathrm{pr}(G) \leqslant 1$ then $d(G) \leqslant d_{ZG}(I_G) + 1$: therefore we may assume $\mathrm{pr}(G) \geqslant 2$. Furthermore it is not restrictive to suppose that $G$ has no solvable proper normal subgroup: otherwise, if $N$ is one, by (2.1)

$$d(G) = d(G/N) \leqslant (\text{by induction}) \max\left(d_{ZG/N}(I_{G/N}) + 1, d_2(G/N)\right) \leqslant \leqslant \max\left(d_{ZG}(I_G) + 1, d_2(G)\right) .$$

Denote with $M$ the socle of $G$: $M \cong \prod_{i=1}^{m} N_i$ where $N_i$, a minimal normal subgroup of $G$, is isomorphic to $\prod_{j=1}^{m_i} S_{i,j}$ with $S_{i,j} \cong S_i$ for all $j$, where $S_i$ is a finite non-abelian simple group and $\{S_{i,j}, 1 \leqslant j \leqslant m_i\}$ is a conjugacy class of subgroups of $G$.

Every non abelian simple group can be generated by 2 elements ([1], th. B), and has a non cyclic Sylow 2-subgroup, so

$$d(M) \leqslant \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} d(S_{i,j}) \leqslant \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} d_2(S_{i,j}) \leqslant d_2(M):$$

hence we may assume $G \neq M$. By (2.3) $M = \langle P, P^x \rangle$ where $x \in M$ and $P \in \mathrm{Syl}_2(M)$ and as remarked in (2.7) $d_2(G) > 2$.

We distinguish the different possibilities:

1) $d(G/M) > d_2(G)$: by (2.6)

$$d(G) = d(G/M) \leqslant (\text{by induction}) \max\left(d_{ZG/M}(I_{G/M}) + 1, d_2(G/M)\right) \leqslant \leqslant \max\left(d_{ZG}(I_G) + 1, d_2(G)\right) .$$

2) $d(G/M) < d_2(G)$: by (2.6, iii), since a perfect group $G$ cannot be written in the form $HM$ with $H$ 2-nilpotent, we get $d(G) \leqslant d_2(G)$.

3) $d(G/M) = d_2(G)$: by (2.6, v) either $d(G) \leqslant d_2(G)$ or $d(G) = d_2(G) + 1$ and in this latter case there exist $g \in G$ and $H$, a 2-nilpotent subgroup of $G$, such that $P$ is a normal subgroup of $H$ and $G = \langle H, g \rangle M$. Suppose, by contradiction, that this may happen: define $L = \bigcap_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} N_G(S_{i,j})$: since for each $g$ in $G$, $(S_{i,j})^g = S_{i,j^*}$ for a suitable $j^*$, $L$ is a normal subgroup.

We prove now that $C_G(P) \leqslant L$: $P = \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} P_{i,j}$ with $P_{i,j} \in \mathrm{Syl}_2\,(S_{i,j})$; if $x \notin L$ then there exists $S_{i,j}$ such that $(S_{i,j})^x \neq S_{i,j}$ but then $(S_{i,j})^x \cap S_{i,j} = 1$ and consequently also $(P_{i,j})^x \cap P_{i,j} = 1$ and so $x \notin C_G(P)$.

Let $F$ be a 2-normal complement of $H$: since both $F$ and $P$ are normal in $H$, $[P, F] \leqslant P \cap F = 1$. Therefore $F \leqslant C_G(P) \leqslant L$. But then, since $H = FQ$ for $Q \in \mathrm{Syl}_2\,(H)$, we get $H \leqslant LQ$; furthermore from $G = M\langle H, g \rangle$ and $M \leqslant L$ it follows $G = \langle Q, g \rangle L$.

By 2.2 $d(G) - \mathrm{pr}\,(G) = d(G/M) - \mathrm{pr}\,(G/M)$: hence $d_2(G) + 1 - \mathrm{pr}\,(G) = d_2(G) - \mathrm{pr}\,(G/M)$ and so $\mathrm{pr}\,(G) = \mathrm{pr}\,(G/M) + 1$; since we are assuming $\mathrm{pr}\,(G) \geqslant 2$, we have that $L/M$ is a normal subgroup of a group $G/M$ whose presentation rank is not zero; furthermore the natural homomorphism from $L$ on the group $\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} \mathrm{Out}\,(S_{i,j})$ has kernel equal to $M$ and so $L/M$ is a subgroup of $\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} \mathrm{Out}\,(S_{i,j})$; since $\mathrm{Out}\,(S)$ is solvable if $S$ is a finite simple group, we conclude that also $L/M$ is solvable. So we can apply (2.1) to obtain

$$d\left(\frac{G}{M}\right) = d\left(\frac{G/M}{L/M}\right) = d\left(\frac{G}{L}\right).$$

Two cases are to be considered:

i) suppose $d(G/L, QL/L) \leqslant d(G/L) - \mathrm{pr}\,(G/L) - 2$: by ([8], 5.1) the set of the augmentation primes for $G/L$ is contained in the set of the primes dividing $QL/L$: since $QL/L$ is a 2-group we get that 2 is an augmentation prime for $G/L$ and then, by 3.7, $d(G/M) = d(G/L) \leqslant d_2(G/L) <$ (by 2.7) $d_2(G)$: this leads to a contradiction since we are supposing $d(G/L) = d(G/M) = d_2(G)$.

ii) suppose $d(G/L, QL/L) > d(G/L) - \mathrm{pr}\,(G/L) - 2$: since $G = \langle Q, g\rangle L$ and $d(G/L) = d_2(G)$ we have: $1 \geqslant d(G/L, QL/L) > d_2(G) - \mathrm{pr}\,(G/L) - 2$ and so $\mathrm{pr}\,(G/L) > d_2(G) - 3$. By (3.6) $\mathrm{pr}\,(G/L) \leqslant d_2(G/L) - 2$ and then $d_2(G) - 3 < \mathrm{pr}\,(G/L) \leqslant d_2(G/L) - 2$ implies $d_2(G) < d_2(G/L) + 1$ hence $d_2(G) \leqslant d_2(G/L)$, in contradiction with 2.7. #

COROLLARY 3.9. *If $G$ is a finite perfect group and* $d = \max\limits_{p||G|} d_p(G)$ *then* $d(G) \leqslant \max\,(d/2 + 2, d_2(G))$.

PROOF. It follows immediately from the previous theorem and remembering that, by 3.2, $d_{ZG}(I_G) + 1 \leqslant d/2 + 2$. #

COROLLARY 3.10. *If $G$ is a finite perfect group with* $\mathrm{pr}\,(G) \geqslant 2$ then $d(G) \leqslant d_2(G)$.

PROOF. It is enough to remark that $d(G) \leqslant \max\,(d_{ZG}(I_G) + 1, d_2(G))$ and that $d(G) = d_{ZG}(I_G) + \mathrm{pr}\,(G) \geqslant d_{ZG}(I_G) + 2$. #

COROLLARY 3.11. *If $G$ is a finite perfect group and* $d = \max\limits_{p||G|} d_p(G)$ *then* $d(G) \leqslant d$.

PROOF. If $\mathrm{pr}\,(G) \geqslant 2$, by corollary 3.10 $d(G) \leqslant d_2(G) \leqslant d$ and if $\mathrm{pr}\,(G) = 0$ the conclusion follows by proposition 3.4. Therefore we may assume $\mathrm{pr}\,(G) = 1$ and complete the proof by induction: let $N$ be a minimal normal subgroup of $G$; since $\mathrm{pr}\,(G) \neq 0$ $G$ is not a simple group and so $N \neq G$. If $N$ is soluble (by 2.1) $d(G) = d(G/N) \leqslant d$ by induction. If $N$ is a direct product of non abelian simple groups, by 2.2, $d(G) - \mathrm{pr}\,(G) = d(G/N) - \mathrm{pr}\,(G/N)$. If $\mathrm{pr}\,(G/N) = 1$ then $d(G) = d(G/N) \leqslant d$ by induction. If $\mathrm{pr}\,(G/N) = 0$ then $d(G) = d(G/N) + 1$: in the case $d(G/N) < d$ we can conclude $d(G) \leqslant d$, while if $d(G/N) = d$ then, from

$$d = d(G/N) \leqslant \max_{p||G/N|} d_p(G/N) \leqslant \max_{p||G|} d_p(G) \leqslant d\,,$$

if follows

$$d = d(G/N) = \max_{p||G/N|} d_p(G/N)\,,$$

hence by 3.4 it must be $d = 2$: but this latter case leads to a contradiction since $d \geqslant d_2(G) \geqslant 3$. #

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