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Parametrization of Carathéodory multifunctions

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Parametrization of Carathéodory Multifunctions.

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1. Introduction.

Let $F: X \to \mathbb{R}^n$ be a multifunction which is Lipschitz with constant l and has values $F(x)$ bounded by m. We show that $\operatorname{co} F(x)$ can be represented as $f(x, U)$, with U the unit closed ball in $\mathbb{R}^n$ and f Lipschitz with constant $6n(2l + m)$. Existing representations were: either with U the unit closed ball in $\mathbb{R}^n$ but f just continuous in (x, u) (Ekeland-Valadier [3]); or with f Lipschitz in (x, u) but U in some infinite dimensional space (LeDonne-Marchi [6]).

More generally, let $F: I \times X \to \mathbb{R}^n$ be a multifunction with $F(\cdot, x)$ measurable and $F(t, \cdot)$ uniformly continuous. We show that $\operatorname{co} F(t, x)$ can be represented as $f(t, x, U)$, where U is either the unit closed ball in $\mathbb{R}^n$ (in case the values $F(t, x)$ are compact) or $U = \mathbb{R}^n$ (in case the values $F(t, x)$ are unbounded). As to f, we obtain $f(\cdot, x, u)$ measurable and $f(t, \cdot, \cdot)$ uniformly continuous (with modulus of continuity equal to that of $F(t, \cdot)$ multiplied by a constant).

A consequence of this is that differential inclusions in $\mathbb{R}^n$ with convex valued multifunctions, continuous in x, do not generalize differential equations with control in $\mathbb{R}^n$. In fact, consider the Cauchy problem in $\mathbb{R}^n$

$$(CP) \qquad x' \in \operatorname{co} F(t, x) \quad \text{a.e. on } I\,, \qquad x(0) = \xi\,,$$

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with $F(t, x)$ measurable in t and continuous in x. As above we can construct a function $f(t, x, u)$ and a convex closed set U in $\mathbb{R}^n$ such that $\operatorname{co} F(t, x) = f(t, x, U)$. Moreover U is compact provided the values $F(t, x)$ are compact, and $f(t, \cdot, u)$ is Lipschitz provided $F(t, \cdot)$ is Lipschitz. Finally by an implicit function lemma of the Filippov type we show that any solution of (CP) also solves the differential equation with control in $\mathbb{R}^n$:

$$(CDE) \qquad x' = f(t, x, u) \quad \text{a.e. on } I\,, \qquad x(0) = \xi\,, \qquad u(t) \in U\,.$$

Reduction of differential inclusions in $\mathbb{R}^n$ (with continuous convex-valued multifunctions) to control differential equations was known, but the regularity conditions were not completely satisfactory. Namely, either f was non-Lipschitz for Lipschitz F (Ekeland-Valadier [3]) or U was infinite dimensional (LeDonne-Marchi [6] or Lojasiewicz-Plis-Suarez [8] added to Ioffe [5]).

General information on multifunctions and differential inclusions can be found in [1].

2. Assumptions.

Let I be a Lebesgue measurable set in $\mathbb{R}^n$ (or, more generally, a separable metrizable space together with a σ-algebra $\mathcal{A}$ which is the completion of the Borel σ-algebra of I relative to a locally finite positive measure μ). Let X be an open or closed set in $\mathbb{R}^n$ (or, more generally, a separable space metrizable complete, with a distance d and Borel σ-algebra $\mathcal{B}$). We consider multifunctions F with values $F(t, x)$ either bounded by a linear growth condition—hypothesis (FLB)—or unbounded—hypothesis (FU).

Hypothesis (FLB). $F: I \times X \to \mathbb{R}^n$ is a multifunction with:

(*a*) values $F(t, x)$ compact;

(*b*) $F(\cdot, x)$ measurable;

(*c*) $\exists \alpha, m: I \to \mathbb{R}^+$ measurable such that

$$y \in F(t, x) \Rightarrow |y| \leqslant \alpha(t)|x| + m(t) \qquad \text{for a.e. } t;$$

(*d*) X is compact, I is σ-compact, $F(t,\cdot)$ is continuous for a.e. t.

HYPOTHESIS (FU). $F\colon I\times X\to\mathbb{R}^n$ is a multifunction with:

(a') values $F(t,x)$ closed;

(b') $F(\cdot,x)$ measurable;

(d') $\exists w\colon I\times\mathbb{R}^+\to\mathbb{R}^+$ such that: $dl(F(t,x),F(t,\boldsymbol{x}))\leqslant w(t,d(x,\boldsymbol{x}))$, with $w(\cdot,r)$ measurable, $w(t,\cdot)$ continuous concave, $w(t,0)=0$ for a.e. t.

We denote by $\operatorname{co}F$ the multifunction such that each value $\operatorname{co}F(t,x)$ is the closed convex hull of $F(t,x)$. It is well known that $\operatorname{co}F$ verifies hypothesis (FLB) or (FU) provided F does (see [4]).

PROPOSITION 1. Let F verify hypothesis (FLB).
Then F verifies hypthesis (FU) also, namely it verifies (d') with

$$w(t,r)\leqslant 2\alpha(t)r+2m(t)\,.$$

3. Parametrization of multifunctions.

THEOREM 1. Let F verify hypothesis (FU). Suppose moreover that each value $F(t,x)$ is compact, and set

$$M(t,x):=\max\{1,\ |y|\colon y\in F(t,x)\}\,.$$

Then there exists a function $f\colon I\times X\times U\to\mathbb{R}^n$, with U the unit closed ball in $\mathbb{R}^n$, such that:

(i) $\operatorname{co}F(t,x)=f(t,x,U)\ \forall x$ for a.e. t;

(ii) $f(\cdot,x,u)$ is measurable;

(iii) $|f(t,x,u)-f(t,\boldsymbol{x},\boldsymbol{u})|\leqslant 12n\ \ w(t,d(x,\boldsymbol{x}))+6n\ \ M(t,x)|u-\boldsymbol{u}|$ for a.e. t.

If moreover F,w are jointly continuous then f is continuous.

COROLLARY 1. – Let F verify hypothesis (FU).

Let U be a convex closed set in $\mathbb{R}^n$ and let $h: I \times X \times U \to \mathbb{R}^n$ verify:

(α) co $F(t, x) \subset h(t, x, U)$ $\forall x$ for a.e. t;

(β) $u \mapsto h(t, x, u)$ has inverse $h^{-1}(t, x, \cdot)$: $h(t, x, u) \mapsto u$ $\forall x, u$ for a.e. t;

(γ) $h(\cdot, x, u)$ and $h^{-1}(\cdot, x, u)$ are measurable;

(δ) $h(t, \cdot, \cdot)$ and $h^{-1}(t, \cdot, \cdot)$ are jointly continuous for a.e. t.

Then there exists a function $f: I \times X \times U \to \mathbb{R}^n$ such that (i), (ii) of Th. 1 hold and:

(iii′) $|f(t, x, u) - f(t, \boldsymbol{x}, \boldsymbol{u})| \leqslant 6nw(t, d(x, \boldsymbol{x})) + 6n\,|h(t, x, u) - h(t, \boldsymbol{x}, \boldsymbol{u})|$ a.e..

COROLLARY 2. Let F verify hypothesis (FU).

Then, setting $h(t, x, u) = u$ in Corollary 1, the conclusions of Theorem 1 hold with $U = \mathbb{R}^n$ and $M(t, x) \equiv 1$. (The final part provided F is jointly h-continuous.)

THEOREM 2. Let F verify hypothesis (FU) and let I be σ-compact Then there exists a σ-compact set E in a Banach space, a function $\varphi: X \times E \to \mathbb{R}^n$ and a multifunction $\mathcal{U}: I \to E$ such that:

(i) co $F(t, x) = \varphi(x, \mathcal{U}(t))$ $\forall x$ for a.e. t;

(ii) $\mathcal{U}(\cdot)$ is measurable with convex closed values;

(iii) $\varphi(x, \cdot)$ is linear nonexpansive;

(iv) $|\varphi(x, u) - \varphi(\boldsymbol{x}, u)| \leqslant 6nw(t, d(x, \boldsymbol{x}))$, $\forall u \in \mathcal{U}(t)$ for a.e. t.

If moreover F is integrably bounded then the values $\mathcal{U}(t)$ are compact for a.e. t.

4. Intermediate results and proofs.

PROOF OF PROPOSITION 1. Apply the Scorza-Dragoni property in 1.2 (ii) to obtain a sequence (I_k) of compact disjoint sets such that $I = I_0 \cup \mathcal{N}$, $\mathcal{N}$ is a null set, $I_0 = \cup I_k$, and $F_k := \text{co}\, F|_{I_k \times X}$, $\alpha|_{I_k}$, $m|_{I_k}$

are continuous. Set $\alpha_k := \max \alpha|_{I_k}$, $m_k := \max m|_{I_k}$ and:

$$v_k(r) := \sup \left\{ dl(F_k(t, x), F_k(t, \boldsymbol{x})) : t \in I_k, \; |x - \boldsymbol{x}| \leqslant r \right\}.$$

It is clear that $v_k(\cdot)$ is nondecreasing and $v_k(r) \leqslant 2\alpha_k r + 2m_k$. Since I_k, X are compact and F_k is jointly h-continuous, we must have $v_k(r) \to 0$ as $r \to 0$, otherwise a contradiction would follow. By a lemma of McShane [9], there exists a continuous concave function $w_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $w_k(0) = 0$, $w_k(r) \geqslant v_k(r)$, hence

$$\mathrm{dl}\,(F_k(t, x), F_k(t, \boldsymbol{x})) \leqslant w_k(|x - \boldsymbol{x}|) \qquad \forall t \in I_k .$$

Set

$$w(t, r) := \min \{w_k(r), 2\alpha(t) r + 2m(t)\} \qquad \text{for } t \in I_k ,$$

$$w(t, r) := 2m(t) + 2\alpha(t) r \qquad \text{for } t \in \mathcal{N} . \qquad \blacklozenge$$

Lemma 1. Let $\mathcal{K}$ be any family of nonempty closed convex sets in $\mathbb{R}^n$ such that $\mathrm{dl}\,(K, \boldsymbol{K}) < \infty$ $\forall K, \boldsymbol{K}$ in $\mathcal{K}$. Let $B(y, K)$ be the closed ball around y with radius $r(y, K) := \sqrt{3}\, d(y, K)$.

Then the map

$$P : \mathbb{R}^n \times \mathcal{K} \to \mathcal{K}, \qquad P(y, K) := K \cap B(y, K)$$

is well defined, verifies $P(y, K) = \{y\}$ whenever $y \in K$, and:

$$\mathrm{dl}\,(P(y, K), P(\boldsymbol{y}, \boldsymbol{K})) \leqslant 3\, \mathrm{dl}\,(K, \boldsymbol{K}) + [1 + \sqrt{3}]\, |y - \boldsymbol{y}| .$$

Remark. This lemma refines and simplifies the construction of LeDonne-Marchi. We have changed the expansion constant from 2 to $\sqrt{3}$ in the definition of the radius r because we believe this value to be the best possible. More precisely, we believe that the Lipschitz constant 3 for the above intersection cannot be improved, and that it is not obtainable unless one uses the expansion constant $\sqrt{3}$.

Moreover, in the definition of the radius r we do not use the Hausdorff distance between two sets, as LeDonne-Marchi, but rather the distance from a point to a set. This is not only conceptually simpler but also seems better fitted for applications (as in Theorem 1).

Proof.

(a) First we fix y_* in $\mathbb{R}^n$ and prove that

$$\mathrm{dl}\,(P(y_*, K), P(y_*, \boldsymbol{K})) \leqslant 3\, \mathrm{dl}\,(K, \boldsymbol{K}) \qquad \forall K, \boldsymbol{K} \in \mathcal{K} .$$

Choose any $K, \boldsymbol{K}$ in $\mathcal{K}$ and any $\boldsymbol{y} \in P(y_*, \boldsymbol{K})$. Set $\varepsilon_* := d(y_*, K)$, $\boldsymbol{\varepsilon} := \mathrm{dl}\,(\boldsymbol{K}, K)$. We may suppose that ε_*, $\boldsymbol{\varepsilon} > 0$, otherwise just take $y := y_*$, $\boldsymbol{y}$ respectively. To prove the above inequality we need only find a point y in $P(y_*, K)$ such that $|y - \boldsymbol{y}| \leqslant 3\boldsymbol{\varepsilon}$.

To find y, choose points y_1, y_2 in K such that

$$|y_* - y_1| \leqslant \varepsilon_* , \qquad |y_2 - \boldsymbol{y}| \leqslant \boldsymbol{\varepsilon} .$$

If $|y_* - y_2| \geqslant \sqrt{3}\,\varepsilon_*$ then take $y := y_2$. Otherwise $y_2 \notin P(y_*, K)$; but in the segment $]y_1, y_2[$ certainly there exists some point y such that $|y_* - y| = \sqrt{3}\,\varepsilon_*$, hence $y \in P(y_*, K)$. If $|y - \boldsymbol{y}| \leqslant 3\boldsymbol{\varepsilon}$ then (a) is proved. Otherwise by the claim below we have

$$|y_* - \boldsymbol{y}| = |y_* - z| + |z - \boldsymbol{y}| > \sqrt{3}\,(\varepsilon_* + \boldsymbol{\varepsilon}) .$$

But this is absurd because $\boldsymbol{y} \in P(y_*, \boldsymbol{K})$ hence

$$|y_* - \boldsymbol{y}| \leqslant \sqrt{3}\, d(y_*, \boldsymbol{K}) \leqslant \sqrt{3}\,(\varepsilon_* + \boldsymbol{\varepsilon}) .$$

Therefore (a) is proved.

Trigonometrical *claim*: If $|y - \boldsymbol{y}| > 3\boldsymbol{\varepsilon}$ then $\exists z \in]y_*, \boldsymbol{y}[$ such that:

$$|y_* - z| > \sqrt{3}\,\varepsilon_* \quad \text{and} \quad |z - \boldsymbol{y}| > \sqrt{3}\,\boldsymbol{\varepsilon} .$$

In fact, as we prove below, in the triangle $\boldsymbol{y}, y, y_*$ the angle $\theta + \pi/2$ at y verifies $\operatorname{sen} \theta > 1/\sqrt{3}$, in particular $\theta > 0$. Therefore in the segment $]y_*, \boldsymbol{y}[$ certainly there exists a point z such that in the triangle y_*, y, z the angle at y is $\pi/2$. This implies that $|y_* - z| > |y_* - y| = \sqrt{3}\,\varepsilon_*$, and since

$$1/\sqrt{3} < \operatorname{sen} \theta \leqslant |z - \boldsymbol{y}|/|y - \boldsymbol{y}| < |z - \boldsymbol{y}|/(3\boldsymbol{\varepsilon}) ,$$

we have $|z - \boldsymbol{y}| > \sqrt{3}\,\boldsymbol{\varepsilon}$.

To prove $\operatorname{sen} \theta > 1/\sqrt{3}$, set

$$0 < \beta_0 := \operatorname{arcsen} 1/3 < \pi/6 < \alpha_0 := \operatorname{arcsen} 1/\sqrt{3} < \pi/4$$

and notice that we only need to show that $\theta > \alpha_0$. Since $\pi - \alpha_0 - \beta_0 = \alpha_0 + \pi/2$, it is enough to prove that $\theta + \pi/2 > \pi - \alpha_0 - \beta_0$. To prove this notice that in the triangle y_*, y, y_1 the angle α at y verifies $\operatorname{sen}\alpha \leqslant \varepsilon_*/(\sqrt{3}\,\varepsilon_*) = 1/\sqrt{3}$, hence $\alpha \leqslant \alpha_0$. In fact we must have $0 \leqslant \alpha \leqslant \alpha_0$ and not $\pi - \alpha_0 \leqslant \alpha \leqslant \pi$ because the later is incompatible with the fact that the angle α has an adjacent side which is larger that the opposite side. Similarly, in the triangle $\mathbf{y}, y, y_2$ the angle β at y verifies $\operatorname{sen}\beta < \boldsymbol{\varepsilon}/3\boldsymbol{\varepsilon} = 1/3$, hence $\beta < \beta_0$. In fact we must have $0 \leqslant \beta < \beta_0$ inside the claim and not $\pi - \beta_0 < \beta \leqslant \pi$ because the later would imply $\beta \geqslant \pi/2$ hence $|\mathbf{y} - y| \leqslant |\mathbf{y} - y_2| \leqslant \boldsymbol{\varepsilon}$. Finally, to show that $\theta + \pi/2 > \pi - \alpha_0 - \beta_0$, we distinguish the following possibilities:

(i) let $\mathbf{y}$ be in the y_*, y_1, y_2-plane, in the same side of the y_1, y_2-line as y_*; then the inequality $\theta + \pi/2 = \pi - \alpha - \beta > \pi - \alpha_0 - \beta_0$ is obvious;

(ii) let $\mathbf{y}$ be in the y_*, y_1, y_2-plane, in the side of the y_1, y_2-line opposite to y_*, and let $0 \leqslant \beta \leqslant \alpha$; then $\theta + \pi/2 = \pi - \alpha + \beta > \pi - \alpha - \beta > \pi - \alpha_0 - \beta_0$;

(iii) as in (ii) but with $\alpha \leqslant \beta < \beta_0$; then $\theta + \pi/2 = \pi - \beta + \alpha > \pi - \alpha_0 - \beta_0$;

(iv) let $\mathbf{y}$ be outside the y_*, y_1, y_2-plane and let the projection $\mathbf{y}'$ of $\mathbf{y}$ onto that plane fall in the side of the y_1, y_2-line opposite to y_* and let the angle β', projection of the angle β on that plane, verify $0 \leqslant \beta' \leqslant \alpha$; then $\theta + \pi/2 > \pi - \alpha_0 > \pi - \alpha_0 - \beta_0$;

(v) as in (iv) but $\alpha \leqslant \beta' < \beta_0$; then $\theta + \pi/2 \geqslant \pi - \beta' - \alpha > \pi - \alpha_0 - \beta_0$;

(vi) as in (iv) but $\mathbf{y}'$ in the same side as y_*; then it is clear that the situation is similar to that in (i), the difference being that $\theta + \pi/2 > \pi - \alpha - \beta$.

This proves the claim.

(*b*) Now consider points $y, \mathbf{y}$ in $\mathbb{R}^n$ and sets $K, \mathbf{K}$ in $\mathcal{K}$. Setting $\varepsilon := \sqrt{3}\,d(y, K)$, $\boldsymbol{\varepsilon} := \sqrt{3}\,d(\mathbf{y}, \mathbf{K})$, and using (*a*) one obtains:

$$\begin{aligned}\mathrm{dl}\,(P(y, K), P(\mathbf{y}, \mathbf{K})) &\leqslant \mathrm{dl}\,(P(y, K), P(\mathbf{y}, K)) + \mathrm{dl}\,(P(\mathbf{y}, K), P(\mathbf{y}, \mathbf{K})) \leqslant \\ &\leqslant \mathrm{dl}\,(B(y, \varepsilon), B(\mathbf{y}, \boldsymbol{\varepsilon})) + 3\,\mathrm{dl}\,(K, \mathbf{K}) \leqslant |y - \mathbf{y}| + |\varepsilon - \boldsymbol{\varepsilon}| + \\ &+ 3\,\mathrm{dl}\,(K, \mathbf{K}) \leqslant |y - \mathbf{y}| + \sqrt{3}\,|y - \mathbf{y}| + 3\,\mathrm{dl}\,(K, \mathbf{K}). \quad \blacklozenge\end{aligned}$$

To prove Theorem 1 we need the following result:

PROPOSITION 2 (Bressan [2]). Denote by $\mathcal{K}^n$ the family of non-empty compact convex sets in $\mathbb{R}^n$. Then there exists a map $\sigma\colon \mathcal{K}^n \to \mathbb{R}^n$ that selects a point $\sigma(K) \in K$ for each K and verifies:

$$|\sigma(K) - \sigma(\mathbf{K})| \leqslant 2n \operatorname{dl}(K, \mathbf{K}) .$$

PROOF OF THEOREM 1. Clearly $M(\cdot, x)$ is measurable and

$$|M(t, x) - M(t, \boldsymbol{x})| \leqslant w(t, d(x, \boldsymbol{x})) .$$

Consider the function $h\colon I \times X \times U \to \mathbb{R}^n$, $h(t, x, u) := M(t, x)u$.

Clearly $h(t, x, \cdot)$ is an homeomorphism between the ball U and the ball of radius $M(t, x)$; let $h^{-1}(t, x, y) := M(t, x)^{-1}y$ be the inverse homeomorphism.

Project now $h(t, x, u)$ into $\operatorname{co} F(t, x)$, i.e. set

$$f(t, x, u) := \sigma \circ P[h(t, x, u), \operatorname{co} F(t, x)] ,$$

where σ is the selection in Proposition 2 and P is the multivalued projection in Lemma 2.

Claim. $f(\cdot, x, u)$ is measurable.

To prove this, notice first that $M_0(\cdot)$ is measurable by Himmelberg [4, Theorem 5.8]. Then $M(\cdot, x)$ and $h(\cdot, x, u)$ are measurable. Consider the closed ball $B(\cdot, x, u)$ of radius

$$r(\cdot, x, u) := \sqrt{3}\, d(h(\cdot, x, u), \operatorname{co} F(\cdot, x))$$

around $h(\cdot, x, u)$. Then $r(\cdot, x, u)$ is measurable by Himmelberg [4, Theorem 3.5, Theorem 6.5], and since

$$d(y, B(\cdot, x, u)) = (|y - h(\cdot, x, u)| - r(\cdot, x, u))^+$$

by Himmelberg [4, Theorem 3.5, Theorem 4.1], $B(\cdot, x, u)$ and its intersection with $\operatorname{co} F(\cdot, x)$ are measurable. Therefore this intersection is a measurable map: $I \to \mathcal{K}^n$; and since $\sigma\colon \mathcal{K}^n \to \mathbb{R}^n$ is continuous, $f(\cdot, x, u)$ is measurable.

It is easy to prove (iii) using the Lipschitz properties of σ and P:

$$\begin{aligned}|f(t, x, u) - f(t, \boldsymbol{x}, \boldsymbol{u})| &\leqslant 6n|M(t, x)u - M(t, x)\boldsymbol{u}| + \\ &+ 6n|M(t, x)\,\boldsymbol{u} - M(t, \boldsymbol{x})\,\boldsymbol{u}| + 6nw(t, d(x, \boldsymbol{x})) \leqslant \\ &\leqslant 6nM(t, x)|u - \boldsymbol{u}| + 6n|M(t, x) - M(t, \boldsymbol{x})| + 6nw(t, d(x, \boldsymbol{x})) \leqslant \\ &\leqslant 12nw(t, d(x, \boldsymbol{x})) + 6nM(t, x)|u - \boldsymbol{u}|\,.\end{aligned}$$

It is clear that if F is jointly h-continuous then $M_0(\cdot)$ is continuous; and if also w is jointly continuous then M is jointly continuous hence h is jointly continuous. Then the ball B is continuous and its intersection with $\operatorname{co} F$ is continuous, by the h-continuity of $\operatorname{co} F$. This means that the intersection is a continuous map: $I \times X \times U \to \mathcal{K}^n$, and since $\sigma\colon \mathcal{K}^n \to \mathbb{R}^n$ is continuous, f is jointly continuous.

To prove (i) fix some $\boldsymbol{t} \in I$, $\boldsymbol{x} \in X$; then for any $\boldsymbol{y} \in \operatorname{co} F(\boldsymbol{t}, \boldsymbol{x})$, set $\boldsymbol{u} := h^{-1}(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y})$, obtaining $\boldsymbol{u} \in U$, $h(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{y}$, hence $f(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{u}) = \sigma \circ P(\boldsymbol{y}, \operatorname{co} F(\boldsymbol{t}, \boldsymbol{x})) = \boldsymbol{y}$ because $\boldsymbol{y} \in \operatorname{co} F(\boldsymbol{t}, \boldsymbol{x})$ already. This means that $\operatorname{co} F(\boldsymbol{t}, \boldsymbol{x}) \subset f(\boldsymbol{t}, \boldsymbol{x}, U)$, and since the opposite inclusion is obvious, (i) is proved. ◆

PROOF OF THEOREM 2. Since I is σ-compact, we can use the Scorza-Dragoni property in [7] to write $I = \mathcal{N} \cup I_0$, $\mathcal{N}$ a null set and $I_0 = \cup I_k$, where (I_k) is a sequence of compact disjoint sets such that $F_k := \operatorname{co} F|_{I_k \times X}$ is *lsc* with closed graph, $w_k := w|_{I_k \times X}$ is continuous. If moreover there exists $m\colon I \to \mathbb{R}^+$ such that $y \in F(t, x) \Rightarrow |y| \leqslant m(t)$, and m is measurable then we may also suppose that $m|_{I_k}$ is continuous. Let $C^0(X, \mathbb{R}^n)$ be the Banach space of continuous bounded maps $u\colon X \to \mathbb{R}^n$ with the usual sup norm. Set, for $t \in I_0$,

$$\begin{aligned}E(t) := \{u \in C^0(X, \mathbb{R}^n)\colon |u(x) - u(\boldsymbol{x})| \leqslant 6nw(t, d(x, \boldsymbol{x})),& \\ \text{and, in case } F \text{ is integrably bounded, } |u(x)| \leqslant m(t)\}\,.&\end{aligned}$$

Set $E_k := \bigcup_{t \in I_k} E(t)$, and let E be the closed convex hull of $\bigcup_{k \in \mathbb{N}} E_k$. Clearly each bounded subset of $E(t)$ is totally bounded, in particular $E(t)$ is compact provided F is integrably bounded; in general $E(t)$ is σ-compact. Since I_k is compact and w_k is jointly continuous, each bounded subset of E_k is totally bounded; in particular E_k is σ-compact, hence E is σ-compact.

Define the function φ to be the evaluation map $\varphi(x, u) := u(x)$; then clearly (iii) holds. Define the multifunction $\mathcal{U}$ by:

$$\mathcal{U}(t) := \{u \in E(t): u(x) \in \operatorname{co} F(t, x) \ \ \forall x \in X\} .$$

Since $\mathcal{U}(t) \subset E(t)$, (iv) holds. Since $\operatorname{co} F(t, x)$ and $E(t)$ are convex closed, $\mathcal{U}(t)$ is convex closed. In particular $\mathcal{U}(t)$ is compact in case F is integrably bounded. Set now $\mathcal{U}_k := \mathcal{U}|_{I_k}$. Since F_k, w_k, m_k have closed graph, one easily shows that $\mathcal{U}_k$ has closed graph. In particular $\mathcal{U}_0 := \mathcal{U}|_{I_0}$ has measurable graph. By Himmelberg [4, Theorem 3.5], $\mathcal{U}_0$ is measurable hence $\mathcal{U}$ is measurable.

Finally, to prove (i), fix any $\boldsymbol{t} \in I_0$, $\boldsymbol{x} \in X$; then, for any $\boldsymbol{y} \in \operatorname{co} F(\boldsymbol{t}, \boldsymbol{x})$, set $\boldsymbol{u}(x) := \sigma \circ P(\boldsymbol{y}, \operatorname{co} F(\boldsymbol{t}, x))$. Clearly $\boldsymbol{u} \in E(\boldsymbol{t})$, and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{t})$; moreover $\varphi(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}$, so that $\operatorname{co} F(\boldsymbol{t}, \boldsymbol{x}) \subset \varphi(\boldsymbol{x}, \mathcal{U}(\boldsymbol{t}))$. Since the opposite inclusion is obvious, (i) is proved. ◆

5. Application to differential inclusions.

Let I be an interval, bounded or unbounded, and let Ω be an open or closed set in $\mathbb{R}^n$. Let $F: I \times \Omega \to \mathbb{R}^n$ be a multifunction with values either bounded by a linear growth condition—hypothesis (FLB)—or unbounded—hypothesis (FU). Notice that hypothesis (FLB) (d) now simply asks the boundedness of I and the continuity of $F(t, \cdot)$; in fact I is already σ-compact, and for X we can take an adequate compact subset of Ω, using an exponential a priori estimate for solutions of (CP) based on Gronwall's inequality (see [1, Theorem 2.4.1] for example), and supposing either Ω large enough or I small enough.

Corollary 3. – Let F verify hypothesis (FU).

Then the Cauchy problem (CP) has the same absolutely continuous solutions as the control differential equation

$$(CDE) \qquad x' = f(t, x, u) \quad \text{a.e. on } I\,, \ x(0) = \xi\,, \ u(t) \in U,$$

where f, U are as in Theorem 1 or Corollary 1 or Corollary 2.

If moreover F, w are jointly h-continuous then for each continuously differentiable solution $\boldsymbol{x}$ of (CP) there exists a continuous control $\boldsymbol{u}: I \to U$ such that

$$\boldsymbol{x}'(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \qquad \forall t .$$

A special case which appears more commonly in applications is covered by the simpler:

COROLLARY 4. Let $F\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a multifunction with compact values $F(t, x)$ bounded by $m(t)$, such that $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is Lipschitz with constant $l(t)$, with $m(\cdot)$ and $l(\cdot)$ integrable.

Then the Cauchy problem (CP) has the same absolutely continuous solutions as the control differential equation

$$x' = f(t, x, u) \quad \text{a.e. on } I, \quad x(0) = \xi\,, \quad |u(t)|\leqslant 1\,,$$

where $f\colon \mathbb{R}\times\mathbb{R}^n\times B \to \mathbb{R}^n$ is measurable in t and Lipschitz in (x, u) with constant $6n[2l(t) + m(t)]$, and B is the unit closed ball in $\mathbb{R}^n$.

PROPOSITION 3. Let F verify hypothesis (FU).

Let f, U be as in Theorem 1 or Corollary 1 or Corollary 2.

Then for each $\boldsymbol{x}\colon I \to X$, $\boldsymbol{y}\colon I \to \mathbb{R}^n$ measurable verifying $\boldsymbol{y}(t) \in \operatorname{co} F(t, \boldsymbol{x}(t))$ a.e. there exists $\boldsymbol{u}\colon I \to U$ measurable such that $\boldsymbol{y}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ a.e.

If moreover F, w are jointly h-continuous and $\boldsymbol{x}$, $\boldsymbol{y}$ are continuous then $\boldsymbol{u}$ is continuous.

PROOF. Consider the homeomorphism h as in Corollary 1 or Corollary 2 or Theorem 1, and set $\boldsymbol{u}(t) := h^{-1}(t, \boldsymbol{x}(t), \boldsymbol{y}(t))$.

PROOF OF COROLLARY 3. For each solution $\boldsymbol{x}$ of (CPR) set $\boldsymbol{y}(t) := \boldsymbol{x}'(t)$ and apply Proposition 3. ◆

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REMARKS ADDED IN PROOF:

(*a*) after sending this paper for publication I have constructed an example showing that the Lipschitz constant 3 for the multivalued projection (Lemma 1) is best possible;

(*b*) four months after sending this paper for publication I have received the preprint [10] which extends my multivalued projection to Hilbert space. Using the same proof as in Lemma 1 the extension to Hilbert space is trivial.

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