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Existence of solutions for a class of nonconvex differential inclusions

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Existence of Solutions for a Class of Nonconvex Differential Inclusions.

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SUMMARY - We prove existence of solutions for the Cauchy problem

$$\dot{x} \in F(x) + f(t, x) , \quad x(0) = \xi ,$$

where F is upper semicontinuous and $F(x)$ is contained in the subdifferential $\partial V(x)$ of a convex continuous function V, while f is a Carathéodory single valued map.

1. Introduction.

It is well known that the initial value problem for the differential inclusion $\dot{x} \in F(x)$ may not have solutions when F is upper semicontinuous but has not necessarily convex values. Bressan, Cellina and Colombo have recently [2] given a condition ensuring existence for the Cauchy problem

$$(1) \qquad \dot{x} \in F(x) , \quad x(0) = \xi \in \mathbf{R}^n .$$

They assume F to be upper semicontinuous with values contained in the subdifferential ∂V of a convex function $V: \mathbf{R}^n \to \mathbf{R}$. This function permits to estimate the L^2-norm of the derivatives of approximat-

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ing polygonals and to obtain their strong L^2-convergence from the weak one.

We extend here the technique of [2] to the differential inclusion

$$\dot{x} \in F(x) + f(t, x) ,$$

where F is as in [2] and f is a Carathéodory single valued map. So we obtain a result that contains Peano's existence theorem as a particular case.

2. The result.

We consider the Cauchy problem

$$\dot{x}(t) \in F\big(x(t)\big) + f\big(t, x(t)\big) , \qquad x(0) = \xi \in \mathbb{R}^n , \tag{2}$$

under the following assumptions:

i) F is an upper semicontinuous multifunction from $\mathbb{R}^n$ into the compact nonempty subsets of $\mathbb{R}^n$ (i.e. for every x and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x'| \leqslant \delta$ implies $F(x') \subseteq F(x) + \varepsilon B$, where B is the unit ball of $\mathbb{R}^n$);

ii) there exists a convex continuous function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$F(x) \subseteq \partial V(x) \qquad \text{for every } x \in \mathbb{R}^n ,$$

where $\partial V(x)$ denotes the subdifferential of V at x;

iii) $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory, i.e. for every $x \in \mathbb{R}^n$, $t \mapsto f(t, x)$ is measurable, for a.e. $t \in \mathbb{R}$, $x \mapsto f(t, x)$ is continuous and there exists $m \in L^2(\mathbb{R})$ such that

$$|f(t, x)| \leqslant m(t) \qquad \text{for a.e. } t \in \mathbb{R} , \qquad \text{for all } x \in \mathbb{R}^n .$$

We recall that, under the assumption i), F satisfies ii) if and only if it is cyclically monotone [2].

By a solution of our Cauchy problem we mean an absolutely continuous function x which satisfies (2) a.e. On the space of solutions we consider the H^1 topology, which coincides with the topology induced by the sup norm on x and the L^2 norm on $\dot{x}$.

The following is our existence result.

THEOREM. Let F and f be maps satisfying i), ii) and iii). Then there exists $T > 0$ such that on $[0, T]$ the Cauchy problem (2) admits a nonempty set of solutions, which is compact in the $H^1(0, T)$ topology.

PROOF. We first define a family of approximate solutions similar to a construction of Tonelli [4, vol. 1 p. 42-45/vol. 2 p. 129-130] and then prove that a subsequence converges to a solution of (2).

By i) there exist $R > 0$ and $M > 0$ such that for every $x \in B(\xi, R)$ and for every $y \in F(x)$ we have $|y| \leqslant M$ [1, Proposition 1.1.3]; by iii) there exists $T > 0$ such that $\int_0^T (m(t) + M)\, dt < R$.

We define on $[0, T]$ a sequence of approximate solutions x_n:

$$x_n(0) = \xi\,,$$

$$x_n(t) = x_n\left(i\frac{T}{n}\right) + \int_{iT/n}^{t} f\left(s, x_n\left(i\frac{T}{n}\right)\right) ds + \left(t - i\frac{T}{n}\right) y_i\,,$$

$$i = 0, \dots, n-1\,, \quad t \in \left[i\frac{T}{n}, (i+1)\frac{T}{n}\right],$$

where $y_i \in F(x_n(iT/n))$.

Set, for $t \in [iT/n, (i+1)T/n[$, $i = 0, \dots, n-1$,

$$f_n(t) = f\left(t, x_n\left(i\frac{T}{n}\right)\right), \qquad g_n(t) = y_i\,. \tag{3}$$

Then, $|x_n(t) - \xi| \leqslant \int_0^t |f_n(s) + g_n(s)|\, ds \leqslant \int_0^T (m(s) + M)\, ds < R$, by our choice of T. Moreover, for all $t, t' \in [0, T]$,

$$|x_n(t') - x_n(t)| \leqslant \left|\int_t^{t'} |\dot{x}_n(s)|\, ds\right| \leqslant \left|\int_t^{t'} (m(s) + M)\, ds\right|,$$

so that the sequence $(x_n(\cdot))_n$ is equiuniformly continuous. Notice also that $\int_0^T |\dot{x}_n(s)|^2\, ds = \int_0^T |f_n(s) + g_n(s)|^2\, ds$ and therefore the sequence

$(\dot{x}_n(\cdot))_n$ is bounded in $L^2(0, T)$. Hence there exists a subsequence, still denoted by $(x_n)_n$, and an absolutely continuous function $x\colon [0, T] \to \mathbf{R}^n$ such that x_n converges to x in the sup norm topology and $\dot{x}_n$ converges to $\dot{x}$ in the weak topology of L^2.

Since $(f_n(\cdot))_n$ converges to $f(\cdot, x(\cdot))$ in L^2 and, for $t \in]i(T/n), (i+1)T/n[$, $i = 0, \ldots, n-1$,

$$\lim_{n\to\infty} d\Big(\big(x_n(t), \dot{x}_n(t) - f_n(t)\big), \text{graph}\,(F)\Big) \leqslant \lim_{n\to\infty} \left| x_n(t) - x_n\left(i\,\frac{T}{n}\right)\right| = 0, \tag{4}$$

by Theorem 1.4.1 in [1] we obtain that x is a solution of the convexified differential inclusion

$$\dot{x} \in \text{co}\,\big(F(x)\big) + f(t, x)\,, \qquad x(0) = \xi\,. \tag{5}$$

By our assumption ii) we then have that

$$\dot{x}(t) - f\big(t, x(t)\big) \in \partial V\big(x(t)\big) \qquad \text{for a.e } t \in [0, T]\,. \tag{6}$$

Since the maps $t \mapsto x(t)$ and $t \mapsto V\big(x(t)\big)$ are absolutely continuous, we obtain from Lemma 3.3 in [3, p. 73] and (6) that $(d/dt)\Big(V\big(x(t)\big)\Big) = \langle \dot{x}(t), \dot{x}(t) - f\big(t, x(t)\big)\rangle$ a.e. on $[0, T]$; therefore,

$$V\big(x(T)\big) - V(\xi) = \int_0^T |\dot{x}(s)|^2\, ds - \int_0^T \langle \dot{x}(s), f\big(s, x(s)\big)\rangle\, ds\,. \tag{7}$$

On the other hand, notice that, by (3),

$$\dot{x}_n(t) - f_n(t) = y_i \in \partial V\left(x_n\left(i\,\frac{T}{n}\right)\right) \qquad \text{for } t \in \left]i\,\frac{T}{n}, (i+1)\,\frac{T}{n}\right[\,, \quad i = 0, \ldots, n-1\,,$$

and so the properties of the subdifferential of a convex function imply, for every $t \in]iT/n, (i+1)T/n[$,

$$V\left(x_n\left((i+1)\,\frac{T}{n}\right)\right) - V\left(x_n\left(i\,\frac{T}{n}\right)\right) \geqslant \left\langle \dot{x}_n(t) - f_n(t), x_n\left((i+1)\,\frac{T}{n}\right) - x_n\left(i\,\frac{T}{n}\right)\right\rangle;$$

since this last expression equals

$$\left\langle y_i, \int_{iT/n}^{(i+1)T/n} \dot{x}_n(s)\, ds \right\rangle = \int_{iT/n}^{(i+1)T/n} \langle y_i, \dot{x}_n(s)\rangle\, ds = \int_{iT/n}^{(i+1)T/n} \langle \dot{x}_n(s) - f_n(s), \dot{x}_n(s)\rangle\, ds =$$

$$= \int_{iT/n}^{(i+1)T/n} |\dot{x}_n(s)|^2\, ds - \int_{iT/n}^{(i+1)T/n} \langle f_n(s), \dot{x}_n(s)\rangle\, ds\,,$$

by adding we obtain

$$V(x_n(T)) - V(\xi) \geqslant \int_0^T |\dot{x}_n(s)|^2\, ds - \int_0^T \langle f_n(s), \dot{x}_n(s)\rangle\, ds\,. \tag{8}$$

The convergence of $(f_n)_n$ in L^2-norm and of $(\dot{x}_n)_n$ in the weak topology of L^2 implies that

$$\lim_{n\to\infty} \int_0^T \langle f_n(s), \dot{x}_n(s)\rangle\, ds = \int_0^T \langle f(s, x(s)), \dot{x}(s)\rangle\, ds\,.$$

By passing to the limit for $n \to \infty$ in (8) and using the continuity of V, a comparison with (7) yields

$$\|\dot{x}\|_2^2 \geqslant \limsup \|\dot{x}_n\|_2^2\,;$$

since, by the weak lower semicontinuity of the norm,

$$\|\dot{x}\|_2^2 \leqslant \liminf \|\dot{x}_n\|_2^2\,,$$

we have that $\|\dot{x}\|_2^2 = \lim_{n\to\infty} \|\dot{x}_n\|_2^2$, i.e. $\dot{x}_n$ converges to $\dot{x}$ strongly in $L^2(0, T)$ [5, p. 124]. Hence there exists a subsequence $\dot{x}_n$ which converges pointwisely a.e. to $\dot{x}$. Recalling (4), we have that

$$d\Big(\big(x(t), \dot{x}(t) - f(t, x(t))\big), \text{graph}\,(F)\Big) = 0 \qquad \text{for a.e. } t \in [0, T];$$

since the graph of F is closed [1, p. 41],

$$\dot{x}(t) \in F(x(t)) + f(t, x(t)) \text{ a.e.}\,,$$

and so problem (2) does have solutions.

Let now $(x_n)_n$ be a sequence of solutions of (2). Using the same argument as for the approximate solutions we obtain that there exist an absolutely continuous function x and a subsequence $(x_n)_n$ such that x_n converges to x in $\mathcal{C}$ and $\dot{x}_n$ converges to $\dot{x}$ weakly in L^2. Both x and the x_n are solutions of the convexified differential inclusion (5) and so formula (7) holds for x as well as for the x_n. By passing to the limit we obtain that $\|\dot{x}\|_2^2 = \lim_{n\to\infty} \|\dot{x}_n\|_2^2$ and so, by the same arguments as before, x is a solution of (2).

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