RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 83 (1990), p. 81-95
<http://www.numdam.org/item?id=RSMUP_1990__83__81_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 83 (1990)

On Bounded Solutions of One-Dimensional Compressible Navier-Stokes Equations.

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SUMMARY - It is shown that if a solution (u, v) of one-dimensional compressible Navier-Stokes equations in Lagrangian mass coordinates is bounded for $t \in R$ (i.e. $u \in L^\infty(R; L^2(0, 1))$ and $0 < \alpha \leqslant v(x, t) \leqslant \beta < \infty$, where u is the velocity and v the specific volume of a fluid), then $u = 0$, $v = v(x)$ (i.e. $(u, v) = (0, v)$ is a stationary solution).

1. - Introduction.

In the present work we study bounded solutions of one-dimensional compressible Navier-Stokes equations in the Lagrangian form:

$$u_t + p(v)_x - \mu \left(\frac{u_x}{v}\right)_x = f\left(\int_0^x v(\xi, t)\, d\xi\right), \tag{1.1}$$

$$v_t = u_x\,, \qquad (x, t) \in Q \equiv (0, 1) \times R\,, \tag{1.2}$$

$$u(0, t) = u(1, t) = 0\,, \tag{1.3}$$

$$\int_0^1 v(x, t)\, dx = 1\,, \qquad t \in R\,, \tag{1.4}$$

$$p = p(v)\,, \qquad \big(p \in C^1((0, \infty)), p'(\xi) < 0, \text{ for } \xi > 0\big)\,. \tag{1.5}$$

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Here u is the velocity, v the specific volume, $\mu = \text{const} > 0$ the viscosity of the fluid, $f = f(x)$, $f \in C([0,1])$ a forcing term, which is supposed to be independent of time.

Global existence theorems for the equations (1.1)-(1.5) with Cauchy data at $t = t_0$ for $t \geqslant t_0$ have been given in [4], [7].

Having global existence of solutions for the problem (1.1)-(1.5) we would like to know some qualitative properties of the solutions. In the system (1.1), (1.2) the equation (1.1) has a dissipative term $-\mu(u_x/v)_x$ which is known to cause a dissipation of energy due to viscosity. This suggests a question: Does the system (1.1)-(1.5) have the usual properties of dissipative equations? To be more specific, we would like to know e.g. whether the following assertions are true:

(*a*) If (u, v) is a bounded solution (see Definitions 2.3, 2.7) then (u, v) is stationary;

(*b*) any solution converges to a stationary one (with a precise rate of convergence) as $t \to \infty$.

Some results close to these assertions have already been proved. See [4] for $f \equiv 0$, [5], [6] for $f = f(x, t)$, $p(v)$ special. These results are reviewed in [7], where (for $f = f(x)$) we have proved that if there exists a sufficiently smooth solution (u, v) of the system (1.1)-(1.5) on $[t_0, \infty)$ which is bounded in the sense that

$$0 < \alpha \leqslant v(x, t) \leqslant \beta < \infty , \tag{1.6}$$

then it converges to the stationary solution in $H^1(0, 1)$ as $t \to \infty$. The result has been obtained by the help of a series of apriori estimates for the solution and its derivatives in L^2-spaces on $[t_0, \infty)$ which allow us to choose a subsequence $u(\cdot, t_n)$, $v(\cdot, t_n)$ converging in $H^1(0, 1)$ to a limit $(\bar{u}, \bar{v})$ of the form $\bar{u} = 0$, $\bar{v} = \bar{v}(x)$. This limit is shown to be a stationary solution to (1.1)-(1.5).

In view of the results of Shelukhin [6], who, as mentioned above, under some special assumptions proved the existence of a bounded solution for $t \in R$, we tried to prove it for more general p. But our result in [7] implies that (for $f = f(x)$) a bounded solution cannot exist on $[t_0, \infty)$ unless there exists a stationary one. This suggests the hypotheses that if the solution satisfies (1.6) on R and

$$\sup\left\{\int_0^1 u(x, t)^2\, dx;\ t \in R\right\} < \infty \tag{1.7}$$

then $u = 0$, $v = \bar{v}(x)$ is stationary. This is exactly what we prove in this paper. The condition (1.7) must be added since for $t \to -\infty$ the boundedness of $\int_0^1 u(x,t)^2\,dx$ does not follow from (1.6) and in general need not hold. One has to remark however that the stationary solution was proved to exist if and only if f and $p(v)$ satisfy suitable compatibility condition (see [1], Theorem 5.3, rewritten here as Lemma 2.9). Hence our result says that a (u, v)-bounded solution on R cannot exist if f and $p(v)$ do not satisfy such a condition.

After this manuscript has been finished the authors acquainted with the most recent results of Beirão de Veiga contained in two preprints [2], [3]. It is proved that any stationary solution is exponentially stable under small perturbations and that there is a positive threshold for the norms of initial conditions u_0, v_0 and the L^∞-norm of the right hand side f under which the solution satisfies estimate (2.1) below for $J = [0, \infty)$. These results improve substantially understanding the large time behavior of the solution especially in the physically significant case $p(v) = Av^{-\gamma}$, $\gamma \in (1, 2)$.

In what follows we adopt the usual notation, namely C_0^∞ for spaces of smooth functions with a compact support, $W^{k,p}$ for the Sobolev spaces, in particular $H^s = W^{s,2}$, $H^0 = L^2$, $\mathring{H}^s = \mathring{W}^{s,2}$, $|\cdot|_s = \|\cdot\|_{H^s}$, $(\mathring{H}^s)' = H^{-s}$; or $H^s(J; B)$ for H^s-functions with values in a Banach space B and the like.

2. – Basic notions and results.

Let the problem (1.1)-(1.5) be given with $p = p(v)$ and $f = f(x)$ satisfying the following assumptions:

(i) $$p \in C^1((0, \infty))\,, \quad p'(s) < 0\,, \quad s \in (0, \infty);$$

(ii) $$f \in L^\infty(0, 1)\,.$$

For $w \in L^1(0,1)$ denote $Iw(x) = \int_0^x w(\xi)\,d\xi$.

2.1 Definition. By a solution of the problem (1.1)-(1.5) on an interval $J \subseteq R$ we mean a couple of functions (u, v) such that

$$u \in C(J; H^0(0,1)) \cap L^2_{\mathrm{loc}}(J; \mathring{H}^1(0,1)),\ u_t \in L^2_{\mathrm{loc}}(J; H^{-1}(0,1))\,,$$

$$v \in C(J; H^1(0,1))\,, \quad v_t \in L^2_{\mathrm{loc}}(J; H^0(0,1))\,, \quad v(x,t) > 0$$

for any $t \in J$ everywhere on $[0, 1]$, u, v satisfy (1.1) in the sense of $H^{-1}(0, 1)$, (1.2) in the sense of $H^0(0, 1)$ and (1.4) for $t \in J$.

2.2 THEOREM. Let the functions p and f satisfy the assumptions (i), (ii). Then for any interval $J \subset R$ bounded from below there exists, a solution to (1.1)-(1.5) (having assigned the initial value $u(\cdot, t_0) \in \in \mathring{H}^1(0, 1)$, $v(\cdot, t_0) \in H^1(0, 1)$ at $t = t_0$ a unique one).

The proof for initial data in $H^1(0, 1)$ can be found in [7]. For $u(\cdot, t_0) \in H^0(0, 1)$, $v(\cdot, t_0) \in H^1(0, 1)$ it is shown in [2], Theorem 4.1, that a solution can be constructed as a limit of solutions with smooth initial data. Uniqueness of such a solution is proved in [2], Theorem 5.3 under the assumption that p' is locally Lipschitz continuous and $f \in L^2(J; L^2(0, 1)) \cap L^1\big(J; C^{(0)+1}([0, 1])\big)$.

2.3 DEFINITION. By a v-bounded solution of (1.1)-(1.5) on J we mean a solution (u, v) of (1.1)-(1.5) on J such that

$$0 < \alpha \leqslant v(x, t) \leqslant \beta < \infty \quad \text{for all } t \in J\,, \quad x \in [0, 1]\,, \tag{2.1}$$

where α, β are constants.

A proof of the following theorem can again be found in [7].

2.4 THEOREM. Under the assumptions (i), (ii) for any bounded interval $J \subset R$ there exists a v-bounded solution of (1.1)-(1.5) on J (having assigned the initial values $u(\cdot, t_0) \in \mathring{H}^1(0, 1)$, $v(\cdot, t_0) \in H^1(0, 1)$ at $t = t_0$ a unique one).

Denote

$$P(\eta) = \int_1^\eta [p(1) - p(\zeta)]\, d\zeta\,, \tag{2.2}$$

$$F(y) = \int_0^y f(\eta)\, d\eta \tag{2.3}$$

and

$$E(t) = \int_0^1 \Big[\tfrac{1}{2} u(t)^2 + P\big(v(t)\big) + \lambda - F\big(Iv(t)\big)\Big]\, dx \tag{2.4}$$

for any solution of (1.1)-(1.5). Here $\lambda \geqslant 0$ is an arbitrary constant to be chosen in what follows so that $\lambda - F(Iv)$ is non-negative. $\left(Iv(t)\right.$ stands for $\left.\int_0^x v(\xi, t)\, d\xi.\right)$ Note that $P(\eta) \geqslant 0$ for $\eta > 0$.

2.5 Lemma. For any v-bounded solution of (1.1)-(1.5) on an interval $J \subseteq R$ the function $E(t)$ is non-increasing on J and for any $t, s \in J$, $t < s$ the inequalities

$$\frac{\mu}{\beta} \int_t^s |u_x(\tau)|_0^2\, d\tau \leqslant E(t) - E(s) \leqslant \frac{\mu}{\alpha} \int_t^s |u_x(\tau)|_0^2\, d\tau \tag{2.5}$$

hold.

Proof. We can prove analogously as in [7], Lemma 2.2 that

$$\dot{E}(t) = -\mu \int_0^1 \frac{u_x(t)^2}{v(t)}\, ds \leqslant 0\,. \tag{2.6}$$

The only distinction is that, handling with the equation (1.1), instead of pairing in $H^0(0,1)$ the pairing between $\mathring{H}^1(0,1)$ and $H^{-1}(0,1)$ must be used. According to (2.6) the inequalities (2.5) follow from (2.1) by integration. ■

2.6 Lemma. Let (u, v) be a v-bounded solution of (1.1)-(1.5) on J, $\lambda > 0$ in (2.4) sufficiently large. Then the following assertions are equivalent:

1° $\quad E(t) \leqslant \text{const} < \infty\,, \quad t \in J;$

2° $\quad |u(t)|_0^2 \leqslant \text{const} < \infty\,, \quad t \in J;$

3° $\quad u \in L^2(J; \mathring{H}^1(0,1))\,.$

Proof. 1° ⇒ 2°: By (2.4), (2.1) we have $\frac{1}{2}|u(t)|_0^2 \leqslant E(t)$ for sufficiently large $\lambda > 0$ (but fixed). The implications 1° ⇒ 3° and 3° ⇒ 1° are easy consequences of (2.5), (2.6). Finally, 2° ⇒ 1° follows from (2.4) and (2.1). ■

2.7 Definition. A v-bounded solution of (1.1)-(1.5) is called (u, v)-bounded if one of the conditions 1°-3° from Lemma 2.6 holds.

2.8 LEMMA. A couple (u, v) is a stationary v-bounded solution of (1.1)-(1.5) if and only if $u(t) \equiv 0$ and $v(t) \equiv v$, where $w = Iv$ is the (unique) solution of the problem

$$w \in W^{2,\infty}([0,1]) \,, \tag{2.7}$$

$$p(w')' - f(w) = 0 \,, \tag{2.8}$$

$$w(0) = 0 \,, \quad w(1) = 1 \,, \tag{2.9}$$

$$0 < \alpha \leqslant w'(x) \leqslant \beta < \infty \tag{2.10}$$

with some constants α, β.

PROOF. By (2.6), for the stationary v-bounded solution we have $u \equiv 0$, $v \in H^1(0,1)$, $0 < \alpha \leqslant v(x) \leqslant \beta < \infty$. Hence

$$w \equiv Iv \in H^2(0,1) \,, \quad w(0) = 0 \,, \quad w(1) = 1$$

and (2.10) holds. Equation (1.1) can be interpreted in $L^2(0,1)$ since $f \in L^\infty(0,1)$. By the assumptions (i) and (ii), (2.7) easily follows from (2.8) and (2.10). The reverse implication is trivial. ■

Define as in [1] (see also [7])

$$\pi(\xi) = -\int_1^\xi s^{-3} p'\left(\frac{1}{s}\right) ds \,, \qquad \xi > 0 \,, \tag{2.11}$$

$$-\infty \leqslant a \equiv \lim_{\xi \to 0+} \pi(\xi) < 0 \,, \qquad 0 < b \equiv \lim_{\xi \to \infty} \pi(\xi) \leqslant \infty \,,$$

$$\Phi = \pi^{-1} \,, \qquad m_0 = \min_{y \in [0,1]} F \,, \qquad M_0 = \max_{y \in [0,1]} F \,. \tag{2.12}$$

2.9 LEMMA ([1], Theorem 5.3). The problem (2.7)-(2.10) has a (unique) solution if and only if

$$a - m_0 < b - M_0 \,, \tag{2.13}$$

and

$$\int_0^1 \Phi(a - m_0 + F(y))\, dy < 1 < \int_0^1 \Phi(b - M_0 + F(y))\, dy \,. \tag{2.14}$$

3. – Main results.

In this section we shall prove some properties of v-bounded and (u, v)-bounded solutions. To this purpose we need some auxiliary results.

3.1 LEMMA. Let $f \in C^0([0,1])$ and $\{v_n\}_{n=1}^{\infty} \subset H^0(0,1)$ be such that $0 < \alpha \leqslant v_n \leqslant \beta < \infty$, $\int_0^1 v_n(x)\,dx = 1$ and

$$(3.1) \qquad \left|\int_0^1 [p(v_n)\psi' + f(Iv_n)\psi]\,dx\right| \leqslant \delta_n |\psi'|_0$$

for $n = 1, 2, \ldots$, $\psi \in C_0^\infty(0,1)$ and some $\delta_n \downarrow 0$. Then there exists a solution w of (2.7)-(2.10) and $v_n \to w'$ in $H^0(0,1)$.

PROOF. From (3.1) we have

$$\left|\int_0^1 [p(v_n) - If(Iv_n)]\psi'\,dx\right| \leqslant \delta_n |\psi'|_0 ,$$

so that

$$\left|\int_0^1 g_n \phi\,dx\right| \leqslant \delta_n |\phi|_0$$

for $\phi \in H^0(0,1)$, $M(\phi) \equiv \int_0^1 \phi\,dx = 0$, where we have denoted $g_n = p(v_n) - If(Iv_n)$. If $h \in H^0(0,1)$ is arbitrary then

$$\left|\int_0^1 [g_n - M(g_n)]\,h\,dx\right| = \left|\int_0^1 [g_n - M(g_n)]\,[h - M(h)]\,dx\right| =$$

$$= \left|\int_0^1 g_n[h - M(h)]\,dx\right| \leqslant \delta_n |h - M(h)|_0 \leqslant \delta_n |h|_0 .$$

This implies that $g_n - M(g_n) \to 0$ in $H^0(0,1)$. As $\{M(g_n)\}_{n=1}^{\infty}$ is a bounded sequence, we can extract a subsequence of g_n (denoted again by g_n) such that $M(g_n)$ converge to some constant k. Hence we have $p(v_n) - If(Iv_n) \to k$ in $H^0(0,1)$. Since $If(Iv_n)$ is compact in $H^0(0,1)$, there exists $z \in H^0(0,1)$ such that $p(v_n) \to z$ in $H^0(0,1)$, and $p(\beta) \leqslant z \leqslant p(\alpha)$. Put $v = p^{-1}(z)$. Then

$$|p(v_n) - p(v)| = \\ = \int_0^1 \Big[- p'(\theta v_n + (1-\theta)v)\Big] d\theta |v_n - v| \geqslant \min_{s\in[\alpha,\beta]} [-p'(s)]|v_n - v| ,$$

from where we get $v_n \to v$ in $H^0(0,1)$. Thus $p(v) - If(Iv) = k$. Differentiating the last equality for $w = Iv$ we get the equation (2.8). The conditions (2.7), (2.9), (2.10) are now easy to verify. Since by Lemma 2.9 the solution of the problem (2.7)-(2.10) is unique we find that not only some subsequence of $\{v_n\}_{n=1}^\infty$ but all the sequence v_n converges to v in $H^0(0,1)$. ■

3.2 Lemma. Let V, H be Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$, V dense in H, $|u|_H \leqslant c|u|_V$ for $u \in V$ with some constant c. Then for any $a, b \in R$, $a < b$ we have

$$\mathcal{H} \equiv H^0(a, b; V) \cap H^1(a, b; V') \hookrightarrow C([a, b]; H)$$

and

$$(3.2) \qquad |u(t)|_H^2 \leqslant \left(\int_a^b |u(s)|_V^2 ds\right)^{\frac12} \left[2\left(\int_a^b |u'(s)|_{V'}^2 ds\right)^{\frac12} + \frac{c^2}{b-a}\left(\int_a^b |u(s)|_V^2 ds\right)^{\frac12}\right]$$

for any $u \in \mathcal{H}$.

Proof. The result is well-known. Let us just prove (3.2). For $u \in \mathcal{H}$ we have

$$\frac{d}{dt}|u(t)|_H^2 = \frac{d}{dt}(u(t), u(t))_H = \frac{d}{dt}\langle u(t), u(t)\rangle_{V,V'} = 2\langle u(t), u'(t)\rangle_{V,V'} .$$

Since there exists a $t^* \in [a, b]$ such that

$$|u(t^*)|_H^2 = \frac{1}{b-a}\int_a^b |u(t)|_H^2 dt ,$$

we get

$$|u(t)|_H^2 = \frac{1}{b-a}\int_a^b |u(t)|_H^2\,dt + \int_{t^*}^{t} 2\langle u(\tau), u'(\tau)\rangle_{V,V'}\,d\tau \leqslant$$

$$\leqslant \frac{1}{b-a}\int_a^b |u(t)|_H^2\,dt + 2\int_a^b |u(t)|_V\,|u'(t)|_{V'}\,dt \leqslant$$

$$\leqslant \frac{c^2}{b-a}\int_a^b |u(t)|_V^2\,dt + 2\left(\int_a^b |u(t)|_V^2\,dt\right)^{\frac12}\left(\int_a^b |u'(t)|_{V'}^2\,dt\right)^{\frac12} =$$

$$= \left(\int_a^b |u(t)|_V^2\,dt\right)^{\frac12}\left[2\left(\int_a^b |u'(t)|_{V'}^2\,dt\right)^{\frac12} + \frac{c^2}{b-a}\left(\int_a^b |u(t)|_V\,dt\right)^{\frac12}\right]. \quad \blacksquare$$

3.3 Remark. If $H = H^0(0,1)$, $V = \mathring{H}^1(0,1)$ with

$$|u|_V^2 = \int_0^1 u_x^2\,dx$$

then the best c is $c = \pi^{-1}$.

The following theorem has been proved in [7], Remark 3.19 for smoother solutions by the method of apriori estimates.

3.4 Theorem. Let (u, v) be a v-bounded solution of (1.1)-(1.5) on (a, ∞), $(a \in R)$, $p \in C^1((0, \infty))$, $f \in C([0,1])$. Then $u(t) \to 0$ in $H^0(0,1)$ as $t \to \infty$, there exists a solution w of the problem (2.7)-(2.10) and $v(t) \to w'$ in $H^0(0,1)$ as $t \to \infty$.

Since the proof of Theorem 3.4 is quite analogous to the proof of the subsequent Theorem 3.5 we do not present it in this place.

3.5 Theorem. Let (u, v) be a (u, v)-bounded solution of (1.1)-(1.5) on $(-\infty, b)$ $(b \in R)$, $p \in C^1((0, \infty))$, $f \in C([0,1])$. Then $u(t) \to 0$ in $H^0(0,1)$ as $t \to -\infty$, there exists a solution w of the problem (2.7)-(2.10) and $v(t) \to w'$ in $H^0(0,1)$ as $t \to -\infty$.

Proof. Let (u, v) be a (u, v)-bounded solution of (1.1)-(1.5) on $(-\infty, b)$. Then by (2.5) we have $u \in L^2((-\infty, b); \mathring{H}^1(0,1))$ and if we put $\sigma(t) = \left(\int_{t-1}^{t} |u(s)|_1^2\, ds\right)^{\frac{1}{2}}$ then

$$\lim_{t \to -\infty} \sigma(t) = 0 . \tag{3.3}$$

From the equation (1.1) we find $(|\cdot|_{V'} \leqslant |\cdot|_H)$

$$|u_t(t)|_{-1} \leqslant |p(v(t))|_0 + \frac{\mu}{\alpha} |u(t)|_1 + |f(Iv(t))|_0 .$$

Hence clearly

$$\int_{t-1}^{t} |u_t(s)|_{-1}^2\, ds \leqslant \gamma < \infty . \tag{3.4}$$

Lemma 3.2 and (3.4) yield

$$|u(t)|_0^2 \leqslant \sigma(t)\left(2\gamma^{\frac{1}{2}} + \pi^{-2}\sigma(t)\right) , \tag{3.5}$$

from where by the help of (3.3) we get $u(t) \to 0$ in $H^0(0, 1)$ as $t \to -\infty$.

To prove the other part of the assertion we shall make use of Lemma 3.1. Obviously, it suffices to show that there exists a non-negative function $\delta(t)$ with $\lim_{t \to -\infty} \delta(t) = 0$ such that

$$\left| \int_0^1 \left[p(v(t))\, \psi' + f(Iv(t))\, \psi\right] \right| \leqslant \delta(t) |\psi'|_0 \tag{3.6}$$

for all $\psi \in C_0^\infty(0, 1)$.

Let $\varphi \in C_0^\infty(-1, 0)$ be a fixed non-negative function with $\int_{-1}^{0} \varphi\, dt = 1$. By (1.1), integration by parts, (3.5), the Schwartz and the Poincaré inequality, (denoting $\langle \cdot, \cdot \rangle$ pairing between $\mathring{H}^1(0, 1)$ and $H^{-1}(0, 1)$), for any $\psi \in C_0^\infty(0, 1)$ we have

$$\left| \int_{t-1}^{t} \varphi(\tau - t) \int_0^1 \left[p(v(\tau))\, \psi' + f(Iv(\tau))\, \psi\right] dx\, d\tau \right| =$$

$$= \left| \int_{t-1}^{t} \varphi(\tau - t) < u_t(\tau) - \mu \left(\frac{u_x(\tau)}{v(\tau)}\right)_x , \psi > d\tau \right| =$$

$$= \left| \int_{t-1} \left[-\varphi'(\tau - t) \int_0^1 u(\tau)\, \psi\, dx + \mu\varphi(\tau - t) \int_0^1 \frac{u_x(\tau)}{v(\tau)}\, \psi'\, dx \right] d\tau \right| \leqslant$$

$$= \left(\int_{-1}^0 \varphi'(s)^2\, ds \right)^{\frac{1}{2}} \left[\int_{t-1}^t \left(\int_0^1 u(\tau)\, \psi\, dx \right)^2 d\tau \right]^{\frac{1}{2}} +$$

$$+ \mu \left(\int_{-1}^0 \varphi(\tau)^2\, d\tau \right)^{\frac{1}{2}} \left[\int_{t-1}^t \left(\int_0^1 \frac{u_x(\tau)}{v(\tau)}\, \psi'\, dx \right)^2 d\tau \right]^{\frac{1}{2}} \leqslant$$

$$\leqslant \left(\int_{-1}^0 \varphi'(s)^2\, ds \right)^{\frac{1}{2}} \left[\left(\int_{t-1}^t \int_0^1 u(\tau)^2\, dx\, d\tau \right)^{\frac{1}{2}} \left(\int_0^1 \psi^2\, dx \right)^{\frac{1}{2}} +\right.$$

$$\left. + \frac{\mu}{\alpha} \left(\int_{t-1}^t \int_0^1 u_x(\tau)^2\, dx\, d\tau \right)^{\frac{1}{2}} \left(\int_0^1 (\psi')^2\, dx \right)^{\frac{1}{2}} \right] \leqslant$$

$$\leqslant (1 + \mu\alpha^{-1}) |\varphi'|_0 \cdot \sigma(t) \cdot |\psi'|_0 \equiv \eta(t) |\psi'|_0 \,.$$

Further, by easy calculations with help of (1.2), we find

$$\left| \int_0^1 [p(v(t))\, \psi' + f(Iv(t))\, \psi]\, dx \right| = \tag{3.7}$$

$$= \left| \int_{t-1}^t \varphi(\tau - t) \int_0^1 [p(v(t))\, \psi' + f(Iv(t))\, \psi]\, dx\, d\tau \right| \leqslant$$

$$\leqslant \left| \int_{t-1}^t \varphi(\tau - t) \int_0^1 [p(v(\tau))\, \psi' + f(Iv(\tau))\, \psi]\, dx\, d\tau \right| +$$

$$+ \left| \int_{t-1}^t \varphi(\tau - t) \int_0^1 [p(v(t)) - p(v(\tau))]\, \psi'\, dx\, d\tau \right| +$$

$$+ \left| \int_{t-1}^t \varphi(\tau - t) \int_0^1 [f(Iv(t)) - f(Iv(\tau))]\, \psi\, dx\, d\tau \right| \leqslant \eta(t) |\psi'|_0 +$$

$$+ \left| \int_{t-1}^t \varphi(\tau - t) \int_\tau^t \int_0^1 p'(v(s))\, v_t(s)\, \psi'\, dx\, ds\, d\tau \right| +$$

$$+\left|\int_{t-1}^{t}\varphi(\tau-t)\int_0^1[f(Iv(t))-f(Iv(\tau))]\psi\,dx\,d\tau\right|\leqslant$$
$$\leqslant\Big[\eta(t)+\sup_{y\in[\alpha,\beta]}|p'(y)|\sigma(t)\Big]\,|\psi'|_0+$$
$$+\left|\int_{t-1}^{t}\varphi(\tau-t)\int_0^1[f(Iv(t))-f(Iv(\tau))]\psi\,dx\,d\tau\right|.$$

Since for $t-1\leqslant\tau\leqslant t$, it is

$$|Iv(t)(x)-Iv(\tau)(x)|\leqslant\int_\tau^t\int_0^1|v_t(s)|\,dx\,ds\leqslant$$
$$\leqslant\int_\tau^t\Big(\int_0^1 v_t(s)^2\,dx\Big)^{\frac12}ds\leqslant\Big(\int_\tau^t|v_t(s)|_0^2\,ds\Big)^{\frac12}(t-\tau)^{\frac12}\leqslant\sigma(t)\,,$$

we have

$$|f(Iv(t))-f(Iv(\tau))|_{L^\infty(0,1)}\leqslant\omega(\sigma(t))\,,$$

where ω is a modulus of continuity of f on $[0, 1]$. Thus

$$(3.8)\qquad\left|\int_0^1[f(Iv(t))-f(Iv(\tau))]\psi\,dx\right|\leqslant\omega(\sigma(t))\int_0^1|\psi|\,dx\leqslant$$
$$\leqslant\omega(\sigma(t))\,|\psi|_0\leqslant\omega(\sigma(t))\,|\psi'|_0\,.$$

By (3.7), (3.8) we get (3.6) with

$$\delta(t)=\eta(t)+\sup_{y\in[\alpha,\beta]}|p'(y)|\sigma(t)+\omega(\sigma(t))\,.\quad\blacksquare$$

3.6 THEOREM. Let $p\in C^1((0,\infty))$, $f\in C([0,1])$. Then any (u, v)-bounded solution of (1.1)-(1.5) on R is stationary.

PROOF. Let (u, v) be such a solution. Then by Theorems 3.4, 3.5 $u(t)\to 0$, $v(t)\to w'$ in $L^2(0,1)$ as $|t|\to\infty$. As w is determined uniquely and (2.10) holds, by the Lebesgue theorem we have

$$E(\infty)\equiv\lim_{|t|\to\infty}E(t)=\int_0^1[P(w')+\lambda-F(w)]\,dx\,.$$

Since E is nonincreasing by Lemma 2.5, it is constant. So by (2.6), $u \equiv 0$ and (1.1) together with Lemma 2.8 yield $v = w'$. ∎

An immediate consequence of Theorem 3.6 and Lemma 2.9 is the following

3.7 THEOREM. Under the assumption of Theorem 3.6 a (u, v)-bounded solution on R exists if and only if the conditions (2.13), (2.14) are satisfied.

3.8 REMARK. If the assumptions of Theorem 3.6 are satisfied, (u, v) is a solution e.g. on $Q_\infty = (0, 1) \times [0, \infty)$,

$$\liminf_{s \to 0+} [-sp'(s)] > 0 \tag{3.9}$$

(as, for instance, in the case $p(s) = As^{-\gamma}$, $A > 0$, $\gamma > 0$) and

$$v(x, t) \leqslant \beta < \infty \quad \text{in } Q_\infty \tag{3.10}$$

then there exists an $\alpha > 0$ such that

$$v(x, t) \geqslant \alpha \quad \text{in } Q_\infty . \tag{3.11}$$

Thus, in this case, v-boundedness follows just from boundedness of v from above. This leads to an obvious version of Theorem 3.4 which the reader can formulate himself easily. The above implication is proved in [2], Theorem 7.5. Before this fact came to our attention we found a proof which, we believe, still has a sense to present in this context. We multiply (1.1) by $(\log v)_x$ and integrate with respect to x over $(0, 1)$. After standard calculations we get

$$\frac{\mu}{2}\frac{d}{dt}\int_0^1 [(\log v)_x]^2\, dx = \frac{d}{dt}\int_0^1 u(\log v)_x\, dx + \int_0^1 \frac{u_x^2}{v}\, dx + \int_0^1 \frac{p'(v)}{v}\, v_x^2\, dx - \int_0^1 f(Iv)(\log v)_x\, dx . \tag{3.12}$$

Put $\varphi(t) = (\mu/2)\int_0^1 [(\log v)_x - u/\mu]^2\, dx$. From (2.4), (2.6) we get

$$\frac{1}{2}\varphi(t) - c_1 \leqslant \frac{\mu}{2}\int_0^1 (\log v)_x^2\, dx \leqslant 2\varphi(t) + c_1 \tag{2.13}$$

with some constant c_1. The identity (3.12) yields

$$\varphi'(t) + \inf[-vp'(v)]\int_0^1 (\log v)_x^2\, dx \leqslant$$

$$\leqslant \int_0^1 \frac{u_x^2}{v}\, dx + \frac{1}{2\mu}\frac{d}{dt}\int_0^1 u^2\, dx + \varepsilon\int_0^1 (\log v)_x^2\, dx + c(\varepsilon)\sup|f|^2 .$$

Since $\inf[-vp'(v)] > 0$ in virtue of (3.9), (3.10), choosing $\varepsilon > 0$ sufficiently small and making use of (3.13) we find

$$\varphi'(t) + c_2\varphi(t) \leqslant c_3 + \int_0^1 \frac{u_x^2}{v}\, dx + \frac{1}{2\mu}\frac{d}{dt}\int_0^1 u^2\, dx \tag{3.14}$$

with some positive constants c_2, c_3. Multiplying (3.14) by $e^{c_2 t}$, integrating over $(0, t)$, multiplying the result by $e^{-c_2 t}$ and integrating by parts the last term we get

$$\varphi(t) \leqslant \varphi(0)\, e^{-c_2 t} + \frac{c_3}{c_2}(e^{c_2 t} - 1)\, e^{-c_2 t} + e^{-c_2 t}\int_0^t e^{c_2 s}\int_0^1 \frac{u_x(x, s)^2}{v(x, s)}\, dx\, ds + \tag{3.15}$$

$$+ \frac{1}{2\mu}\, e^{-c_2 t}\int_0^t e^{c_2 s}\frac{d}{ds}\int_0^1 u(s)^2\, dx\, ds \leqslant \varphi(0) + \frac{c_3}{c_2} + \int_0^t\int_0^1 \frac{u_x^2}{v}\, dx\, ds +$$

$$+ \frac{1}{2\mu}\int_0^1 u(t)^2\, dx - \frac{e^{-c_2 t}}{2\mu}\int_0^1 u(0)^2\, dx - \frac{c_2\, e^{-c_2 t}}{2\mu}\int_0^t e^{c_2 s}\int_0^1 u(s)^2\, dx\, ds \leqslant c_4 < \infty .$$

In the last inequality we have used (2.4), (2.6). Now, from (3.13), (3.15) the estimate

$$\int_0^1 (\log v)_x^2 \, dx \leqslant c_5 < \infty \tag{3.16}$$

follows. From (3.16) the estimate (3.11) is obtained in a standard manner (see e.g. [7], proof of Lemma 2.3). The estimate (3.11) for generalized solutions in the sense of Definition 2.1 can be obtained via weak* limit of smoother solutions.

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Manoscritto pervenuto in redazione il 18 marzo 1989.